Research Article

The Essential Norm of the Generalized Hankel Operators on the Bergman Space of the Unit Ball in $\mathbb{C}^n$

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In 1993, Peloso introduced a kind of operators on the Bergman space $A^2(B)$ of the unit ball that generalizes the classical Hankel operator. In this paper, we estimate the essential norm of the generalized Hankel operators on the Bergman space $A^p(B)$ ($p > 1$) of the unit ball and give an equivalent form of the essential norm.

1. Introduction

Let $B$ be the open unit ball in $\mathbb{C}^n$, $m$ the Lebesgue measure on $\mathbb{C}^n$ normalized so that $m(B) = 1$, $H(B)$ denotes the class of all holomorphic functions on $B$. The Bergman space $A^2(B)$ is the Banach space of all holomorphic functions $f$ on $B$ such that $\int_B |f(z)|^2 dm(z) < \infty$. It is easy to show that $A^2(B)$ is a closed subspace of $L^2(B, dm)$.

There is an orthogonal projection of $L^2(B, dm)$ onto $A^2(B)$, denoted by $P$ and

$$ Pf(z) = \int_B K(z, w)f(w)dm(w), \quad (1.1) $$

where $K(z, w) = 1/(1 - \langle z, w \rangle)^{n+1}$ is the Bergman kernel on $B$.

For a function $f \in H(B)$, define the Hankel operator $H_f : A^2(B) \to A^2(B)^\perp$ with symbol $f$ by

$$ H_fg = (I - P)(\overline{f}g) = \int_B (f(z) - \overline{f(w)})K(z, w)g(w)dm(w), \quad (1.2) $$

where $I$ is the identity operator.
Since the Hankel operator $H_f$ is connected with the Toeplitz operator, the commutator, the Bloch space, and the Besov space, it has been extensively studied. Important papers in this context are [1, 2] for the case $n = 1$ and [3–5] for the case $n > 1$. It is known that $H_f$ is bounded on $A^2(B)$ if and only if $f \in \beta(B)$ and $H_f$ is compact $A^2(B)$ if and only if $f \in \beta_0(B)$, where

$$\beta(B) = \left\{ f \in H(B) : \sup_{z \in B} \left( 1 - |z|^2 \right) |Rf(z)| < \infty \right\},$$

$$\beta_0(B) = \left\{ f \in H(B) : \left( 1 - |z|^2 \right) |Rf(z)| \rightarrow 0, \text{ as } |z| \rightarrow 1 \right\}.$$ (1.3)

$Rf$ is the radial derivative of $f$ defined by

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f(z)}{\partial z_j}. \quad (1.4)$$

$\beta(B)$ is called the Bloch space, and $\beta_0(B)$ is called the little Bloch space.

For $n = 1$, $f \in H(D)$ ($D$ is the open unit disc), $H_f$ is in the Schatten class $S_p$ ($1 < p < \infty$) if and only if $f \in B_p(D)$; $H_f \in S_p$ ($0 < p \leq 1$) if and only if $f$ is a constant, where

$$B_p(D) = \left\{ f \in H(D) : f'(z) \left( 1 - |z|^2 \right) \in L^p(d\lambda) \right\}, \quad p > 1,$$ (1.5)

and $d\lambda(z) = (1 - |z|^2)^{-2} dm(z)$ is the invariant volume measure on $D$, $B_p(D)$ is called the Besov space on $D$. This theorem expresses that there is a cutoff of $H_f$ at $p = 1$.

For $n > 1$, $f \in H(B)$, $H_f \in S_p$ ($2n < p < \infty$) if and only if $f \in B_p(B)$, $H_f \in S_p$ ($0 < p \leq 2n$) if and only if $f$ is a constant, where

$$B_p(B) = \left\{ f \in H(B) : \left( 1 - |z|^2 \right) Rf(z) \in L^p(d\lambda) \right\}, \quad p > n,$$ (1.6)

and $d\lambda(z) = (1 - |z|^2)^{-(n+1)} dm(z)$ is the invariant volume measure on $B$. $B_p(B)$ is called the Besov space on $B$. Then, the cutoff phenomenon of $H_f$ appears at $p = 2n$. If $c(n)$ denotes the value of “cutoff,” then

$$c(n) = \begin{cases} 1, & n = 1, \\ 2n, & n > 1. \end{cases} \quad (1.7)$$

Obviously, $c(n)$ depends on the dimension $n$ of the unit ball.

