Research Article


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We introduce a new iterative scheme based on extragradient method and viscosity approximation method for finding a common element of the solutions set of a system of equilibrium problems, fixed point sets of an infinite family of nonexpansive mappings, and the solution set of a variational inequality for a relaxed cocoercive mapping in a Hilbert space. We prove strong convergence theorem. The results in this paper unify and generalize some well-known results in the literature.

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $C$ be a nonempty, closed, and convex subset of $H$. Let $\{F_k\}_{k \in \Gamma}$ be a countable family of bifunctions from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Combettes and Hirstoaga [1] considered the following system of equilibrium problems:

Find $x \in C$ such that $(\forall k \in \Gamma), (\forall y \in C), F_k(x, y) \geq 0$. \hspace{1cm} (1.1)

If $\Gamma$ is a singleton, problem (1.1) becomes the following equilibrium problem:

Finding $x \in C$ such that $F(x, y) \geq 0, \hspace{1cm} \forall y \in C$. \hspace{1cm} (1.2)
The solutions set of (1.2) is denoted by \( \text{EP}(F) \). And clearly the solutions set of problem (1.1) can be written as \( \bigcap_{k \in \mathcal{E}} \text{EP}(F_k) \).

Problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; see for instance, [1–4].

Recall that a mapping \( S \) of a closed and convex subset \( C \) into itself is nonexpansive if

\[
\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.
\] (1.3)

We denote fixed-points set of \( S \) by \( \text{Fix}(S) \). A mapping \( f : C \to C \) is called contraction if there exists a constant \( \alpha \in (0, 1) \) such that

\[
\|fx - fy\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.
\] (1.4)

A bounded linear operator \( B \) on \( H \) is strongly positive, if there is a constant \( \gamma > 0 \) such that \( \langle Bx, x \rangle \geq \gamma \|x\|^2 \) for all \( x \in H \).


Several algorithms for problem (1.2) have been proposed (see [5–20]). S. Takahashi and W. Takahashi [5] introduced and studied the following iterative scheme by the viscosity approximation method for finding a common element of the solutions set of problem (1.2) and fixed-points set of a nonexpansive mapping in a Hilbert space. Let an arbitrary \( x_1 \in H \) define sequences \( \{x_n\} \) and \( \{u_n\} \) by

\[
F(u_n, y) + \frac{1}{\beta_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,
\] (1.5)

\[
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad \forall n \in \mathbb{N}.
\]

Shang et al. [6] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the solutions set of problem (1.2) and fixed-points
set of a nonexpansive mapping in a Hilbert space. Let an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$
F(u_n, y) + \frac{1}{\beta_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,
$$

$$
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Su_n, \quad \forall n \in N.
$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\beta_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.6) converge strongly to the unique solution of the variational inequality

$$
\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(S) \cap \text{EP}(F),
$$

which is the optimality condition for the minimization problem

$$
\min_{x \in \text{Fix}(S) \cap \text{EP}(F)} \frac{1}{2} \langle Bx, x \rangle - h(x),
$$

where $h$ is a potential function for $\gamma f$ (i.e., $h(x) = \gamma f(x)$ for $x \in H$). If $C = H$, the algorithm (1.6) was also studied by Plubtieng and Punpaeng [7].

Let $A : C \to H$ be a monotone mapping. The variational inequality problem is to find a point $x \in C$ such that

$$
\langle Ax, y - x \rangle \geq 0
$$

for all $y \in C$. The solutions set of the variational inequality problem is denoted by $\text{VI}(C, A)$. Qin et al. [8] introduced the following general iterative scheme for finding a common element of the solutions set of problem (1.2), the solutions set of a variational inequality and fixed-points set of a nonexpansive mapping in a Hilbert space. Let an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$
F(u_n, y) + \frac{1}{\beta_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,
$$

$$
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)SP_C(I - s_n A)u_n, \quad \forall n \in N.
$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{s_n\}$ and $\{\beta_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.10) converge strongly to the unique solution of the variational inequality

$$
\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(S) \cap \text{VI}(C, A) \cap \text{EP}(F).
$$

Qin et al. [9] introduced the following general iterative scheme for finding a common element of the solutions set of problem (1.2) and fixed-points set of a finite family of
nonexpansive mappings in a Hilbert space. Let an arbitrary \( x_1 \in H \), define sequences \( \{x_n\} \) and \( \{u_n\} \) by

\[
F(u_n, y) + \frac{1}{\beta_n} (y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C,
\]

\[
x_{n+1} = \alpha_n f(W_n x_n) + (1 - \alpha_n B) W_n P_C(I - s_n A) u_n, \quad \forall n \in N,
\]

where \( W_n \) is the \( W \)-mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \lambda_n, \lambda_{n1}, \ldots, \lambda_{nN} \). They proved that under certain appropriate conditions imposed on \( \{\alpha_n\}, \{s_n\} \) and \( \{\beta_n\} \), the sequences \( \{x_n\} \) and \( \{u_n\} \) generated by (1.12) converge strongly to the unique solution of the variational inequality

\[
\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} \text{Fix}(T_i) \cap \text{VI}(C, A) \cap \text{EP}(F).
\]

A typical problem is to minimize a quadratic function over the fixed-points set of a nonexpansive mapping \( S \) on a real Hilbert space \( H \), that is,

\[
\min_{x \in \text{Fix}(S)} \frac{1}{2} \langle Bx, x \rangle - \langle x, b \rangle,
\]

where \( b \) is a given point in \( H \). In 2003, Xu [21] proved that the sequence \( \{x_n\} \) defined by the iterative method below, with the initial point \( x_0 \in H \), chosen arbitrarily:

\[
x_{n+1} = (I - \alpha_n B) S x_n + \alpha_n u, \quad n \geq 0,
\]

converges strongly to the unique solution of the minimization problem (1.15) provided the sequence \( \{\alpha_n\} \) satisfies certain conditions. Marino and Xu [22] combine the iterative method (1.15) with the viscosity approximation in [23] and consider the following general iterative method: with the initial point \( x_0 \in H \), chosen arbitrarily:

\[
x_{n+1} = (1 - \alpha_n B) S x_n + \alpha_n f(x_n), \quad n \geq 0.
\]

They proved that if the sequence \( \{\alpha_n\} \) satisfies appropriate conditions, then the sequence \( \{x_n\} \) generated by (1.16) converges strongly to the unique solution of the variational inequality

\[
\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in \text{Fix}(S)
\]

which is the optimality condition for the minimization problem

\[
\min_{x \in \text{Fix}(S)} \frac{1}{2} \langle Bx, x \rangle - h(x),
\]

where \( h \) is a potential function for \( \gamma f \).
Recently, Qin et al. [24] introduced the following general iterative process: with the initial point $x_1 \in C$, chosen arbitrarily:

$$y_n = P_C(I - s_nA)x_n,$$

$$x_{n+1} = a_nf(W_nx_n) + (I - a_nB)W_nP_C(I - r_nA)y_n, \quad \forall n \in N,$$  \hspace{1cm} (1.19)

where $W_n$ is the $W$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nN}$. They proved that if the sequences of parameters $\{a_n\}, \{r_n\}$ and $\{s_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}, \{y_n\}$ generated by (1.19) converge strongly to a point $x^*$ which is the unique solution of the variational inequality

$$\langle (B - yf)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A).$$  \hspace{1cm} (1.20)

Inspired and motivated by above works, we introduce a new iterative scheme based on extragradient method and viscosity approximation method for finding a common element of the solutions set of a system of equilibrium problems, fixed-points set of a family of infinitely nonexpansive mappings and the solutions set of a variational inequality for a relaxed cocoercive mapping in a Hilbert space. We prove strong convergence theorem. The results in this paper unify, generalize and extend some well-known results in [6–9, 21, 22, 24].

2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $C$ be a nonempty, closed, and convex subset of $H$. Let symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively. It is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$  \hspace{1cm} (2.1)

for all $x, y \in H$ and $\lambda \in [0,1]$.

For any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping $P_C$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is a nonexpansive mapping from $H$ onto $C$; $P_C(x) \in C$ and

$$\langle x - P_C(x), P_C(x) - y \rangle \geq 0$$  \hspace{1cm} (2.2)

for all $x, y \in H$.

