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Research Article

On a Higher-Order Nonlinear Difference Equation

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This paper shows that all positive solutions of a higher-order nonlinear difference equation are bounded, extending some recent results in the literature.

1. Introduction

There is a considerable interest in studying nonlinear difference equations nowadays; see, for example, [1–40] and numerous references listed therein.

The investigation of the higher-order nonlinear difference equation

$$x_n = A + \frac{x_{n-m}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $A, r > 0$ and $p \geq 0$, and $k, m \in \mathbb{N}$, $k \neq m$, was suggested by Stević at numerous talks and in papers (see, e.g., [20, 28, 30, 34–38] and the related references therein).

In this paper we show that under some conditions on parameters A, r, and p all positive solutions of the difference equation

$$x_n = A + \frac{x_{n-1}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $k \in \mathbb{N} \setminus \{1\}$, are bounded. To do this we modify some methods and ideas from Stević's papers [30, 35–37]. Our motivation stems from these four papers.

The reader can find results for some particular cases of (1.2), as well as on some closely related equations treated in, for example, [1, 2, 5–11, 18–20, 26, 30, 33–35, 38, 40].

2. Main Result

Here we investigate the boundedness of the positive solutions to (1.2) for the case $0 < p < (rk^k/(k-1)^{k-1})^{1/k}$. The following result completely describes the boundedness of positive solutions to (1.2) in this case. The result is an extension of one of the main results in [35].

Theorem 2.1. *Assume, $p, r > 0$ and $k \in \mathbb{N} \setminus \{1\}$. Then every positive solution of* (1.2) *is bounded if*

$$0 < p < \left(\frac{rk^k}{(k-1)^{k-1}}\right)^{1/k}. \tag{2.1}$$

Proof. First note that from (1.2) it directly follows that

$$x_n > A, \quad \text{for } n \in \mathbb{N}_0. \tag{2.2}$$

Using (1.2), it follows that

$$\begin{aligned}
x_n &= A + \frac{x_{n-1}^p}{x_{n-k}^r} \\
&= A + \left(\frac{x_{n-1}}{x_{n-k}^{r/p}}\right)^p \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \frac{x_{n-2}^p}{x_{n-k}^{r/p} x_{n-k-1}^r}\right)^p \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{x_{n-2}}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right)^p\right)^p \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \frac{x_{n-3}^p}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p} x_{n-k-2}^r}\right)^p\right)^p \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{x_{n-3}}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right)^p\right)^p\right)^p.
\end{aligned} \tag{2.3}$$

After k steps we obtain the following formula

$$\begin{aligned}
x_n &= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p}} + \frac{x_{n-k}^{p}}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p} x_{n-(2k-1)}^{r}} \right)^p \cdots \right)^p \right. \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p}} + \frac{x_{n-k}^{p-(r/p^{k-1})}}{x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p} x_{n-(2k-1)}^{r}} \right)^p \cdots \right)^p \right. .
\end{aligned} \tag{2.4}$$

Two subcases can be considered now.

Case 1 ($r \geq p^k$). If $r \geq p^k$, then by (2.2) equality (2.4) implies that

$$\begin{aligned}
x_n &< A + \left(\frac{A}{A^{r/p}} + \left(\frac{A}{A^{r/p^2+r/p}} + \left(\frac{A}{A^{r/p^3+r/p^2+r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{A^{r/p^{k-1}+r/p^{k-2}+\cdots+r/p}} + \frac{1}{A^{r/p^{k-1}+r/p^{k-2}+\cdots+r/p+r-p}} \right)^p \cdots \right)^p \right. < \infty,
\end{aligned} \tag{2.5}$$

for $n \geq 2k-1$. This means that (x_n) is a bounded sequence.

Case 2 ($p^k > r$). In this case we have

$$p - \frac{r}{p^{k-1}} > 0. \tag{2.6}$$

From (2.4) and (1.2) we further obtain

$$
\begin{aligned}
x_n &= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p}} + \frac{x_{n-k}^{p-r/p^{k-1}}}{x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p} x_{n-(2k-1)}^{r}} \right)^p \cdots \right)^p \right. \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-j}^{z_0^{(j)}}} + \frac{x_{n-k}^{p-z_0^{(0)}}}{\left(\prod_{j=1}^{k-2} x_{n-k-j}^{z_0^{(j)}}\right) x_{n-(2k-1)}^{r}} \right)^p \cdots \right)^p \right. \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-j}^{z_0^{(j)}}} + \left(\frac{x_{n-k}}{\left(\prod_{j=1}^{k-2} x_{n-k-j}^{z_0^{(j)}/p-z_0^{(0)}}\right) x_{n-(2k-1)}^{r/(p-z_0^{(0)})}} \right)^{p(p-z_0^{(0)})} \right)^p \cdots \right)^p \right. \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-1-j}^{z_1^{(j)}}} + \frac{x_{n-k-1}^{p-z_1^{(0)}}}{\prod_{j=1}^{k-2} x_{n-k-1-j}^{z_1^{(j)}} x_{n-2k}^{r}} \right)^{p(p-z_0^{(0)})} \right)^p \cdots \right)^p \\
&= \cdots = A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-m-j}^{z_m^{(j)}}} + \frac{x_{n-k-m}^{p-z_m^{(0)}}}{\left(\prod_{j=1}^{k-2} x_{n-k-m-j}^{z_m^{(j)}}\right) x_{n-2k+1-m}^{r}} \right)^{p\prod_{i=0}^{m-1}(p-z_i^{(0)})} \right)^p \cdots \right)^p ,
\end{aligned}
\tag{2.7}
$$