In 1993, Peloso [3] replaced $f(z) - f(w)$ with

$$\Delta_j f(w, z) = f(w) - \sum_{|\alpha| < j} \frac{D^\alpha f(z)}{\alpha!} (w - z)^\alpha$$ (1.8)
to define a kind of generalized Hankel operator:

\[ H_{f,j}g(z) = \int_B \Delta_j f(w,z) K(z,w) g(w) dm(w), \]
\[ H'_{f,j}g(z) = \int_B \bar{\Delta}_j f(z,w) K(z,w) g(w) dm(w). \]  

(1.9)

Here, \((D^a f)(z) = (\partial^{|\alpha|} f(z)) / (\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n})\). Clearly, if \( j = 1 \), \( H_{f,1} \) and \( H'_{f,1} \) are just the classical Hankel operator \( H_f \). He proved that \( H_{f,j} \) has the same boundedness and compactness properties as \( H_f \), but the Schatten class property of \( H_{f,j} \) is different from that of \( H_f \). If \( n \geq 2 \), \( f \in H(B), H_{f,j} \in S_p((2n/j) < p < \infty) \) if and only if \( f \in B_p(B) \); if \( 0 < p \leq (2n/j) \), \( H_{f,j} \in S_p \) if and only if \( f \) is a polynomial of degree at most \( j - 1 \). So the value of “cutoff” of \( H_{f,j} \) is \( 2n/j \); this means that the cutoff constant \( c(n) \) depends not only on the dimension but also on the degree of the polynomial

\[ \sum_{|\alpha|<j} \frac{D^a f(z)}{a!} (w-z)^a, \]  

(1.10)

and we are able to lower the cutoff constant by increasing \( j \).

The cutoff phenomenon expressed that the generalized Hankel operator \( H_{f,j} \) defined by Peloso and the classical Hankel operator \( H_f \) are different.

In the present paper, we will consider the generalized Hankel operators \( H_{f,j} \) defined by Peloso on the Bergman space \( A^p(B) \) which is the Banach space of all holomorphic functions \( f \) on \( B \) such that \( \int_B |f(z)|^p dm(z) < \infty \), for \( p > 1 \).

For \( f(z) \in H(B) \), \( j \) is a positive integer, and we define the generalized Hankel operators \( H_{f,j} \) and \( H'_{f,j} \) of order \( j \) with symbol \( f \) by

\[ H_{f,j}g(z) = \int_B \Delta_j f(w,z) K(z,w) g(w) dm(w), \]
\[ H'_{f,j}g(z) = \int_B \bar{\Delta}_j f(z,w) K(z,w) g(w) dm(w), \]  

(1.11)

where \( g \in A^p(B) \),

\[ \Delta_j f(w,z) = f(w) - \sum_{|\alpha|<j} \frac{D^a f(z)}{a!} (w-z)^a, \]
\[ \bar{\Delta}_j f(z,w) = f(w) - \sum_{|\alpha|<j} \frac{D^a f(w)}{a!} (z-w)^a. \]  

(1.12)

Luo and Ji-Huai [6] studied the boundedness, compactness, and the Schatten class property of the generalized Hankel operator \( H_{f,j} \) on the Bergman space \( A^p(B) \) \((p > 1)\), which extended the known results.
We will study the essential norm of this kind of generalized Hankel operators $H_{f,j}$ and $H'_{f,j}$. We recall that the essential norm of a bounded linear operator $T$ is the distance from $T$ to the compact operators; that is,

$$\|T\|_{\text{ess}} = \inf\{\|T - K\| : K \text{ is a compact operator}\}.$$  (1.13)

The essential norm of a bounded linear operator $T$ is connected with the compactness of the operator $T$ and the spectrum of the operator $T$.