It is easy to see that (2.2) is equivalent to

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$$  \hspace{1cm} (2.3)

for all $x, y \in H$. It is also known that $P_C$ has the following firmly nonexpansive property:

$$\langle x - y, P_Cx - P_Cy \rangle \geq \|P_Cx - P_Cy\|^2$$  \hspace{1cm} (2.4)

for all $x, y \in H$. 
Recall also that a mapping $A$ of $C$ into $H$ is called monotone if
\[ \langle Ax - Ay, x - y \rangle \geq 0, \] (2.5)
for all $x, y \in C$. $A$ is said to be $\mu$-cocoercive, if for each $x, y \in C$, we have
\[ \langle Ax - Ay, x - y \rangle \geq \mu \| Ax - Ay \|^2, \] (2.6)
for a constant $\mu > 0$. $A$ is said to be relaxed $(u, v)$-cocoercive, if there exist two constants $u, v > 0$ such that
\[ \langle Ax - Ay, x - y \rangle \geq (-u) \| Ax - Ay \|^2 + v \| x - y \|^2, \quad \forall x, y \in C. \] (2.7)

Let $A$ be a monotone mapping of $C$ into $H$. In the context of the variational inequality problem the characterization of projection (2.2) implies the following:
\[ u \in \text{VI}(C, A) \Rightarrow u = P_C(u - \lambda Au), \quad \lambda > 0, \]
\[ u = P_C(u - \lambda Au) \quad \text{for some } \lambda > 0 \Rightarrow u \in \text{VI}(C, A). \] (2.8)

It is also known that $H$ satisfies the Opial’s condition [25], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality
\[ \liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \] (2.9)
holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ of $T$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let $A$ be a monotone and $k$-Lipschitz-continuous mapping of $C$ into $H$ and let $N_Cv$ be normal cone to $C$ at $v \in C$, that is, $N_Cv = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$. Define
\[ Tv = \begin{cases} 
Av + N_Cv & \text{if } v \in C, \\
\emptyset & \text{if } v \notin C.
\end{cases} \] (2.10)

Then $T$ is maximal monotone and $0 \in Tv$ if and only if $v \in \text{VI}(C, A)$ (see [26]).

For solving the problem (1.1), let us assume that the bifunction $F$ satisfies the following condition:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) $F$ is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
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(A3) for each \(x, y, z \in C\),
\[
\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);
\]  \hfill (2.11)

(A4) for each \(x \in C, y \mapsto F(x, y)\) is convex;

(A5) for each \(x \in C, y \mapsto F(x, y)\) is lower semicontinuous.

We recall some lemmas needed later.

**Lemma 2.1** \(\text{see} \ [1, 10]\). Let \(C\) be a nonempty, closed, and convex subset of \(H\), and let \(F\) be a bifunction from \(C \times C\) to \(\mathbb{R}\) which satisfies conditions \((A1)–(A5)\). For \(\beta > 0\) and \(x \in H\), define the mapping \(T_{\beta}^F : H \to C\) as follows:
\[
T_{\beta}^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{\beta} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}
\]  \hfill (2.12)

for all \(x \in H\). Then, the following statements hold:

1. \(T_{\beta}^F(x) \neq \emptyset\);  
2. \(T_{\beta}^F\) is single-valued;  
3. \(T_{\beta}^F\) is firmly nonexpansive, that is, for any \(x, y \in H\),
\[
\|T_{\beta}^F(x) - T_{\beta}^F(y)\|^2 \leq \langle T_{\beta}^F(x) - T_{\beta}^F(y), x - y \rangle;
\]  \hfill (2.13)

4. \(\text{Fix}(T_{\beta}^F) = \text{EP}(F)\);  
5. \(\text{EP}(F)\) is closed and convex.

**Lemma 2.2** \(\text{see} \ [27]\). Assume that \(\{s_n\}\) is a sequence of nonnegative real numbers such that
\[
s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\beta_n + \delta_n, \quad n \geq 1,
\]  \hfill (2.14)

where \(\{\alpha_n\}, \{\beta_n\}\) and \(\{\delta_n\}\) are sequences of numbers which satisfy the conditions:

(i) \(\{\alpha_n\} \subset [0, 1], \sum_{n=1}^{\infty} \alpha_n = \infty, \text{or equivalently,} \prod_{i=1}^{\infty} (1 - \alpha_n) = 0\);

(ii) \(\lim \sup_{n \to \infty} \beta_n \leq 0\);

(iii) \(\delta_n \geq 0(n \geq 1), \sum_{n=1}^{\infty} \delta_n < \infty\);

Then, \(\lim_{n \to \infty} s_n = 0\).

**Lemma 2.3.** In a real Hilbert space \(H\), the following inequality holds:
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle
\]  \hfill (2.15)

for all \(x, y \in H\).
Lemma 2.4 (see [22]). Assume that $A$ is a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\gamma > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho \gamma$.

Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of $C$ into itself and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 \leq \xi_i \leq 1$ for every $i \in N$. For any $n \in N$, define a mapping $W_n$ of $C$ into $C$ as follows:

\[
U_{n,1} = I, \quad U_{n,n+1} = I, \quad U_{n,n} = \xi_n S_n U_{n,n+1} + (1 - \xi_n)I, \quad U_{n,n-1} = \xi_{n-1} S_{n-1} U_{n,n} + (1 - \xi_{n-1})I, \\
\vdots \\
U_{n,k} = \xi_k S_k U_{n,k+1} + (1 - \xi_k)I, \quad U_{n,k-1} = \xi_{k-1} S_{k-1} U_{n,k} + (1 - \xi_{k-1})I, \\
\vdots \\
U_{n,2} = \xi_2 S_2 U_{n,3} + (1 - \xi_2)I, \quad W_n = U_{n,1} = \xi_1 S_1 U_{n,2} + (1 - \xi_1)I.
\]

Such a mapping $W_n$ is called the $W$-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$; see [28, 29].

Lemma 2.5 (see [28]). Let $C$ be a nonempty, closed, and convex subset of a Banach space $E$. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of $C$ into itself such that $\bigcap_{i=1}^{\infty} \text{Fix}(S_i)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. For any $n \in N$, let $W_n$ be the $W$-mapping of $C$ into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Then $W_n$ is asymptotically regular and nonexpansive. Further, if $E$ is strictly convex, then $F(W_n) = \bigcap_{i=1}^{\infty} \text{Fix}(S_i)$.

Lemma 2.6 (see [29]). Let $C$ be a nonempty, closed, and convex subset of a strictly convex Banach space $E$. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of $C$ into itself such that $\bigcap_{i=1}^{\infty} \text{Fix}(S_i)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. Then for every $x \in C$ and $k \in N$, the limit $\lim_{n \to \infty} U_{n,k} x$ exists.

Remark 2.7. Using Lemma 2.6, one can define mappings $U_{\infty,k}$ and $W$ of $C$ into itself as follows:

\[
U_{\infty,k} x = \lim_{n \to \infty} U_{n,k} x, \quad W x = \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x 
\]

and $W x = \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x$ for every $x \in C$. Such a mapping $W$ is called the $W$-mapping generated by $S_1, S_2, \ldots$ and $\xi_1, \xi_2, \ldots$. Since $W_n$ is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$

\[
\|W x - W y\| = \lim_{n \to \infty} \|W_n x - W_n y\| \leq \|x - y\|. \quad (2.18)
\]
If \( \{x_n\} \) is a bounded sequence in \( C \), then we have
\[
\lim_{n \to \infty} ||Wx - W_n x|| = 0. \tag{2.19}
\]

**Lemma 2.8** (see [29]). Let \( C \) be a nonempty, closed and convex subset of a strictly convex Banach space \( E \). Let \( S_1, S_2, \ldots \) be an infinite family of nonexpansive mappings of \( C \) into itself such that \( \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \) is nonempty, and let \( \xi_1, \xi_2, \ldots \) be real numbers such that \( 0 < \xi_i \leq d < 1 \) for every \( i \in N \). Then \( \text{Fix}(W) = \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \).

### 3. Strong Convergence Theorem

In this section, we prove strong convergence theorem which solve the problem of finding a common element of the solutions set of a system of equilibrium problems, fixed-points set of a family of infinitely nonexpansive mappings, and the solutions set of a variational inequality for a relaxed cocoercive mapping in Hilbert space.

