Research Article

On a Higher-Order Nonlinear Difference Equation

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This paper shows that all positive solutions of a higher-order nonlinear difference equation are bounded, extending some recent results in the literature.

1. Introduction

There is a considerable interest in studying nonlinear difference equations nowadays; see, for example, [1–40] and numerous references listed therein.

The investigation of the higher-order nonlinear difference equation

\[ x_n = A + \frac{x_{n-m}}{x_{n-k}}, \quad n \in \mathbb{N}_0, \]  

(1.1)

where \( A, r > 0 \) and \( p \geq 0 \), and \( k, m \in \mathbb{N}, k \neq m \), was suggested by Stević at numerous talks and in papers (see, e.g., [20, 28, 30, 34–38] and the related references therein).

In this paper we show that under some conditions on parameters \( A, r, \) and \( p \) all positive solutions of the difference equation

\[ x_n = A + \frac{x_{n-1}}{x_{n-k}}, \quad n \in \mathbb{N}_0, \]  

(1.2)

where \( k \in \mathbb{N} \setminus \{1\} \), are bounded. To do this we modify some methods and ideas from Stević’s papers [30, 35–37]. Our motivation stems from these four papers.
The reader can find results for some particular cases of (1.2), as well as on some closely related equations treated in, for example, [1, 2, 5–11, 18–20, 26, 30, 33–35, 38, 40].

2. Main Result

Here we investigate the boundedness of the positive solutions to (1.2) for the case \(0 < p < (rk^k/(k-1)^{k-1})^{1/k}\). The following result completely describes the boundedness of positive solutions to (1.2) in this case. The result is an extension of one of the main results in [35].

**Theorem 2.1.** Assume, \(p, r > 0\) and \(k \in \mathbb{N} \setminus \{1\}\). Then every positive solution of (1.2) is bounded if

\[
0 < p < \left(\frac{rk^k}{(k-1)^{k-1}}\right)^{1/k}.
\]  

**Proof.** First note that from (1.2) it directly follows that

\[x_n > A, \quad \text{for } n \in \mathbb{N}_0.\]  

Using (1.2), it follows that

\[
x_n = A + \frac{x_{n-1}^p}{x_{n-k}^r} = A + \left(\frac{x_{n-1}}{x_{n-k}^{r/p}}\right)^p = A + \left(\frac{A}{x_{n-k}^{r/p}} + \frac{x_{n-2}^p}{x_{n-k}^{r/p}x_{n-k-1}^x}\right)^p = A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{x_{n-2}}{x_{n-k}^{r/p}x_{n-k-1}^x}\right)^p\right)^p
\]  

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\]  

\[
= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{x_{n-2}}{x_{n-k}^{r/p}x_{n-k-1}^x}\right)^p\right)^p.
\]  

This completes the proof.
After $k$ steps we obtain the following formula

$$
x_n = A + \left( \frac{A}{x_n^{r/p}} \right)^n_k + \left( \frac{A}{x_n^{r/p^2}} \right)^n_k + \left( \frac{A}{x_n^{r/p^3}} \right)^n_k + \cdots
$$

Two subcases can be considered now.

Case 1 ($r \geq p^k$). If $r \geq p^k$, then by (2.2) equality (2.4) implies that

$$
x_n < A + \left( \frac{A}{A^{r/p}} \right)^n + \left( \frac{A}{A^{r/p^{2}+r/p^k}} \right)^n + \cdots
$$

for $n \geq 2k - 1$. This means that $(x_n)$ is a bounded sequence.

Case 2 ($p^k > r$). In this case we have

$$
p - \frac{r}{p^{k-1}} > 0.
$$
From (2.4) and (1.2) we further obtain

\[
x_n = A + \left( \frac{A}{x_{n-k}} \right) + \left( \frac{A}{x_{n-k}^2 x_{n-k-1}} \right) + \left( \frac{A}{x_{n-k}^3 x_{n-k-2}} \right)
\]

\[
+ \cdots + \left( \frac{A}{x_{n-k}^{p-1} x_{n-k-1} \cdots x_{n-(2k-2)}} \right) \left( \frac{A}{x_{n-k}^{p-2} x_{n-k-2} \cdots x_{n-(2k-2)}^{p-2}} \right) \left( \frac{A}{x_{n-k}^{p-3} x_{n-k-3} \cdots x_{n-(2k-2)}^{p-3}} \right) \cdots \right)^p
\]

\[
= A + \left( \frac{A}{x_{n-k}^p} \right) + \left( \frac{A}{x_{n-k}^r x_{n-k}} \right) + \left( \frac{A}{x_{n-k}^r x_{n-k}^2 x_{n-k-1}} \right) + \left( \frac{A}{x_{n-k}^r x_{n-k}^3 x_{n-k-2}} \right)
\]

\[
+ \cdots + \left( \frac{A}{x_{n-k}^{p-z_{0}^{(0)}}} \right) + \left( \frac{x_{n-k}}{\prod_{j=0}^{k-2} x_{n-k-j}^{z_{0}^{(0)}}} \left( \prod_{j=1}^{k-2} x_{n-k-j}^{z_{0}^{(0)}} \right) x_{n-k}^{p-z_{0}^{(0)}} \right) \cdots \right)^p
\]

\[
= A + \left( \frac{A}{x_{n-k}^p} \right) + \left( \frac{A}{x_{n-k}^r x_{n-k}} \right) + \left( \frac{A}{x_{n-k}^r x_{n-k}^2 x_{n-k-1}} \right) + \left( \frac{A}{x_{n-k}^r x_{n-k}^3 x_{n-k-2}} \right)
\]