We know that $\|T\|_{\text{ess}} = 0$ if and only if $T$ is compact, so that estimates on $\|T\|_{\text{ess}}$ lead to conditions for $T$ to be compact. Thus, we will obtain a different proof of the compactness of the generalized Hankel operators $H_{f,j}$ and $H'_{f,j}$.

Throughout the paper, $C$ denotes a positive constant, whose value may change from one occurrence to the next one.

2. Preliminaries

For any fixed point $a \in B - \{0\}$, $z \in B$, define the Möbius transformation $\varphi_a$ by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle},$$  (2.1)

where $s_a = \sqrt{1 - |a|^2}$ and $P_a$ is the orthogonal projection from $C^n$ onto the one-dimensional subspace $[a]$ generated by $a$, $Q_a$ is the orthogonal projection from $C^n$ onto $C^n![a]$. It is clear that

$$P_a(z) = \frac{\langle z, a \rangle}{|a|^2} a, \quad z \in C^n,$$

$$Q_a(z) = z - \frac{\langle z, a \rangle}{|a|^2} a, \quad z \in B.$$

Lemma 2.1. For every $a \in B$, $\varphi_a$ has the following properties:

1. $\varphi_a(0) = a$ and $\varphi_a(a) = 0$,
2. $\varphi_a \circ \varphi_a(z) = z, \quad z \in B$,
3. $1/(1 - \langle \varphi_a(z), a \rangle) = (1 - \langle z, a \rangle)/(1 - |a|^2), \quad z \in B$.

Proof. The proofs can be found in [7].

Lemma 2.2. For $s > -1$, $t$ real, define

$$I_t(z) = \int_B \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{n+1+st}} dm(w), \quad z \in B.$$  (2.3)
Then,

(1) $t < 0$, $I_t(z)$ is bounded in $B$,
(2) $t = 0$, $I_t(z) \sim \log(1/(1 - |z|^2))$ as $|z| \to 1^-$,
(3) $t > 0$, $I_t(z) \sim (1 - |z|^2)^{-t}$ as $|z| \to 1^-$.

Here, the notation $a(z) \sim b(z)$ means that the ratio $a(z)/b(z)$ has a positive finite limit as $|z| \to 1^-$.

**Proof.** This is in [7, Theorem 1.12].

**Lemma 2.3.** Let $k(z) = K(z, \xi)/\|K\|_{L^p(d\mu)}$, where $K(z) = K(z, \xi) = 1/(1 - \langle z, \xi \rangle)^{n+1}$, then $k(z)$ has the following properties:

(1) $\|k\|_{L^p(d\mu)} = 1$,
(2) $k(z) \to 0$ at every point $z \in B$ as $|\xi| \to 1^-$.

**Proof.** It is obvious.

**Lemma 2.4.** Let $K(z) = K(z, \xi)$. Then, for any positive integer $j$,

(1) $H_{f,k} = -\Delta f_{k}^\ast K_k$,
(2) $H_{f,k} = \Delta f_{k}^\ast K_k$.

**Proof.** The proof is obtained by the definition of $H_{f,k}$ and $H_{f,k}$ and the reproducing property of $K(z, \xi)$, through the direct computation to get them.

**Lemma 2.5.** Let $j$ be any positive integer, $f \in H(B)$, and $0 < q < \infty$, then there is a constant $C$ independent of $f$, such that

(1) $\left| f_j(z) \right|^2 \left| R^j f(z) \right| \leq C \int_B \left| D_j f(z, w) \right|^q d\mu(w)$,
(2) $\left| f_j(z) \right|^2 \left| R^j f(z) \right| \leq C \int_B \left| D_j f(z, w) \right|^q d\mu(w)$,

where $R^j f$ is the $j$th order radial derivative of $f$,

$$R^j f(z) = \sum_{k=1}^{\infty} k^j f_k(z), \quad (2.4)$$

and $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion.