**Theorem 3.1.** Let \( C \) be a nonempty, closed, and convex subset of \( H \). Let \( F_1, F_2, \ldots, F_m \) be bifunctions from \( C \times C \) to \( \mathbb{R} \) which satisfies conditions (A1)–(A5). Let \( A : C \to H \) be relaxed \((u, v)\)-cocoercive and \( \mu \)-Lipschitz continuous and \( B \) a strongly positive linear bounded operator on \( H \) with coefficient \( \tilde{\gamma} > 0 \). Assume that \( 0 < \gamma < \tilde{\gamma} / \alpha \). Let \( S_1, S_2, \ldots \) be a family of infinitely nonexpansive mappings of \( C \) into itself such that \( \Omega = \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \cap \text{VI}(C, A) \cap \bigcap_{i=1}^{m} \text{EP}(F_i) \neq \emptyset \) and let \( \xi_1, \xi_2, \ldots \) be real numbers such that \( 0 < \xi_i \leq \delta < 1 \) for every \( i \in N \), and let \( W_n \) be the \( W \)-mapping of \( C \) into itself generated by \( S_n, S_{n-1}, \ldots, S_1 \) and \( \xi_n, \xi_{n-1}, \ldots, \xi_1 \). Let \( f : C \to C \) be a contraction with coefficient \( \alpha \) \((0 < \alpha < 1)\) and \( \{x_n\}, \{u_n\}, \text{ and } \{y_n\} \) be sequences generated by
\[
x_1 = x \in H,
\]
\[
u_n = T_{\beta_n}^{-1} \cdots T_{\beta_n}^{-1} \cdot T_{\beta_n}^{-1} \cdot T_{\beta_n}^{-1} \cdot T_{\beta_n}^{-1} \cdot x_n, \tag{3.1}
\]
\[
y_n = P_C(I - r_n A)u_n,
\]
\[
x_{n+1} = \alpha_n y f(W_n x_n) + (1 - \alpha_n B) W_n P_C(I - r_n A) y_n
\]
for every \( n = 1, 2, \ldots \), where \( \{\alpha_n\}, \{\beta_n\}, \{r_n\}, \text{ and } \{s_n\} \) are sequences of numbers which satisfy the conditions:

- (C1) \( \{\alpha_n\} \subset [0, 1] \) with \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), and \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \);
- (C2) \( \{r_n\} \subset [a, b] \) and \( \{s_n\} \subset [a, b] \) for some \( a, b \) with \( 0 \leq a \leq b \leq 2(v - u \mu^2) / \mu^2 \), \( \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty \), and \( \sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty \);
- (C3) \( \liminf_{n \to \infty} r_n > 0 \) and \( \sum_{n=1}^{\infty} |r_{n+1} - \beta_n| < \infty \).

Then, \( \{x_n\}, \{y_n\}, \text{ and } \{u_n\} \) converge strongly to a point \( q \in \Omega \) which solves the following variational inequality:
\[
\langle y f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \Omega. \tag{3.2}
\]

Equivalently, one has \( q = P_\Omega(\gamma f + (I - B))(q) \).
Proof. Since $\alpha_n \to 0$ from condition (C1), we may assume, with no loss of generality, that
$\alpha_n \leq \|B\|^{-1}$ for all $n$. Lemma 2.4 implies $\|I - \alpha_n B\| \leq 1 - \alpha_n \tilde{\gamma}$. Next, we will assume that
$\|I - B\| \leq 1 - \tilde{\gamma}$. Now, we show that the mappings $I - s_n A$ and $I - r_n A$ are nonexpansive. Indeed, from the relaxed $(u, v)$-cocoercivity and $\mu$-Lipschitz continuity of $A$ and condition (C2), we have

$$
\|(I - s_n A)x - (I - s_n A)y\|^2 = \|(x - y) - s_n (Ax -Ay)\|^2
$$

$$
= \|x - y\|^2 - 2s_n \langle x - y, Ax - Ay \rangle + s_n^2 \|Ax - Ay\|^2
$$

$$
\leq \|x - y\|^2 - 2s_n \left[ -u \|Ax - Ay\|^2 + \nu \|x - y\|^2 \right] + s_n^2 \|Ax - Ay\|^2
$$

$$
\leq \|x - y\|^2 + 2s_n \mu^2 u \|x - y\|^2 - 2s_n \nu \|x - y\|^2 + \mu^2 s_n^2 \|x - y\|^2
$$

$$
= \left( 1 + 2s_n \mu^2 u - 2s_n \nu + \mu^2 s_n^2 \right) \|x - y\|^2
$$

$$
\leq \|x - y\|^2,
$$

(3.3)

which implies the mapping $I - s_n A$ is nonexpansive, so is $I - r_n A$.

For $k \in \{0, 1, 2, \ldots, m\}$, and for any positive integer number $n$, we define the operator
$\Theta^k_{\beta_n} : H \to C$ as follows:

$$
\Theta^0_{\beta_n} x = x,
$$

$$
\Theta^k_{\beta_n} x = T_{\beta_n}^{T_{\beta_n} x} T_{\beta_n}^{T_{\beta_n} x} \ldots T_{\beta_n}^{T_{\beta_n} x} x, \quad k = 1, 2, \ldots, m.
$$

(3.4)

Next, we show that the sequence $\{x_n\}$ is bounded. Let $p \in \Omega$. Then from Lemma 2.1(3),
we know that for $k \in \{1, 2, \ldots, m\}$, $T_{\beta_n}^{T_{\beta_n} x}$ is nonexpansive and $p = T_{\beta_n}^{T_{\beta_n} x} p$, and

$$
\|u_n - p\| = \|\Theta^m_{\beta_n} x_n - p\| = \|\Theta^m_{\beta_n} x_n - \Theta^m_{\beta_n} p\| \leq \|x_n - p\|
$$

(3.5)

for all $n = 1, 2, \ldots$. By $p = P_{\Omega}(I - s_n A)p$ and (3.5), we have

$$
\|y_n - p\| = \|P_{\Omega}(I - s_n A)u_n - P_{\Omega}(I - s_n A)p\|
$$

$$
\leq \|(I - s_n A)u_n - (I - s_n A)p\| \leq \|u_n - p\| \leq \|x_n - p\|.
$$

(3.6)

Since $x_{n+1} = \alpha_n f(W_n x_n) + (I - \alpha_n B)W_n P_{\Omega}(I - r_n A)y_n$ and $p = W_n p$, we have

$$
\|x_{n+1} - p\| = \|\alpha_n (\gamma f(W_n x_n) - Bp) + (I - \alpha_n B)(W_n P_{\Omega}(I - r_n A)y_n - p)\|
$$

$$
\leq \alpha_n \|\gamma f(W_n x_n) - Bp\| + (1 - \alpha_n \tilde{\gamma}) \|P_{\Omega}(I - r_n A)y_n - p\|
$$

$$
\leq \alpha_n \|\gamma f(W_n x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \tilde{\gamma}) \|y_n - p\|
$$

$$
\leq \left[ 1 - \alpha_n (\tilde{\gamma} - \gamma) \right] \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\|.
$$

(3.7)
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By inductions, we have

$$
\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|rf(p) - Bp\|}{\bar{y} - \alpha y} \right\},
$$

(3.8)

which proves that the sequence \( \{x_n\} \) is bounded. It follows from (3.5) and (3.6) that \( \{y_n\} \) and \( \{u_n\} \) are also bounded.

Since \( \Theta^k\beta_n x_n = T^k\beta_n \Theta^{k-1}x_n \) and \( \Theta^k\beta_{n+1} x_{n+1} = T^k\beta_{n+1} \Theta^{k-1}x_{n+1} \) for each \( k = 1, 2, \ldots, m \), by Lemma 2.1, we have

$$
F_k \left( \Theta^k\beta_n x_n, y \right) + \frac{1}{\beta_n} \left( y - \Theta^k\beta_n x_n, \Theta^k\beta_n x_n - \Theta^{k-1}x_n \right) \geq 0 \quad \forall y \in C,
$$

(3.9)

$$
F_k \left( \Theta^k\beta_{n+1} x_{n+1}, y \right) + \frac{1}{\beta_{n+1}} \left( y - \Theta^k\beta_{n+1} x_{n+1}, \Theta^k\beta_{n+1} x_{n+1} - \Theta^{k-1}x_{n+1} \right) \geq 0 \quad \forall y \in C,
$$

(3.10)

Setting \( y = \Theta^k\beta_n x_n \) in (3.9) and \( y = \Theta^k\beta_{n+1} x_{n+1} \) in (3.10), we have

$$
F_k \left( \Theta^k\beta_n x_n, \Theta^k\beta_{n+1} x_{n+1} \right) + \frac{1}{\beta_n} \left( \Theta^k\beta_n x_n - \Theta^k\beta_{n+1} x_{n+1}, \Theta^k\beta_n x_n - \Theta^{k-1}x_n \right) \geq 0,
$$