\[
+ \cdots + \left( \frac{A}{x_{n-k}^{p-z_{1}^{(0)}}} \right) + \left( \frac{x_{n-k}^{1-z_{1}^{(0)}}}{\prod_{j=0}^{k-2} x_{n-k-j}^{1-z_{1}^{(0)}}} \left( \prod_{j=1}^{k-2} x_{n-k-j}^{1-z_{1}^{(0)}} \right) x_{n-k-1}^{p-z_{1}^{(0)}} \right) \cdots \right)^p
\]

\[
= \cdots = A + \left( \frac{A}{x_{n-k}^p} \right) + \left( \frac{A}{x_{n-k}^r x_{n-k}} \right) + \left( \frac{A}{x_{n-k}^r x_{n-k}^2 x_{n-k-1}} \right) + \left( \frac{A}{x_{n-k}^r x_{n-k}^3 x_{n-k-2}} \right)
\]

\[
+ \cdots + \left( \frac{A}{x_{n-k}^{p-z_{m}^{(0)}}} \right) + \left( \frac{x_{n-k}^{m-z_{m}^{(0)}}}{\prod_{j=0}^{k-2} x_{n-k-j}^{m-z_{m}^{(0)}}} \left( \prod_{j=1}^{k-2} x_{n-k-j}^{m-z_{m}^{(0)}} \right) x_{n-k-m}^{p-z_{m}^{(0)}} \right) \cdots \right)^p,
\]

(2.7)
for each $k \in \mathbb{N} \setminus \{1\}$ and every $n \geq 2k + m - 1$, where the sequences $(z_m^{(j)})$, $j = 0, 1, \ldots, k - 2$, satisfy the system

$$
z_{m+1}^{(0)} = \frac{z_{m}^{(1)} z_{m+1}^{(1)}}{p - z_{m}^{(0)}}, \quad z_{m+1}^{(1)} = \frac{z_{m}^{(2)} z_{m+1}^{(2)}}{p - z_{m}^{(0)}}, \ldots, \quad z_{m+1}^{(k-2)} = \frac{z_{m}^{(k-2)} z_{m+1}^{(k-2)}}{p - z_{m}^{(0)}}, \quad z_{m+1}^{(k-1)} = \frac{r}{p - z_{m}^{(0)}} \quad (2.8)
$$

and the initial values are given by

$$
z_0^{(j)} = r p^{j+1-k}, \quad j = 0, 1, \ldots, k-2. \quad (2.9)
$$

Note that $p^k > r$ implies that $z_0^{(0)} < p$. Assume $z_0^{(0)} < p$ for every $m \in \mathbb{N}_0$.

By a direct calculation it follows that $z_0^{(j)} < z_1^{(j)}$, $j = 0, 1, \ldots, k-2$, which, along with (2.8) implies that $(z_m^{(j)})$, $j = 0, 1, \ldots, k-2$, are strictly increasing sequences.

From system (2.8), we have,

$$
z_{m+1}^{(0)} = \frac{r}{(p - z_m^{(0)})(p - z_{m-1}^{(0)}) \cdots (p - z_{m-k+2}^{(0)})}, \quad m \geq k - 2. \quad (2.10)
$$

If it were $z_m^{(0)} < p$, $m \in \mathbb{N}_0$, then there was

$$
\lim_{m \to \infty} z_m^{(0)} = z \in (0, p]. \quad (2.11)
$$

Clearly $z$ is a solution of the equation

$$
f(x) = x(p - x)^{k-1} - r = 0. \quad (2.12)
$$

Since

$$
f(0) = f(p) = -r, \quad (2.13)
$$

and

$$
f'(x) = (p - x)^{k-2}(p - kx), \quad (2.14)
$$

we see that the function $f$ attains its maximum at the point $x = p/k$.

Further, by assumption (2.1) we get

$$
f\left(\frac{p}{k}\right) = \frac{(k-1)^{k-1}}{k^k} \left( p^k - r \frac{k^k}{(k-1)^{k-1}} \right) < 0, \quad (2.15)
$$

which along with (2.13) implies that (2.12) does not have solutions on $(0, p]$, arriving at a contradiction.
This implies that there is a fixed index $m_0 \in \mathbb{N}$ such that
\begin{equation}
-z_{m_0-1}^{(0)} < p, \quad z_{m_0}^{(0)} \geq p.
\end{equation}

From this, inequality (2.2), and identity (2.7) with $m = m_0$, it follows that
\begin{align}
x_n &= A + \left( \frac{A}{x_{n-k}^{\frac{r}{p}}} + \left( \frac{A}{x_{n-k-1}^{\frac{r}{p}}} + \left( \frac{A}{x_{n-k-2}^{\frac{r}{p}}} + \cdots + \left( \frac{A}{\prod_{j=0}^{k-2} x_{n-k-m_j}^{\frac{r}{p}}} + \left( \frac{A}{\prod_{j=1}^{k-2} x_{n-k-m_j}^{\frac{r}{p}}} \right) \right) \right) \right) \right) \cdots \\
&\leq A + \left( \frac{A}{A^{\frac{r}{p}}} + \left( \frac{A}{A^{\frac{r}{p}+r/p}} + \left( \frac{A}{A^{\frac{r}{p}+r/p}} + \cdots + \left( \frac{A}{A^{\prod_{j=0}^{m_j-1} (p-z_{j}^{(0)})}} + \left( \frac{A}{A^{\prod_{j=1}^{m_j-1} (p-z_{j}^{(0)})}} \right) \right) \right) \right) \right) \cdots < \infty
\end{align}

for $n \geq 2k + m_0 - 1$.

From (2.17) the boundedness of the sequence $(x_n)$ directly follows, as desired. \qed

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