**Proof.** This is in [3, Proposition 3.2].

**Lemma 2.6.** Let $j$ be any positive integer, $f \in H(B)$, and $0 < \rho < 1$, $p > 1$, then

(1) $\int_B \left| D_j f(z, \omega) \right|^p \left| (1 - |\omega|^2)^{-\rho} / |1 - \langle z, \omega \rangle|^{n+1} d\mu(\omega) \leq C (1 - |z|^2)^{-\rho} (\sup_{z \in B} (1 - |z|^2)|R^j f(z)|)^p$,
(2) $\int_B \left| D_j f(z, \omega) \right|^p \left| (1 - |\omega|^2)^{-\rho} / |1 - \langle z, \omega \rangle|^{n+1} d\mu(\omega) \leq C (1 - |z|^2)^{-\rho} (\sup_{z \in B} (1 - |z|^2)|R^j f(z)|)^p$.
Proof. (1) Write \( F(w, z) \) for \( \Delta_j f(w, z) \). Using the change of variables \( w = q(z, \xi) \), we obtain

\[
\int_B |F(w, z)|^p \frac{(1 - |w|^2)^{-p}}{|1 - \langle z, w \rangle|^{n+1}} dm(w)
\]

\[
= \int_B |F(q_z(\xi), z)|^p \frac{(1 - |q_z(\xi)|^2)^{-p}}{|1 - \langle z, q_z(\xi) \rangle|^{n+1} |1 - \langle \xi, z \rangle|^{2(n+1)}} dm(\xi)
\]

\[
= (1 - |z|^2)^{-p} \int_B |F(q_z(\xi), z)|^p \frac{(1 - |\xi|^2)^{-p}}{|1 - \langle \xi, z \rangle|^{n+2p}} dm(\xi). \tag{\star}
\]

Let

\[
1 < q' < \min \left( \frac{1}{\rho'}, \frac{n+1}{n+1-\rho} \right) \tag{2.5}
\]

and set \( q = q'/(q' - 1) \). Then, applying Hölder’s inequality to \( \star \), we obtain

\[
\int_B |F(w, z)|^p \frac{(1 - |w|^2)^{-p}}{|1 - \langle z, w \rangle|^{n+1}} dm(w)
\]

\[
\leq (1 - |z|^2)^{-p} \left( \int_B |F(q_z(\xi), z)|^{pq} dm(\xi) \right)^{1/q} \left( \int_B \frac{(1 - |\xi|^2)^{-p}}{|1 - \langle \xi, z \rangle|^{n+2p}} dm(\xi) \right)^{1/q'.} \tag{2.6}
\]

Because of our choice of \( q' \), it follows that \(-pq' > -1\) and \((n+1 - 2\rho)q' < n+1 - \rho q'\). Now, Lemma 2.2 implies that

\[
\int_B \frac{(1 - |\xi|^2)^{-pq}}{|1 - \langle \xi, z \rangle|^{n+2p}} dm(\xi) \tag{2.7}
\]

is bounded by a constant. Therefore, applying [3, Theorem 3.4], we get

\[
\int_B |\Delta_j f(w, z)|^p \frac{(1 - |w|^2)^{-p}}{|1 - \langle z, w \rangle|^{n+1}} dm(w) \leq C \left( \sup_{z \in B} \left( 1 - |z|^2 \right)^{-p} \right)^p \left( R^1 f(z) \right)^p. \tag{2.8}
\]

(2) The proof of (2) is similar to that of (1). \(\square\)
3. The Main Result and Its Proof

Theorem 3.1. Let \( f \in H(B) \), \( j \) any positive integer, \( p > 1 \), and the generalized Hankel operators \( H_{f,j}, H'_{f,j} \) defined on \( A^p(B) \) by

\[
H_{f,j}g(z) = \int_B -\Delta_j f(w,z)K(z,w)g(w)dm(w),
\]

\[
H'_{f,j}g(z) = \int_B \overline{\Delta_j f(z,w)}K(z,w)g(w)dm(w).
\]

Suppose that \( H_{f,j} \) and \( H'_{f,j} \) are bounded on \( A^p(B) \), then the following quantities are equivalent:

1. \( \|H_{f,j}\|_{ess} \) and \( \|H'_{f,j}\|_{ess} \).
2. \( \lim_{|z| \to 1} (1 - |z|^2)^j |R^j f(z)| \).
3. \( \lim_{|z| \to 1} (1 - |z|^2)^j |R f(z)| \).