(3.11)

$$
F_k \left( \Theta^k\beta_{n+1} x_{n+1}, \Theta^k\beta_n x_n \right) + \frac{1}{\beta_{n+1}} \left( \Theta^k\beta_{n+1} x_{n+1} - \Theta^k\beta_n x_n, \Theta^k\beta_{n+1} x_{n+1} - \Theta^{k-1}x_{n+1} \right) \geq 0.
$$

(3.12)

Adding the two inequalities and from the monotonicity of \( F \), we get

$$
\left( \Theta^k\beta_{n+1} x_{n+1} - \Theta^k\beta_n x_n \right) \left( \Theta^k\beta_n x_n - \Theta^{k-1}x_n \right) \geq 0
$$

and hence

$$
\left\| \Theta^k\beta_{n+1} x_{n+1} - \Theta^k\beta_n x_n \right\|^2
\leq \left( \Theta^k\beta_{n+1} x_{n+1} - \Theta^k\beta_n x_n \right) \left( \Theta^{k-1}x_n - \Theta^{k-1}x_{n+1} \right) + \left( 1 - \frac{\beta_n}{\beta_{n+1}} \right) \left( \Theta^k\beta_{n+1} x_{n+1} - \Theta^k\beta_n x_n \right).
$$

(3.13)

Without loss of generality, let us assume that there exists a real number \( d > 0 \) for all \( n = 1, 2, \ldots \). Hence, for each \( k = 1, 2, \ldots, m \) we have

$$
\left\| \Theta^k\beta_n x_n - \Theta^k\beta_{n+1} x_{n+1} \right\| \leq \left\| \Theta^{k-1}x_n - \Theta^{k-1}x_{n+1} \right\| + \frac{1}{\beta_{n+1}} \left| \beta_{n+1} - \beta_n \right| \left\| \Theta^k\beta_{n+1} x_{n+1} - \Theta^k\beta_n x_n \right\|
\leq \left\| \Theta^{k-1}x_n - \Theta^{k-1}x_{n+1} \right\| + \frac{1}{d} \left| \beta_{n+1} - \beta_n \right| M_0,
$$

(3.14)
where $M_0$ is an approximate constant such that

$$
M_0 \geq \max \left\{ \sup_{n \geq 1} \left\| \Theta_{\rho_{n+1}}^k x_{n+1} - \Theta_{\rho_n}^{k-1} x_{n+1} \right\|, \quad k = 1, 2, \ldots, m \right\}.
$$

(3.15)

It follows from (3.14) that

$$
\| u_{n+1} - u_n \| = \left\| \Theta_{\rho_{n+1}}^m x_{n+1} - \Theta_{\rho_n}^m x_n \right\| \leq \| x_{n+1} - x_n \| + \frac{m}{d} |\beta_{n+1} - \beta_n| M_0.
$$

(3.16)

Put $\rho_n = P_C (I - r_n A) y_n$. We have

$$
\| y_n - y_{n+1} \| = \| P_C (I - s_n A) u_n - P_C (I - s_{n+1} A) u_{n+1} \|
\leq \| (I - s_n A) u_n - (I - s_{n+1} A) u_{n+1} \|
= \| (u_n - s_n A u_n) - (u_{n+1} - s_n A u_{n+1}) + (s_{n+1} - s_n) A u_{n+1} \|
\leq \| u_n - u_{n+1} \| + |s_{n+1} - s_n| M_1,
$$

(3.17)

where $M_1$ is an approximate constant such that $M_1 \geq \max\{\sup_{n \geq 1} \{\| A u_n \|\}, M_0\}$. Substituting (3.16) into (3.17), we have

$$
\| y_n - y_{n+1} \| \leq \| x_{n+1} - x_n \| + \left[ \frac{m}{d} |\beta_{n+1} - \beta_n| + |s_{n+1} - s_n| \right] M_1.
$$

(3.18)

It follows from (3.18) that

$$
\| \rho_n - \rho_{n+1} \| = \| P_C (I - r_n A) y_n - P_C (I - r_{n+1} A) y_{n+1} \|
\leq \| (I - r_n A) y_n - (I - r_{n+1} A) y_{n+1} \|
= \| (y_n - r_n A y_n) - (y_{n+1} - r_n A y_{n+1}) + (r_{n+1} - r_n) A y_{n+1} \|
\leq \| y_n - y_{n+1} \| + |r_{n+1} - r_n| M_2
\leq \| x_n - x_{n+1} \| + \left[ \frac{m}{d} |\beta_{n+1} - \beta_n| + |s_{n+1} - s_n| + |r_{n+1} - r_n| \right] M_2,
$$

(3.19)

where $M_2$ is an approximate constant such that $M_2 \geq \max\{M_1, \sup_{n \geq 1} \{\| A y_{n+1} \|\}\}$. Observe that

$$
x_{n+1} = \alpha_n g f(W_n x_n) + (I - \alpha_n B) W_n \rho_n,
$$

$$
x_{n+2} = \alpha_{n+1} g f(W_{n+1} x_{n+1}) + (I - \alpha_{n+1} B) W_{n+1} \rho_{n+1},
$$

(3.20)

we have

$$
x_{n+2} - x_{n+1} = \alpha_{n+1} g \left[ f(W_{n+1} x_{n+1}) - f(W_n x_n) \right] + (I - \alpha_{n+1} B) (W_{n+1} \rho_{n+1} - W_n \rho_n)
+ (\alpha_{n+1} - \alpha_n) \left[ g f(W_n x_n) - B W_n \rho_n \right].
$$

(3.21)
It follows that

\[
\begin{align*}
\|x_{n+2} - x_{n+1}\| & \leq \alpha_{n+1} \gamma(\|W_{n+1}x_{n+1} - W_n x_n\| + (1 - \alpha_{n+1})\|W_{n+1}\rho_{n+1} - W_n \rho_n\|) \\
& \quad + |\alpha_{n+1} - \alpha_n|\|y_f(W_n x_n) - BW_n \rho_n\| \\
& \leq \alpha_{n+1} \gamma(\|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_n x_n\|) \\
& \quad + (1 - \alpha_{n+1})\|W_{n+1}\rho_n - W_n \rho_n\| \quad (3.22)
\end{align*}
\]

Next we estimate \(\|W_{n+1}x_n - W_n x_n\|\) and \(\|W_{n+1}\rho_n - W_n \rho_n\|\). It follows from the definition of \(W_n\) and nonexpansiveness of \(S_i\) that

\[
\begin{align*}
\|W_{n+1}x_n - W_n x_n\| &= \|U_{n+1,1}x_n - U_{n,1}x_n\| \\
& = \|\xi_1 S_1 U_{n+1,2}x_n + (1 - \xi_1)x_n - \{\xi_1 S_1 U_{n,2}x_n + (1 - \xi_1)x_n\}\| \\
& = \xi_1 \|S_1 U_{n+1,2}x_n - S_1 U_{n,2}x_n\| \\
& \leq \xi_1 \|U_{n+1,2}x_n - U_{n,2}x_n\| \\
& = \xi_1 \|\xi_2 S_2 U_{n+1,3}x_n + (1 - \xi_2)x_n - \{\xi_2 S_2 U_{n,3}x_n + (1 - \xi_2)x_n\}\| \\
& = \xi_1 \xi_2 \|S_2 U_{n+1,3}x_n - S_2 U_{n,3}x_n\| \\
& \leq \xi_1 \xi_2 \|U_{n+1,3}x_n - U_{n,3}x_n\| \\
& \quad \vdots
\end{align*}
\]

\[
\begin{align*}
& \quad \leq \prod_{i=1}^{n} \xi_i \|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
& = \prod_{i=1}^{n} \xi_i \|\xi_{n+1} S_{n+1}x_n + (1 - \xi_{n+1})x_n - x_n\| \\
& = \prod_{i=1}^{n+1} \xi_i \|S_{n+1}x_n - x_n\| \\
& \leq \prod_{i=1}^{n+1} \xi_i M_3 \\
\end{align*}
\]

where \(M_3\) is an approximate constant such that

\[
M_3 \geq \max \left\{ M_2, \sup_{n \geq 1} \|S_{n+1}x_n - x_n\|, \sup_{n \geq 1} \|S_{n+1}\rho_n - \rho_n\| \right\}. \quad (3.24)
\]

