Research Article

Existence and Nonexistence Results for Classes of Singular Elliptic Problem

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1. Introduction and Main Results

In this paper, we study the existence or the nonexistence of solutions to the following singular semilinear elliptic problem

\[-\Delta u + k(x)u^{-\gamma} = \lambda u^p, \quad \text{in} \quad \Omega,\]

\[u > 0, \quad \text{in} \quad \Omega,\]

\[u = 0, \quad \text{on} \quad \partial \Omega,\]

where \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is a bounded domain with \(C^{2+\alpha}\) boundary for some \(\alpha \in (0,1)\), \(k \in C^\alpha_{\text{loc}}(\Omega) \cap C(\overline{\Omega})\), and \(\gamma, p, \lambda\) are three nonnegative constants. This problem arises in the study of non-Newtonian fluids, chemical heterogeneous catalysts, in the theory of heat conduction in electrically conducting materials (see [1–7] and their references).

Many authors have considered this problem. For examples, when \(k(x) < 0\) in \(\Omega\), problem (1.1) was studied in [3, 8–11]; when \(k(x) > 0\) in \(\Omega\), problem (1.1) was considered in [12–14]. Particularly, when \(k(x) \equiv 1\), it has been established in Zhang [14] that there exists \(\overline{\lambda} > 0\) such that problem (1.1) has at least one solution in \(C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})\) for all \(\lambda > \overline{\lambda}\) and
has no solution in \( C^2(\Omega) \cap C(\overline{\Omega}) \) if \( \lambda < \lambda_0 \). After that Shi and Yao in [13] have also obtained the same results with \( k \in C^{2,a}(\overline{\Omega}) \) and \( k(x) > 0 \) in \( \overline{\Omega} \). Recently, Ghergu and Rădulescu in [12] considered more general sublinear singular elliptic problem with \( k \in C^{s}(\overline{\Omega}) \).

In this paper, we consider the case that \( k \in C^{s}_{loc}(\Omega) \cap C(\overline{\Omega}) \), and \( k \) may have zeros in \( \overline{\Omega} \). The following main results are obtained by the sub-supersolution method with restriction on the boundary in Cui [15].

**Theorem 1.1.** Suppose that \( k \in C^{s}_{loc}(\Omega) \cap C(\overline{\Omega}), k \geq 0, \) and \( k \neq 0 \). Assume that \( 0 < p < 1 \) and \( 0 < \gamma < 1 \) such that problem (1.1) has at least one solution \( u_\lambda \in C^{2+a}(\Omega) \cap C(\overline{\Omega}) \) and \( u_\lambda^{+} \in L^1(\Omega) \) for all \( \lambda > \lambda_0 \), and problem (1.1) has no solution in \( C^2(\Omega) \cap C(\overline{\Omega}) \) if \( \lambda < \lambda_0 \). Moreover, problem (1.1) has a maximal solution \( v_\lambda \) which is increasing with respect to \( \lambda \) for all \( \lambda > \lambda_0 \).

**Remark 1.2.** Theorem 1.1 generalizes Theorem 1.2 in [13] in coefficient \( k(x) \) of the singular term. Consequently, it also generalizes Theorem 1 in [14]. Moreover, there are functions \( k \) satisfying our Theorem 1.1 and not satisfying Theorem 1.2 in [13]. For example, let

\[
k(x) = \begin{cases} 
\frac{1}{\ln(|x - x_0|/(2d))}, & x \in \overline{\Omega} \setminus \{x_0\}, \\
0, & x = x_0,
\end{cases}
\]

where \( x_0 \in \partial \Omega \), and

\[
d = \text{diam}(\Omega) \triangleq \max \left\{ |x - y| \mid x, y \in \overline{\Omega} \right\}.
\]

Certainly, this example does not satisfy Theorem 1.2 in [12] yet.

**Theorem 1.3.** Suppose that \( k \in C^a_{loc}(\Omega) \cap C(\overline{\Omega}) \) and \( k(x) > 0 \) in \( \overline{\Omega} \). If \( \gamma \geq 1 \), problem (1.1) has no solution in \( C^2(\Omega) \cap C(\overline{\Omega}) \) for all \( \lambda > 0 \) and \( p > 0 \).