Particularly, \( H_{f,j} \) and \( H'_{f,j} \) are compact on \( A^p(B) \) if and only if \( \lim_{|z| \to 1} (1 - |z|^2)^j |R f(z)| = 0 \).

Proof. First, we will prove that \( \|H_{f,j}\|_{ess} \geq C \lim_{|z| \to 1} (1 - |z|^2)^j |R^j f(z)| \). By the definition of \( k_\xi(z) \) of Lemmas 2.3 and 2.4, we have

\[
\|H_{f,j}k_\xi(z)\|_{L^p(dm)}^p = \int_B |H_{f,j}k_\xi(z)|^p dm(z)
\]

\[
= \int_B \left| H_{f,j} \right|_{L^p(dm)} \left| K(z,\xi) \right|^p dm(z)
\]

\[
= \frac{1}{\|K_\xi\|_{L^p(dm)}} \int_B |H_{f,j}K(z,\xi)|^p dm(z)
\]

\[
= \frac{1}{\|K_\xi\|_{L^p(dm)}} \int_B |\Delta_j f(\xi, z)|^p |K(z,\xi)|^p dm(z)
\]

\[
= \frac{1}{\|K_\xi\|_{L^p(dm)}} \cdot I,
\]

where \( I = \int_B |\Delta_j f(\xi, z)|^p |K(z,\xi)|^p dm(z) \).

Use the change of variables \( z = \phi_\xi(\tau) \) in the integral \( I \), and recall that

\[
dm(z) = \left( \frac{1 - |\xi|^2}{|1 - \langle \tau, \xi \rangle|^2} \right)^{n+1} dm(\tau).
\]
Thus

\[ I = \int_B \frac{|\Delta_j f(\xi, \varphi_\xi(\tau))|}{|1 - \langle \varphi_\xi(\tau), \xi \rangle|^{p(n+1)}} \cdot \left( \frac{1 - |\xi|^2}{|1 - \langle \tau, \xi \rangle|^2} \right)^{n+1} \ dm(\tau) \]

\[ = \frac{1}{(1 - |\xi|^2)^{(n+1)(p-1)}} \int_B \frac{|\Delta_j f(\xi, \varphi_\xi(\tau))|^p}{|1 - \langle \tau, \xi \rangle|^{(2-p)(n+1)}} \ dm(\tau) \]

\[ \geq \frac{1}{(1 - |\xi|^2)^{(n+1)(p-1)}} \left( \int_B |\Delta_j f(\xi, \varphi_\xi(\tau))| \ dm(\tau) \right)^p 
\times \left( \int_B \frac{1}{|1 - \langle \tau, \xi \rangle|^{(2-p)(n+1)/(1-p)}} \ dm(\tau) \right)^{1-p} \geq \frac{C}{(1 - |\xi|^2)^{(n+1)(p-1)}} \left( \int_B |\Delta_j f(\xi, \varphi_\xi(\tau))| \ dm(\tau) \right)^p 
\geq \frac{C}{(1 - |\xi|^2)^{(n+1)(p-1)}} \left[ \left( 1 - |\xi|^2 \right)^j |R^j f(\xi)| \right]^p. \]

(3.4)

Here, we have used (3) of Lemma 2.1, Hölder’s inequality for the indexes \( p \) and \( p/(p-1) \), (1) of Lemma 2.2, and (2) of Lemma 2.5.