Similarly, we have

\[
\|W_{n+1}\rho_n - W_n \rho_n\| \leq \prod_{i=1}^{n+1} \xi_i M_3. \quad (3.25)
\]
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Substituting (3.19), (3.23), and (3.25) into (3.22) yields that

$$\|x_{n+2} - x_{n+1}\|
\leq \alpha_{n+1} \gamma \alpha \left( \|x_{n+1} - x_n\| + \Pi_{i=1}^{n+1} \xi_i M_3 \right)
+ (1 - \alpha_{n+1} \gamma) \left( \|x_{n+1} - x_n\| + \left\| \frac{m}{d} |\beta_{n+1} - \beta_n| + |s_{n+1} - s_n| + |r_{n+1} - r_n| \right\| M_2 + \Pi_{i=1}^{n+1} \xi_i M_3 \right)
+ |\alpha_{n+1} - \alpha_n| \left\| \gamma f(W_n x_n) - BW_n \rho_n \right\|
\leq \left[ 1 - \alpha_{n+1} (\bar{\gamma} - \gamma \alpha) \right] \|x_{n+1} - x_n\| + M_4 \left( \frac{m}{d} |\beta_{n+1} - \beta_n| + |s_{n+1} - s_n| + |r_{n+1} - r_n| + |\alpha_{n+1} - \alpha_n| \right),$$

$$+ \Pi_{i=1}^{n+1} \xi_i$$

(3.26)

where $M_4$ is an approximate constant such that

$$M_4 \geq \max \left\{ M_3, \sup_{n \geq 1} \left\{ \left\| \gamma f(W_n x_n) - BW_n \rho_n \right\| \right\}. \right.$$ 

(3.27)

It follows from conditions (C1)–(C3) and $\Pi_{i=1}^{n+1} \xi_i \leq \delta_{n+1}$ and Lemma 2.2 that

$$\|x_{n+1} - x_n\| \longrightarrow 0.$$ 

(3.28)

Observe that

$$x_{n+1} - W_n \rho_n = \alpha_n (\gamma f(W_n x_n) - BW_n \rho_n),$$

(3.29)

it follows from (C1) that

$$\lim_{n \to \infty} \|W_n \rho_n - x_{n+1}\| = 0.$$ 

(3.30)

For $p \in \Omega$, we have

$$\|y_n - p\|^2 = \|P_C (I - s_n A) u_n - P_C (I - s_n A) p\|^2
\leq \| (u_n - p) - s_n (Au_n - Ap) \|^2
= \| u_n - p \|^2 - 2 s_n (u_n - p, Au_n - Ap) + s_n^2 \| Au_n - Ap \|^2
\leq \| x_n - p \|^2 - 2 s_n \left[ -u \| Au_n - Ap \|^2 + v \| u_n - p \|^2 \right] + s_n^2 \| Au_n - Ap \|^2
\leq \| x_n - p \|^2 + \left( 2 s_n u + s_n^2 - \frac{2 s_n v}{\mu^2} \right) \| Au_n - Ap \|^2.$$

(3.31)
Similarly, we have

\[
\|\rho_n - p\|^2 \leq \|x_n - p\|^2 + \left(2r_nu + r_n^2 - \frac{2r_n^2v}{\mu^2}\right)\|Ay_n - Ap\|^2. \tag{3.32}
\]

On the other hand, we have

\[
\|x_{n+1} - p\|^2 = \|a_n(\gamma f(W_nx_n) - Bp) + (I - \alpha_nB)(W_n\rho_n - p)\|^2
\leq (a_n\|\gamma f(W_nx_n) - Bp\|^2 + (1 - \alpha_n\gamma)\|\rho_n - p\|^2
\leq a_n\|\gamma f(W_nx_n) - Bp\|^2 + \|\rho_n - p\|^2 + 2\alpha_n\|\rho_n - p\||\gamma f(W_nx_n) - Bp|.
\]

Substituting (3.32) into (3.33), we have

\[
\|x_{n+1} - p\|^2 \leq a_n\|\gamma f(W_nx_n) - Bp\|^2 + \|x_n - p\|^2 + \left(2r_nu + r_n^2 - \frac{2r_n^2v}{\mu^2}\right)\|Ay_n - Ap\|^2
+ 2\alpha_n\|\rho_n - p\||\gamma f(W_nx_n) - Bp|. \tag{3.34}
\]

It follows from condition (C2) that

\[
\left(\frac{2\alpha u}{\mu^2} - 2bu - b^2\right)\|Ay_n - Ap\|^2
\leq a_n\|\gamma f(W_nx_n) - Bp\|^2 + \|x_n - p\|^2
- \|x_{n+1} - p\|^2 + 2\alpha_n\|\rho_n - p\||\gamma f(W_nx_n) - Bp|
\leq a_n\|\gamma f(W_nx_n) - Bp\|^2 + (\|x_n - p\|^2 + \|x_{n+1} - p\|)\|x_{n+1} - x_n\|
+ 2\alpha_n\|\rho_n - p\||\gamma f(W_nx_n) - Bp|.
\]

As \(\|x_{n+1} - x_n\| \to 0\) and \(\lim_{n \to \infty} \alpha_n = 0\), we have

\[
\lim_{n \to \infty} \|Ay_n - Ap\| = 0 \tag{3.36}
\]

It is easy to see that \(\|\rho_n - p\| \leq \|y_n - p\|\). Using (3.33) again, we have

\[
\|x_{n+1} - p\|^2 \leq a_n\|\gamma f(W_nx_n) - Bp\|^2 + \|y_n - p\|^2 + 2\alpha_n\|\rho_n - p\||\gamma f(W_nx_n) - Bp|. \tag{3.37}
\]

Substituting (3.31) into (3.37), we can obtain

\[
\|x_{n+1} - p\|^2 \leq a_n\|\gamma f(W_nx_n) - Bp\|^2 + \|x_n - p\|^2 + \left(2s_nu + s_n^2 - \frac{2s_n^2v}{\mu^2}\right)\|Au_n - Ap\|^2
+ 2\alpha_n\|\rho_n - p\||\gamma f(W_nx_n) - Bp|. \tag{3.38}
\]

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It follows from (C2) that

\[
\left(\frac{2av}{\mu^2} - 2bv - b^2\right) \|Au_n - Ap\|^2 \\
\leq \alpha_n \|\gamma f(W_nx_n) - Bp\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
+ 2\alpha_n \|\rho_n - p\| \|\gamma f(W_nx_n) - Bp\| \\
\leq \alpha_n \|\gamma f(W_nx_n) - Bp\|^2 + \left(\|x_n - p\| - \|x_{n+1} - p\|\right) \|x_{n+1} - x_n\| \\
+ 2\alpha_n \|\rho_n - p\| \|\gamma f(W_nx_n) - Bp\|.
\]

(3.39)

As \(\|x_{n+1} - x_n\| \to 0\) and \(\lim_{n \to \infty} \alpha_n = 0\), we have

\[
\lim_{n \to \infty} \|Au_n - Ap\| = 0.
\]

(3.40)

Observe that

\[
\|\rho_n - p\|^2 = \|P_C(I - r_nA)y_n - P_C(I - r_nA)p\|^2 \\
\leq \langle (I - r_nA)y_n - (I - r_nA)p, \rho_n - p \rangle \\
= \frac{1}{2} \left\{ \| (I - r_nA)y_n - (I - r_nA)p \|^2 + \|\rho_n - p\|^2 \\
- \| (I - r_nA)y_n - (I - r_nA)p - (\rho_n - p) \|^2 \right\} \\
\leq \frac{1}{2} \left\{ \| y_n - p \|^2 + \|\rho_n - p\|^2 - \| (y_n - \rho_n) - r_n(Ay_n - Ap) \|^2 \right\} \\
\leq \frac{1}{2} \left\{ \| x_n - p \|^2 + \|\rho_n - p\|^2 - \| y_n - \rho_n \|^2 - r_n^2 \| Ay_n - Ap \|^2 \\
+ 2r_n \langle y_n - \rho_n, Ay_n - Ap \rangle \right\},
\]

which yields that

\[
\|\rho_n - p\|^2 \leq \| x_n - p \|^2 - \| y_n - \rho_n \|^2 + 2r_n \| y_n - \rho_n \| \| Ay_n - Ap \|. 
\]

(3.42)

Substituting (3.42) into (3.33) we have

\[
\| x_{n+1} - p \|^2 \leq \alpha_n \|\gamma f(W_nx_n) - Bp\|^2 + \| x_n - p \|^2 - \| y_n - \rho_n \|^2 \\
+ 2r_n \| y_n - \rho_n \| \| Ay_n - Ap \| + 2\alpha_n \|\rho_n - p\| \|\gamma f(W_nx_n) - Bp\|. 
\]

(3.43)
which implies that

\[
\|y_n - \rho_n\|^2 \leq \alpha_n \|y f(W_n x_n) - B p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
+ 2r_n \|y_n - \rho_n\| \|A y_n - A p\| + 2\alpha_n \|\rho_n - p\| \|y f(W_n x_n) - B p\|
\]

\[
\leq \alpha_n \|y f(W_n x_n) - B p\|^2 + \|x_n - p\|^2 + \|x_{n+1} - p\| \|x_{n+1} - x_n\| \\
+ 2r_n \|y_n - \rho_n\| \|A y_n - A p\| + 2\alpha_n \|\rho_n - p\| \|y f(W_n x_n) - B p\|
\]

(3.44)

It follows from (C1), \(\|x_{n+1} - x_n\| \to 0\), and \(\|A y_n - A p\| \to 0\) that \(\lim_{n \to \infty} \|y_n - \rho_n\| = 0\).