**Remark 1.4.** Obviously, Theorem 1.3 is a generalization of Theorem 2 in [14]. There are also functions \( k(x) \) satisfying our Theorem 1.3 and not satisfying Theorem 2 in [14] and Theorem 1.1 in [12]. For example, let

\[
k(x) = \begin{cases} 
\frac{1}{\ln(|x - x_0|/(2d))} + \varepsilon, & x \in \overline{\Omega} \setminus \{x_0\}, \\
\varepsilon, & x = x_0,
\end{cases}
\]

where \( x_0 \in \partial \Omega, \varepsilon \) is any positive constant and \( d = \text{diam}(\Omega) \) is the diameter of \( \Omega \).
2. Proof of Theorems

Consider the more general semilinear elliptic problem

\[-\Delta u = f(x,u), \quad \text{in } \Omega,\]
\[u > 0, \quad \text{in } \Omega,\]
\[u = 0, \quad \text{on } \partial \Omega,\]

(2.1)

where the function \(f(x,s)\) is locally \(H^\alpha\) continuous in \(\Omega \times (0, \infty)\) and continuously differentiable with respect to the variable \(s\). A function \(\underline{u}\) is called to be a subsolution of problem (2.1) if \(\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})\), and

\[-\Delta \underline{u} \leq f(x,\underline{u}), \quad \text{in } \Omega,\]
\[\underline{u} > 0, \quad \text{in } \Omega,\]
\[\underline{u} = 0, \quad \text{on } \partial \Omega.\]

(2.2)

A function \(\overline{u}\) is called to be a supersolution of problem (2.1) if \(\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})\), and

\[-\Delta \overline{u} \geq f(x,\overline{u}), \quad \text{in } \Omega,\]
\[\overline{u} > 0, \quad \text{in } \Omega,\]
\[\overline{u} = 0, \quad \text{on } \partial \Omega.\]

(2.3)

According to Lemma 3 in the study of Cui [15], we can easily have the following basic existence of classical solution to problem (2.1).

Lemma 2.1. Let \(f \in C^\alpha_{\text{loc}}(\Omega \times (0, \infty))\) be continuously differentiable with respect to the variable \(s\). Suppose that problem (2.1) has a supersolution \(\overline{u}\) and a subsolution \(\underline{u}\) such that

\[\underline{u}(x) \leq \overline{u}(x), \quad \text{in } \Omega,\]

(2.4)

then problem (2.1) has at least one solution \(u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})\) satisfying

\[\underline{u}(x) \leq u(x) \leq \overline{u}(x), \quad \text{in } \overline{\Omega}.\]

(2.5)

Let \(\lambda_1\) be the first eigenvalue of the eigenvalue problem

\[-\Delta u = \lambda u, \quad \text{in } \Omega,\]
\[u = 0, \quad \text{on } \partial \Omega,\]

(2.6)

and \(\varphi_1 > 0\) in \(\Omega\) the corresponding eigenfunction. Then \(\varphi_1 \in C^{2+\alpha}(\overline{\Omega})\). Moreover one has the following lemma.
Lemma 2.2 (see [10]). One has
\[ \int_\Omega \varphi_1^r \, dx < \infty \] (2.7)
if and only if \( r > -1 \).

Now we give the proof of our theorems.

Proof of Theorem 1.1. Let \( p \in (0, 1) \), and let \( u^* \) denote the unique solution of
\[ -\Delta u = u^p, \quad \text{in } \Omega, \]
\[ u > 0, \quad \text{in } \Omega, \]
\[ u = 0, \quad \text{on } \partial \Omega, \] (2.8)
where \( u^* \) belongs to \( C^2(\bar{\Omega}) \) (see [16]). Then \( u = \lambda^{1/(1-p)} u^* \) is a solution of
\[ -\Delta u = \lambda u^p, \quad \text{in } \Omega, \]
\[ u > 0, \quad \text{in } \Omega, \]
\[ u = 0, \quad \text{on } \partial \Omega, \] (2.9)
where \( 0 < p < 1 \) and \( \lambda > 0 \). Then fix \( \lambda > 0 \) and set
\[ \overline{u} = \lambda^{1/(1-p)} u^*, \] (2.10)
thus we can easily obtain that \( \overline{u} \) is a supersolution of problem (1.1).