Therefore,

\[ \|H_{f,j}k_\xi\|^p_{L^p(dm)} \geq \frac{1}{\|K_\xi\|_{L^p(dm)}} \frac{C}{(1 - |\xi|^2)^{(n+1)(p-1)}} \left[ \left( 1 - |\xi|^2 \right)^j |R^j f(\xi)| \right]^p 
\geq C \left( 1 - |\xi|^2 \right)^{(n+1)(p-1)} \frac{1}{(1 - |\xi|^2)^{(n+1)(p-1)}} \left[ \left( 1 - |\xi|^2 \right)^j |R^j f(\xi)| \right]^p \geq C \left[ \left( 1 - |\xi|^2 \right)^j |R^j f(\xi)| \right]^p. \]

(3.5)

So \( \|H_{f,j}k_\xi\|_{L^p(dm)} \geq C \left( 1 - |\xi|^2 \right)^j |R^j f(\xi)|. \)
Abstract and Applied Analysis

For any compact operator $T$, by (2) of Lemma 2.3, we have $\|Tk_k\|_{L^p(dm)} \to 0$ as $|\xi| \to 1$. Then,

$$\|H_{f,j} - T\| \geq \lim_{|\xi| \to 1} \| (H_{f,j} - T) k_k \|_{L^p(dm)} $$

$$\geq \lim_{|\xi| \to 1} \left( \|H_{f,j} k_k\|_{L^p(dm)} - \|Tk_k\|_{L^p(dm)} \right) $$

$$= \lim_{|\xi| \to 1} \|H_{f,j} k_k\|_{L^p(dm)} $$

$$\geq C \lim_{|\xi| \to 1} \left( 1 - |\xi|^2 \right)^{1/p} |R^f(\xi)|. $$

Thus, $\|H_{f,j}\|_{\text{ess}} \geq C \lim_{|z| \to 1} (1 - |z|^2)^{1/p} |R^f(z)|$.

Now, we will prove that $\|H_{f,j}\|_{\text{ess}} \leq C \lim_{|z| \to 1} (1 - |z|^2)^{1/p} |R^f(z)|$.

Write $F(z, w)$ for $-\Delta f(z, w)$. For $0 < \rho < 1$ and $g \in A^p(B)$, let $B(0, \rho)$ and $B(0, \rho, 1)$ denote the ball $|z| \leq \rho$ and the ring $\rho < |z| < 1$, respectively, then we have

$$H_{f,j}g(z) = \chi_{B(0, \rho)}(z) \int_B \overline{F(w, z)} K(z, w) g(w) dm(w) $$

$$+ \chi_{B(0, \rho, 1)}(z) \int_B \overline{F(w, z)} K(z, w) g(w) dm(w) $$

$$= T_1g(z) + T_2g(z). $$

Here,

$$T_1g(z) = \chi_{B(0, \rho)}(z) \int_B \overline{F(w, z)} K(z, w) g(w) dm(w), $$

$$T_2g(z) = \chi_{B(0, \rho, 1)}(z) \int_B \overline{F(w, z)} K(z, w) g(w) dm(w). $$

We first show that $T_1$ is compact. Let $\{g_n\}$ be a sequence weakly converging to 0 and $p' = p/(p - 1)$, by Hölder’s inequality, then we have

$$|T_1g(z)|^p = \left| \chi_{B(0, \rho)}(z) \int_B \overline{F(w, z)} K(z, w) g(w) dm(w) \right|^p $$

$$\leq \chi_{B(0, \rho)}(z) \left( \int_B |F(w, z)| \frac{|g(w)|}{|1 - (z, w)|^{n+1}} dm(w) \right)^p $$

$$\leq \chi_{B(0, \rho)}(z) \left( \int_B \frac{|F(w, z)|^p (1 - |w|^2)^{-1/p}}{|1 - (z, w)|^{n+1}} dm(w) \right)^{p/p'} $$

$$\times \int_B |g(w)|^p (1 - |w|^2)^{1/p'} $$

$$\frac{1}{|1 - (z, w)|^{n+1}} dm(w). $$
By Lemma 2.6, we get

$$
|T_1g_i(z)|^p \leq C_{\chi_B(0,p)}(z) \left( \sup_{z \in B} (1 - |z|^2)^j |R^j f(z)| \right)^p
$$

$$
\times \left( 1 - |z|^2 \right)^{-1/p'} \int_B |g_i(w)|^p \left( 1 - |w|^2 \right)^{1/p'} \frac{1}{|1 - \langle z, w \rangle|^{n+1}} dm(w).
$$