For \(p \in \Omega\), we have

\[
\|y_n - p\|^2 \\
= \|P_C(I - s_n A) u_n - P_C(I - s_n A) p\|^2 \\
\leq \langle P_C(I - s_n A) u_n - P_C(I - s_n A) p, (I - s_n A) u_n - (I - s_n A) p \rangle \\
= \frac{1}{2} \left( \|y_n - p\|^2 - \|u_n - p - s_n (A u_n - A p)\|^2 \right) \\
\leq \frac{1}{2} \left( \|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - p - [u_n - p - s_n (A u_n - A p)]\|^2 \right) \\
= \frac{1}{2} \left( \|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2s_n \langle y_n - u_n, A u_n - A p \rangle - s_n^2 \|A u_n - A p\|^2 \right).
\]

(3.45)

This implies that

\[
\|y_n - p\|^2 \leq \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2s_n \langle y_n - u_n, A u_n - A p \rangle - s_n^2 \|A u_n - A p\|^2 \\
\leq \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2s_n \|y_n - u_n\| \|A u_n - A p\|
\]

(3.46)

By (3.46), (3.37), and (3.5), we obtain

\[
\|x_{n+1} - p\|^2 \leq \alpha_n \|y f(W_n x_n) - B p\|^2 + \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2s_n \|y_n - u_n\| \|A u_n - A p\| \\
+ 2\alpha_n \|\rho_n - p\| \|y f(W_n x_n) - B p\|
\]

\[
\leq \alpha_n \|y f(W_n x_n) - B p\|^2 + \|x_n - p\|^2 - \|y_n - u_n\|^2 \\
+ 2s_n \|y_n - u_n\| \|A u_n - A p\| + 2\alpha_n \|\rho_n - p\| \|y f(W_n x_n) - B p\|
\]

(3.47)
It follows that

\[
\|y_n - u_n\|^2 \leq a_n \|\gamma f(W_n x_n) - B p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2s_n \|y_n - u_n\| \|A u_n - A p\| \\
+ 2a_n \|\rho_n - p\| \|\gamma f(W_n x_n) - B p\| \\
\leq a_n \|\gamma f(W_n x_n) - B p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)(\|x_n - x_{n+1}\|) \\
+ 2s_n \|y_n - u_n\| \|A u_n - A p\| + 2a_n \|\rho_n - p\| \|\gamma f(W_n x_n) - B p\|.
\]  

(3.48)

It follows from (C1), \(\|A u_n - A p\| \to 0\), and \(\|x_{n+1} - x_n\| \to 0\) that \(\|y_n - u_n\| \to 0\). It follows from \(\|\rho_n - u_n\| \leq \|\rho_n - y_n\| + \|y_n - u_n\|\) that \(\lim_{n \to \infty} \|u_n - \rho_n\| = 0\).

We now show that

\[
\lim_{n \to \infty} \left\| \Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n \right\| = 0, \quad k = 1, 2, \ldots, m.
\]  

(3.49)

Indeed, let \(p \in \Omega\), it follows from the firmly nonexpansiveness of \(T_{\beta_n}^{F_k}\), we have for each \(k \in \{1, 2, \ldots, m\}\),

\[
\left\| \Theta_{\beta_n}^k x_n - p \right\|^2 = \left\|T_{\beta_n}^{F_k} \Theta_{\beta_n}^{k-1} x_n - T_{\beta_n}^{F_k} p \right\|^2 \leq \langle \Theta_{\beta_n}^k x_n - p, \Theta_{\beta_n}^{k-1} x_n - p \rangle \\
= \frac{1}{2} \left( \left\| \Theta_{\beta_n}^k x_n - p \right\|^2 + \left\| \Theta_{\beta_n}^{k-1} x_n - p \right\|^2 - \left\| \Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n \right\|^2 \right).
\]  

(3.50)

Thus, we get

\[
\left\| \Theta_{\beta_n}^k x_n - p \right\|^2 \leq \left\| \Theta_{\beta_n}^{k-1} x_n - p \right\|^2 - \left\| \Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n \right\|^2, \quad k = 1, 2, \ldots, m.
\]  

(3.51)

This implies that for each \(k \in \{1, 2, \ldots, m\}\),

\[
\left\| \Theta_{\beta_n}^k x_n - p \right\|^2 \leq \left\| \Theta_{\beta_n}^0 x_n - p \right\|^2 - \left\| \Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n \right\|^2 \\
- \left\| \Theta_{\beta_n}^{k-1} x_n - \Theta_{\beta_n}^{k-2} x_n \right\|^2 - \cdots - \left\| \Theta_{\beta_n}^1 x_n - \Theta_{\beta_n}^0 x_n \right\|^2 - \left\| \Theta_{\beta_n}^0 x_n - \Theta_{\beta_n}^0 x_n \right\|^2.
\]  

(3.52)

It follows from \(u_n = \Theta_{\beta_n}^m x_n\) that for each \(k = 1, 2, \ldots, m\)

\[
\left\| u_n - p \right\|^2 \leq \left\| x_n - p \right\|^2 - \left\| \Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n \right\|^2.
\]  

(3.53)
By (3.37), (3.6), and (3.53), we have that for each \( k = 1, 2, \ldots, m \)
\[
\| x_{n+1} - p \|^2 \leq \alpha_n \| y f(W_n x_n) - B p \|^2 + \| u_n - p \|^2 + 2 \alpha_n \| \rho_n - p \| \| y f(W_n x_n) - B p \|
\]
\[
\leq \alpha_n \| y f(W_n x_n) - B p \|^2 + \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + 2 \alpha_n \| \rho_n - p \| \| y f(W_n x_n) - B p \|.
\] (3.54)

Thus, we have that for each \( k = 1, 2, \ldots, m \)
\[
\| \Theta^k_{\rho_n} x_n - \Theta^{k-1}_{\rho_n} x_n \|^2 \leq \alpha_n \| y f(W_n x_n) - B p \|^2 + \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + 2 \alpha_n \| \rho_n - p \| \| y f(W_n x_n) - B p \|.
\] (3.55)

It follows from (C1) and \( \| x_{n+1} - x_n \| \to 0 \) that for each \( k = 1, 2, \ldots, m \)
\[
\| \Theta^k_{\rho_n} x_n - \Theta^{k-1}_{\rho_n} x_n \| \to 0.
\] (3.56)

Since
\[
\| W_n \rho_n - \rho_n \| \leq \| W_n \rho_n - x_{n+1} \| + \| x_{n+1} - x_n \| + \| x_n - \Theta^1_{\rho_n} x_n \| + \| \Theta^1_{\rho_n} x_n - \Theta^2_{\rho_n} x_n \|
\]
\[
+ \cdots + \| \Theta^{m-1}_{\rho_n} x_n - \Theta^m_{\rho_n} x_n \| + \| u_n - y_n \| + \| y_n - \rho_n \|.
\] (3.57)

It follows from (3.56) that
\[
\lim_{n \to \infty} \| W_n \rho_n - \rho_n \| = 0.
\] (3.58)

Observe that
\[
\| W \rho_n - \rho_n \| \leq \| W \rho_n - W_n \rho_n \| + \| W_n \rho_n - \rho_n \|.
\] (3.59)

It follows from Remark 2.7 that
\[
\lim_{n \to \infty} \| W \rho_n - \rho_n \| = 0.
\] (3.60)
We show that $P_Ω(γf + (I - B))$ is a contraction. Indeed, for all $x, y \in H$, we have
\[
\begin{align*}
\|P_Ω(γf + (I - B))(x) - P_Ω(γf + (I - B))(y)\| &\leq \|γf + (I - B)(x) - (γf + (I - B))(y)\| \\
&\leq γ\|f(x) - f(y)\| + \|I - B\|\|x - y\| \\
&\leq γα\|x - y\| + (1 - \tilde{γ})\|x - y\| \\
&= (γα + 1 - \tilde{γ})\|x - y\|.
\end{align*}
\]  
(3.61)

The Banach’s Contraction Mapping Principle guarantees that $P_Ω(γf + (I - B))$ has a unique fixed point, say $q ∈ H$. That is, $q = P_Ω(γf + (I - B))(q)$.