for each $k \in \mathbb{N} \setminus \{1\}$ and every $n \geq 2k + m - 1$, where the sequences $(z_m^{(j)})$, $j = 0, 1, \ldots, k-2$, satisfy the system

$$z_{m+1}^{(0)} = \frac{z_m^{(1)}}{p - z_m^{(0)}}, z_{m+1}^{(1)} = \frac{z_m^{(2)}}{p - z_m^{(0)}}, \ldots, z_{m+1}^{(k-3)} = \frac{z_m^{(k-2)}}{p - z_m^{(0)}}, z_{m+1}^{(k-2)} = \frac{r}{p - z_m^{(0)}}, \tag{2.8}$$

and the initial values are given by

$$z_0^{(j)} = rp^{j+1-k}, \quad j = 0, 1, \ldots, k-2. \tag{2.9}$$

Note that $p^k > r$ implies that $z_0^{(0)} < p$. Assume $z_m^{(0)} < p$ for every $m \in \mathbb{N}_0$.

By a direct calculation it follows that $z_0^{(j)} < z_1^{(j)}$, $j = 0, 1, \ldots, k-2$, which, along with (2.8) implies that $(z_m^{(j)})$, $j = 0, 1, \ldots, k-2$, are strictly increasing sequences.

From system (2.8), we have,

$$z_{m+1}^{(0)} = \frac{r}{\left(p - z_m^{(0)}\right)\left(p - z_{m-1}^{(0)}\right) \cdots \left(p - z_{m-k+2}^{(0)}\right)}, \quad m \geq k-2. \tag{2.10}$$

If it were $z_m^{(0)} < p$, $m \in \mathbb{N}_0$, then there was

$$\lim_{m \to \infty} z_m^{(0)} = z \in (0, p]. \tag{2.11}$$

Clearly z is a solution of the equation

$$f(x) = x(p - x)^{k-1} - r = 0. \tag{2.12}$$

Since

$$f(0) = f(p) = -r, \tag{2.13}$$

and

$$f'(x) = (p - x)^{k-2}(p - kx), \tag{2.14}$$

we see that the function f attains its maximum at the point $x = p/k$.

Further, by assumption (2.1) we get

$$f\left(\frac{p}{k}\right) = \frac{(k-1)^{k-1}}{k^k}\left(p^k - r\frac{k^k}{(k-1)^{k-1}}\right) < 0, \tag{2.15}$$

which along with (2.13) implies that (2.12) does not have solutions on $(0, p]$, arriving at a contradiction.

This implies that there is a fixed index $m_0 \in \mathbb{N}$ such that

$$z_{m_0-1}^{(0)} < p, \quad z_{m_0}^{(0)} \geq p. \tag{2.16}$$

From this, inequality (2.2), and identity (2.7) with $m = m_0$, it follows that

$$\begin{aligned}
x_n &= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-m_0-j}^{z_{m_0}^{(j)}}} + \frac{x_{n-k-m_0}^{p-z_{m_0}^{(0)}}}{\left(\prod_{j=1}^{k-2} x_{n-k-m_0-j}^{z_{m_0}^{(j)}}\right) x_{n-2k+1-m_0}^{r}} \right)^{p\prod_{i=0}^{m_0-1}(p-z_i^{(0)})} \right)^{p} \cdots \right)^{p} \\
&\leq A + \left(\frac{A}{A^{r/p}} + \left(\frac{A}{A^{r/p^2+r/p}} + \left(\frac{A}{A^{r/p^3+r/p^2+r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{A^{\sum_{j=0}^{k-2} z_{m_0}^{(j)}}} + \frac{1}{A^{r+z_{m_0}^{(0)}-p+\sum_{j=1}^{k-2} z_{m_0}^{(j)}}} \right)^{p\prod_{i=0}^{m_0-1}(p-z_i^{(0)})} \right)^{p} \cdots \right)^{p} < \infty
\end{aligned} \tag{2.17}$$

for $n \geq 2k + m_0 - 1$.

From (2.17) the boundedness of the sequence (x_n) directly follows, as desired. □

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