Thus,

$$
\|T_1g_i\|_{L^p(d\mu)}^p = \int_B |T_1g_i(z)|^p d\mu(z)
$$

$$
\leq C \int_{|z| < \rho} \left( \sup_{z \in B} (1 - |z|^2)^j |R^j f(z)| \right)^p \left( 1 - |z|^2 \right)^{-1/p'}
$$

$$
\times \int_B |g_i(w)|^p \left( 1 - |w|^2 \right)^{1/p'} \frac{1}{|1 - \langle z, w \rangle|^{n+1}} dm(w) dm(z)
$$

$$
\leq C \left( 1 - |\rho|^2 \right)^{-1/p'} \left( \sup_{z \in B} (1 - |z|^2)^j |R^j f(z)| \right)^p
$$

$$
\times \int_B |g_i(w)|^p \left( 1 - |w|^2 \right)^{1/p'} \int_B \frac{1}{|1 - \langle z, w \rangle|^{n+1}} dm(z) dm(w)
$$

$$
\leq C \left( 1 - |\rho|^2 \right)^{-1/p'} \left( \sup_{z \in B} (1 - |z|^2)^j |R^j f(z)| \right)^p
$$

$$
\times \int_B |g_i(w)|^p \left( 1 - |w|^2 \right)^{1/p'} \log(1 - |w|^2) dm(w)
$$

$$
\rightarrow 0, \quad \text{as} \ l \rightarrow \infty.
$$

(3.11)

So, $T_1$ is compact.

For $g \in A^p$ and $p' = p/(p - 1)$, by Hölder’s inequality,

$$
|T_2g(z)|^p = \left| \chi_{\mathbb{B}(0,p,1)}(z) \int_B \frac{F(w, z)K(z, w)g(w) dm(w)}{|F(w, z)|^{n+1}} \right|^p
$$

$$
\leq \left( \int_B \chi_{\mathbb{B}(0,p,1)}(z)|F(w, z)| \frac{|g(w)|}{|1 - \langle z, w \rangle|^{n+1}} dm(w) \right)^p
$$
\begin{align*}
&\leq \left( \int_B \mathcal{X}_{B(0,1)}(z) \frac{|F(w, z)|^{p'} (1 - |w|^2)^{-1/p}}{|1 - \langle z, w \rangle|^{n+1}} dm(w) \right)^{p/p'} \\
&\times \int_B |g(w)|^{p} \frac{(1 - |w|^2)^{1/p'}}{|1 - \langle z, w \rangle|^{n+1}} dm(w).
\end{align*}

(3.12)

So

\begin{align*}
\|T_2 g\|_{L^p(B,dm)}^p &= \int_B |T_2 g(z)|^p dm(z) \\
&\leq \int_B \left( \int_B \mathcal{X}_{B(0,1)}(z) \frac{|F(w, z)|^{p'} (1 - |w|^2)^{-1/p}}{|1 - \langle z, w \rangle|^{n+1}} dm(w) \right)^{p/p'} \\
&\times \int_B |g(w)|^{p} \frac{(1 - |w|^2)^{1/p'}}{|1 - \langle z, w \rangle|^{n+1}} dm(w) dm(z).
\end{align*}

(3.13)