Next, we show that
\[
\limsup_{n → ∞}(γf(q) - Bq,x_n - q) ≤ 0.
\]  
(3.62)
To show that, we choose a subsequence $\{x_{n_k}\}$ of $x_n$ such that
\[
\limsup_{n → ∞}(γf(q) - Bq,x_n - q) = \lim_{i → ∞}(γf(q) - Bq,x_{n_i} - q).
\]  
(3.63)

As $\{x_n\}$ is bounded, we know that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $p$. We may assume, without loss of generality, that $x_{n_k} → p$. From $\|Ω^k\rho_n x_n - Ω^k-1\rho_n x_n\| → 0$ for each $k = 1, 2, \ldots, m$, we obtain that $Ω^k\rho_n x_n → p$ for $k = 1, 2, \ldots, m$. From $\|u_n - ρ_n\| → 0$, we also obtain that $ρ_{n_k} → p$. Since $\{u_{n_k}\} ⊂ C$ and $C$ is closed and convex, we obtain $p ∈ C$.

Now we show that $p ∈ Ω$. Indeed, let us first show that $p ∈ VI(C,A)$. Put
\[
Tw_1 = \begin{cases} 
Aw_1 + NCw_1 & \text{if } w_1 ∈ C, \\
0 & \text{if } w_1 ∉ C.
\end{cases}
\]  
(3.64)

Since $A$ is relaxed $(u,v)$-cocoercive, we have
\[
\langle Ax - Ay, x - y \rangle ≥ (-u)\|Ax - Ay\|^2 + v\|x - y\|^2 ≥ (v - uμ^2)\|x - y\|^2 ≥ 0,
\]  
(3.65)

which yields that $A$ is monotone. Thus $T$ is maximal monotone. Let $(w_1, w_2) ∈ G(T)$. Since $w_2 - Aw_1 ∈ NCw_1$ and $ρ_n ∈ C$, we have
\[
\langle w_1 - ρ_n, w_2 - Aw_1 \rangle ≥ 0.
\]  
(3.66)

On the other hand, from $ρ_n = P_C(I - r_nA)y_n$, we have
\[
\langle w_1 - ρ_n, ρ_n - (I - r_nA)y_n \rangle ≥ 0
\]  
(3.67)
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and hence

\[ \left< w_1 - \rho_n, \frac{\rho_n - y_n}{r_n} + Ay_n \right> \geq 0. \] (3.68)

It follows that

\[ \left< w_1 - \rho_n, w_2 \right> \geq \left< w_1 - \rho_n, Aw_1 \right> \]
\[ \geq \left< w_1 - \rho_n, Aw_1 - \frac{\rho_n - y_n}{r_n} - Ay_n \right> \]
\[ = \left< w_1 - \rho_n, Aw_1 - A\rho_n \right> + \left< w_1 - \rho_n, A\rho_n - Ay_n \right> \]
\[ - \left< w_1 - \rho_n, \frac{\rho_n - y_n}{r_n} \right> \]
\[ \geq \left< w_1 - \rho_n, A\rho_n - Ay_n \right> - \left< w_1 - \rho_n, \frac{\rho_n - y_n}{r_n} \right>, \] (3.69)

which implies that \( \left< w_1 - p, w_2 \right> \geq 0 \). We have \( p \in T^{-1}0 \) and hence \( p \in VI(C, A) \).

We next show that \( p \in \bigcap_{k=1}^{m} EP(F_k) \). Indeed, by Lemma 2.1, we have that for each \( k = 1,2,\ldots,m \),

\[ F_k\left( \Theta^k_{\beta_n} x_n, y \right) + \frac{1}{\beta_n} \left< y - \Theta^k_{\beta_n} x_n, \Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n \right> \geq 0, \quad \forall y \in C \] (3.70)

It follows from (A2) that

\[ \frac{1}{\beta_n} \left< y - \Theta^k_{\beta_n} x_n, \Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n \right> \geq F_k\left( y, \Theta^k_{\beta_n} x_n \right), \quad \forall y \in C. \] (3.71)

Hence,

\[ \left< y - \Theta^k_{\beta_n} x_n, \frac{\Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n}{\beta_n} \right> \geq F_k\left( y, \Theta^k_{\beta_n} x_n \right), \quad \forall y \in C. \] (3.72)

It follows from (A4), (A5), \( (\Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n)/\beta_n \to 0 \), and \( \Theta^k_{\beta_n} x_n \rightharpoonup p \) that for each \( k = 1,2,\ldots,m \),

\[ F_k(y, p) \leq 0, \quad \forall y \in C. \] (3.73)
For \( t \) with \( 0 < t \leq 1 \) and \( y \in C \), let \( y_t = ty + (1 - t)p \). Since \( y \in C \) and \( p \in C \), we obtain \( y_t \in C \) and hence \( F_k(y_t, p) \leq 0 \). So by (A4), we have

\[
0 = F_k(y_t, y_t) \leq t F_k(y_t, y) + (1 - t) F_k(y_t, p) \leq t F_k(y_t, y). \tag{3.74}
\]

Dividing by \( t \), we get that for each \( k = 1, 2, \ldots, m \),

\[
F_k(y_t, y) \geq 0. \tag{3.75}
\]

Letting \( t \to 0 \), it follows from (A3) that for each \( k = 1, 2, \ldots, m \),

\[
F_k(p, y) \geq 0 \tag{3.76}
\]

for all \( y \in C \) and hence \( p \in EP(F_k) \) for \( k = 1, 2, \ldots, m \). That is, \( p \in \bigcap_{k=1}^{m} EP(F_k) \).

We now show that \( p \in Fix(W) \). Assume that \( p \not\in Fix(W) \). Since \( \rho_{n_i} \to p \) and \( p \not\in Wp \), from (3.60) and the Opial condition we have

\[
\liminf_{i \to \infty} \| \rho_{n_i} - p \| < \liminf_{i \to \infty} \| \rho_{n_i} - Wp \|
\]

\[
\leq \liminf_{i \to \infty} \left( \| \rho_{n_i} - Wp \| + \| Wp - p \| \right)
\]

\[
\leq \liminf_{i \to \infty} \| \rho_{n_i} - p \|,
\tag{3.77}
\]

which is a contradiction. So, we get \( p \in Fix(W) = \bigcap_{i=1}^{\infty} Fix(S_i) \). This implies that \( p \in \Omega \).

Since \( q = P_{\Omega}(yf + (I - B))(q) \), we have

\[
\limsup_{n \to \infty} \langle yf(q) - Bq, x_n - q \rangle = \lim_{i \to \infty} \langle yf(q) - Bq, x_{n_i} - q \rangle
\]

\[
= \langle yf(q) - Bq, p - q \rangle \leq 0.
\tag{3.78}
\]

That is, (3.62) holds. Next, we consider

\[
\| x_{n+1} - q \|^2 = \| \alpha_n (yf(W_n x_n) - Bq) + (1 - \alpha_n B)(W_n \rho_n - q) \|^2
\]

\[
\leq (1 - \alpha_n \bar{\gamma})^2 \| W_n \rho_n - q \|^2 + 2 \alpha_n \bar{\gamma} (yf(W_n x_n) - Bq, x_{n+1} - q)
\]

\[
\leq (1 - \alpha_n \bar{\gamma})^2 \| x_n - q \|^2 + 2 \alpha_n \bar{\gamma} (f(W_n x_n) - f(q), x_{n+1} - q)
\]

\[
+ 2 \alpha_n \bar{\gamma} (yf(q) - Bq, x_{n+1} - q)
\tag{3.79}
\]

\[
\leq (1 - \alpha_n \bar{\gamma})^2 \| x_n - q \|^2 + \alpha_n \bar{\gamma} \left( \| x_n - q \|^2 + \| x_{n+1} - q \|^2 \right)
\]

\[
+ 2 \alpha_n \bar{\gamma} (yf(q) - Bq, x_{n+1} - q)
\]
So, we can obtain

\[
\|x_{n+1} - q\|^2 \leq \left( 1 - \frac{\alpha_n \gamma}{1} \right)^2 + \frac{\alpha_n \gamma}{1 - \alpha_n \gamma} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma} \langle y f(q) - Bq, x_{n+1} - q \rangle
\]

\[
= \left( 1 - \frac{2\alpha_n \gamma}{1 - \alpha_n \gamma} \right) \|x_n - q\|^2 + \frac{\alpha_n \gamma^2}{1 - \alpha_n \gamma} \|x_n - q\|^2
\]

\[
+ \frac{\alpha_n}{1 - \alpha_n \gamma} \langle y f(q) - Bq, x_{n+1} - q \rangle
\]

\[
\leq \left[ 1 - \frac{2\alpha_n \gamma}{1 - \alpha_n \gamma} \right] \|x_n - q\|^2
\]

\[
+ \frac{\alpha_n \gamma^2}{2(1 - \alpha_n \gamma)} \left( \frac{1}{\gamma - \alpha \gamma} \langle y f(q) - Bq, x_{n+1} - q \rangle + \frac{\alpha_n \gamma^2}{2(1 - \alpha_n \gamma)} M \right),
\]

where \( M \) is an approximate constant such that \( M \geq \sup_{n \geq 1} \|x_n - q\|^2 \).

Put \( l_n = 2\alpha_n \gamma / (1 - \alpha_n \gamma) \) and \( t_n = (1 / (\gamma - \alpha \gamma)) \langle y f(q) - Bq, x_{n+1} - q \rangle + (\alpha_n \gamma^2 / 2(\gamma - \alpha \gamma)) M \). That is,

\[
\|x_{n+1} - q\|^2 \leq (1 - l_n) \|x_n - q\|^2 + l_n t_n.
\]

From condition (C1) and Lemma 2.2, we concluded that \( x_n \to q \in \Omega \). It is easy to see that \( u_n \to q \) and \( y_n \to q \). This completes the proof. \( \square \)

**Corollary 3.2.** Let \( C \) be a nonempty, closed and convex subset of \( H \). Let \( F \) be a bifunction from \( C \times C \) to \( \mathbb{R} \) satisfies conditions (A1)–(A5). Let \( A : C \to H \) be relaxed \((u,v)\)-cocoercive and \( \mu \)-Lipschitz continuous and \( B \) a strongly positive linear bounded operator on \( H \) with coefficient \( \gamma \geq 0 \). Assume that \( 0 < \gamma < \gamma/\alpha \). Let \( S_1, S_2, \ldots \) be a family of infinitely nonexpansive mappings of \( C \) into itself such that \( \Gamma = \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \cap \text{VI}(C,A) \cap \text{EP}(F) \neq \emptyset \), let \( \xi_0, \xi_1, \ldots \) be real numbers such that \( 0 <\xi_i \leq \delta < 1 \) for every \( i \in \mathbb{N} \) and \( W_n \) be the \( W \)-mapping of \( C \) into itself generated by \( S_n, S_{n-1}, \ldots, S_1 \) and \( \xi_0, \xi_1, \ldots, \xi_1 \). Let \( f : C \to C \) be a contraction with coefficient \( \alpha \) \((0 < \alpha < 1)\) and \( \{x_n\}, \{u_n\} \) and \( \{y_n\} \) be sequences generated by

\[
x_1 = x \in H,
\]

\[
F(u_n, y) + \frac{1}{\beta_n} (y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C,
\]

\[
y_n = P_C(I - \delta_n A)u_n,
\]

\[
x_{n+1} = \alpha_n f(W_n x_n) + (I - \alpha_n B)W_n P_C(I - r_n A)y_n
\]
for every $n = 1, 2, \ldots$, where $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$ and $\{s_n\}$ are sequences of numbers satisfying the conditions:

(C1) $\{\alpha_n\} \subset [0, 1]$ with $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\{r_n\} \subset [a, b]$ and $\{s_n\} \subset [a, b]$ for some $a, b$ with $0 \leq a < b < 2(\nu - \mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, and $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(C3) $\lim_{n \to \infty} \beta_n > 0$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \Gamma$, which solves the following variational inequality:

\[
\langle yf - Bq, p - q \rangle \leq 0, \quad \forall p \in \Gamma.
\] (3.83)

Proof. Let $m = 1$, by Theorem 3.1, we obtain the desired result. \qed

Corollary 3.3. Let $C$ be a nonempty, closed, and convex subset of $H$. Let $A : C \to H$ be relaxed $(u, v)$-cocoercive and $\mu$-Lipschitz continuous and let $B$ be a strongly positive linear bounded operator on $H$ with coefficient $\tilde{\gamma} > 0$. Assume that $0 < \gamma < \tilde{\gamma}/\alpha$. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of $C$ into itself such that $\Delta = \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \cap \text{VI}(C, A) \neq \emptyset$, let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i < 1$ for every $i \in \mathbb{N}$, and let $W_n$ be the $W$-mapping of $C$ into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Let $f : C \to C$ be a contraction with coefficient $\alpha$ ($0 < \alpha < 1$) and $\{x_n\}, \{u_n\}, \{y_n\}$ be sequences generated by

\[
x_1 = x \in C,
\]

\[
y_n = P_C(I - s_n A)x_n,
\]

\[
x_{n+1} = \alpha_n y f(W_n x_n) + (I - \alpha_n B)W_n P_C(I - r_n A)y_n
\] (3.84)

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$, and $\{s_n\}$ are sequences of numbers satisfying the conditions:

(C1) $\{\alpha_n\} \subset [0, 1]$ with $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\{r_n\} \subset [a, b]$ and $\{s_n\} \subset [a, b]$ for some $a, b$ with $0 \leq a < b < 2(\nu - \mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ and $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$.

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \Delta$, which solves the following variational inequality:

\[
\langle yf - Bq, p - q \rangle \leq 0, \quad \forall p \in \Delta.
\] (3.85)

Proof. Let $F(x, y) = 0$ for $x, y \in C$, by Corollary 3.2 we obtain the desired result. \qed

Corollary 3.4. Let $C$ be a nonempty, closed, and convex subset of $H$. Let $F_1, F_2, \ldots, F_m$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfies conditions (A1)-(A5). Let $A : C \to H$ be relaxed $(u, v)$-cocoercive and $\mu$-Lipschitz continuous and $B$ a strongly positive linear bounded operator on $H$ with coefficient $\tilde{\gamma} > 0$.
such that $\Xi = \bigcap_{k=1}^{m} EP(F_k) \cap VI(C,A) \neq \emptyset$. Let $f : C \to C$ be a contraction with coefficient $\alpha$ ($0 < \alpha < 1$) and $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by

$$
    x_1 = x \in H,
    u_n = T_{\beta_n}^{F_n} T_{\beta_{n-1}}^{F_{n-1}} \cdots T_{\beta_1}^{F_1} x_n,
    y_n = P_C (I - s_n A) u_n,
    x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n B) P_C (I - r_n A) y_n
$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$ and $\{s_n\}$ are sequences of numbers satisfying the conditions:

(C1) $\{\alpha_n\} \subset [0, 1]$ with $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\{r_n\} \subset [a, b]$ and $\{s_n\} \subset [a, b]$ for some $a, b$ with $0 \leq a \leq b \leq 2(b-a \mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, and $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(C3) $\lim \inf_{n \to \infty} \beta_n > 0$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}, \{y_n\}$ and $\{u_n\}$ converge strongly to $q \in \Omega$, which solves the following variational inequality:

$$
    \langle y f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \Xi.
$$

Remark 3.5. (i) If $s_n = 0$ for all $n \geq 0$, by Corollary 3.2, we get Theorem 2.1 in [9]. If $s_n = 0$ and $S_i = I$ for all $n \geq 0$, by Corollary 3.2, we get Theorem 2.1 in [8] with $S = I$. If $S_i = I, r_n = 0$ and $S_i = I$ for all $n \geq 0$, by Corollary 3.2, we get Theorem 3.1 in [6] with $S = I$ and Theorem 3.3 in [7] with $S = I$ and $C = H$.

(ii) Corollary 3.3 extends, generalizes and improves the main results in [21, 22, 24].

(iii) It is easy to see that Theorem 3.1 is different from the main results in [1–4].

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