Change the variables \( w = \varphi_z(\xi) \), let

\begin{equation}
1 < q' < \min\left(p, \frac{n + 1}{n + 1 - 1/p}\right),
\end{equation}

and set \( q = q'(q' - 1) \), by Lemmas 2.1 and 2.2, then we obtain

\begin{align*}
\int_B \mathcal{X}_{B(0,1)}(z) \frac{|F(w, z)|^{p'} (1 - |w|^2)^{-1/p}}{|1 - \langle z, w \rangle|^{n+1}} dm(w) \\
&\leq \int_B \mathcal{X}_{B(0,1)}(z) \frac{|F(\varphi_z(\xi), z)|^{p'} (1 - |\varphi_z(\xi)|^2)^{-1/p}}{|1 - \langle z, \varphi_z(\xi) \rangle|^{n+1}} \left(1 - |\xi|^2\right)^{n+1} \frac{(1 - |\xi|^2)^{-1/p}}{|1 - \langle \xi, z \rangle|^{2(n+1)}} dm(\xi) \\
&\leq \left(1 - |z|^2\right)^{-1/p} \left( \int_B \mathcal{X}_{B(0,1)}(z) |F(\varphi_z(\xi), z)|^{p'} dm(\xi) \right)^{1/q} \\
&\times \left( \int_B \frac{(1 - |\xi|^2)^{-q/p}}{|1 - \langle \xi, z \rangle|^{(n+1-2/p)q}} dm(\xi) \right)^{1/q} \\
&\leq C \left(1 - |z|^2\right)^{-1/p} \left( \int_B \mathcal{X}_{B(0,1)}(z) |F(\varphi_z(\xi), z)|^{p'q} dm(\xi) \right)^{1/q}.
\end{align*}

(3.15)
By the same argument of [3, Theorem 3.4], we know that

$$\left( \int_B \chi_B(z) |F(\varphi_z(\xi), z)|^{p'/q} \, dm(\xi) \right)^{1/q} \leq C \left( \sup_{\rho<|z|<1} \left( 1 - |z|^2 \right) \left| R^i f(z) \right| \right)^{p'}. \tag{3.16}$$

Applying Fubini’s theorem and Lemma 2.2, we have

$$\| T_2 g \|_{L^p(dm)}^p \leq C \left( \sup_{\rho<|z|<1} \left( 1 - |z|^2 \right) \left| R^i f(z) \right| \right)^p$$

$$\times \int_B \left( 1 - |z|^2 \right)^{-1/p'} \left( \frac{|g(\omega)|^p \left( 1 - |\omega|^2 \right)^{1/p'}}{|1 - (z, \omega)|^{n+1}} \right) \, dm(\omega) \, dm(z) \tag{3.17}$$

$$\leq C \left( \sup_{\rho<|z|<1} \left( 1 - |z|^2 \right) \left| R^i f(z) \right| \right)^p \| g \|_{L^p(dm)}^p.$$ 

So

$$\| T_2 \| \leq C \sup_{\rho<|z|<1} \left( 1 - |z|^2 \right)^{1} \left| R^i f(z) \right|. \tag{3.18}$$

Thus, by the definition of the essential norm, we have

$$\| H_{f,j} \|_{ess} \leq \| T_1 + T_2 \|_{ess} \leq \| T_2 \| \leq C \sup_{\rho<|z|<1} \left( 1 - |z|^2 \right) \left| R^i f(z) \right|. \tag{3.19}$$

As $\rho \to 1$, we have

$$\| H_{f,j} \|_{ess} \leq \lim_{|z| \to 1^-} \left( 1 - |z|^2 \right)^{1} \left| R^i f(z) \right|. \tag{3.20}$$

Similarly, we get the equality of $\| H_{f,j} \|_{ess} = \lim_{|z| \to 1^-} \left( 1 - |z|^2 \right)^{1} \left| R^i f(z) \right|$. By [7, Theorems 3.4 and 3.5], we obtain the equality of $\lim_{|z| \to 1^-} \left( 1 - |z|^2 \right)^{1} \left| R^i f(z) \right|$ and $\lim_{|z| \to 1^-} \left( 1 - |z|^2 \right) |Rf(z)|$.

We complete the proof of Theorem 3.1. \qed

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References


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