Now, we want to find a subsolution of problem (1.1). Let
\[ \underline{u} = M \varphi_1^{2/(1+\gamma)}, \] (2.11)
where \( M \) is a positive constant; now we will prove that \( \underline{u} \) is a subsolution of problem (1.1). By Hopf's maximum principle in [17], there exist \( \delta > 0 \) and \( \varepsilon_0 > 0 \) such that
\[ |\nabla \varphi_1| \geq \delta, \quad \text{on } \Omega \setminus \Omega', \]
\[ \varphi_1 \geq \delta, \quad \text{on } \Omega', \] (2.12)
where \( \Omega' = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \varepsilon_0 \} \). On \( \Omega' \), we choose \( M \geq M_1 = \frac{\lambda_1 M}{\lambda_1 \delta^2} \) \( |\kappa|_{\infty} (1 + \gamma) \), then we have
\[ \frac{k(x)}{M^\gamma \varphi_1^{2/(1+\gamma)}} \leq \frac{\lambda_1 M}{\lambda_1 \delta^2} \frac{2/(1+\gamma)}{\varphi_1^{2/(1+\gamma)}}, \] (2.13)
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where \( \| k \|_\infty = \max \{ |k(x)| \ | x \in \overline{\Omega} \} \) for \( k \in C(\overline{\Omega}) \). On \( \Omega \setminus \Omega' \), we choose \( M \geq M_2 \overset{\Delta}{=} (\|k\|_\infty (1 + \gamma)^2 / 2(1 - \gamma) \delta^2)^{1/(1 + \gamma)} \), then one obtains

\[
\frac{k(x)}{M \varphi_1^{2/(1 + \gamma)}} \leq \frac{2(1 - \gamma)M |\nabla \varphi_1|^2}{(1 + \gamma)^2 \varphi_1^{2/(1 + \gamma)}}. \tag{2.14}
\]

Thus, we choose \( M \geq \max \{ M_1, M_2 \} \), then fixing \( M \), let \( \lambda > \lambda' \overset{\Delta}{=} (3\lambda_1M^{1 - p}) / (1 + \gamma) \| \varphi_1 \|_\infty^{2(1 - p)/(1 + \gamma)} \), it follows from (2.13) and (2.14) that

\[
-\Delta u + k(x)u_\alpha^{-\gamma} = -M \Delta \varphi_1^{2/(1 + \gamma)} + \frac{k(x)}{M \varphi_1^{2/(1 + \gamma)}}
\]

\[
= -M \left( \frac{2(1 - \gamma)}{1 + \gamma} |\nabla \varphi_1|^2 \varphi_1^{2/(1 + \gamma)} + \frac{2}{1 + \gamma} \varphi_1^{1 - 1/(1 + \gamma)} \Delta \varphi_1 \right) + \frac{k(x)}{M \varphi_1^{2/(1 + \gamma)}}
\]

\[
= \frac{2\lambda_1M}{1 + \gamma} \varphi_1^{2/(1 + \gamma)} + \frac{k(x)}{M \varphi_1^{2/(1 + \gamma)}} - \frac{2(1 - \gamma)M |\nabla \varphi_1|^2}{(1 + \gamma)^2 \varphi_1^{2/(1 + \gamma)}}
\]

\[
\leq \frac{3\lambda_1M}{1 + \gamma} \varphi_1^{2/(1 + \gamma)}
\]

\[
\leq \lambda \left( M \varphi_1^{2/(1 + \gamma)} \right)^p
\]

\[
= \lambda u_\alpha^p. \tag{2.15}
\]

Thus we proved that \( u = M \varphi_1^{2/(1 + \gamma)} \) is a subsolution of problem (1.1) for all \( \lambda > \lambda' \). According to Lemma 4 in [14], there exists a positive constant \( C \) such that

\[
\varphi_1(x) \leq Cu^*(x), \quad \text{in } \overline{\Omega}. \tag{2.16}
\]

Set \( \lambda \geq \lambda'' \overset{\Delta}{=} (MC\| \varphi_1 \|_\infty^{(1 - p)/(1 + \gamma)})^{1 - p} \), then we have

\[
\overline{u} = \lambda^{1/(1 - p)}u^* \geq u = M \varphi_1^{2/(1 + \gamma)}, \quad \text{in } \Omega. \tag{2.17}
\]

Thus we choose \( \lambda^* = \max \{ \lambda', \lambda'' \} \); via Lemma 2.1, problem (1.1) has at least one solution \( u_1 \in C^{2 + \delta}(\Omega) \cap C(\overline{\Omega}) \) and satisfying

\[
\underline{u}(x) \leq u_1(x) \leq \overline{u}(x), \quad \text{in } \overline{\Omega}, \tag{2.18}
\]

for all \( \lambda \geq \lambda^* \).
Since \( u_1 \geq M \phi_1^{2/(1+\gamma)} \) in \( \overline{\Omega} \) for all \( \lambda \geq \lambda^* \) and \(-2\gamma/(1+\gamma) > -1\), according to Lemma 2.2 one has

\[
\int_{\Omega} u_1^{-\gamma}(x) \, dx \leq \frac{1}{M^\gamma} \int_{\Omega} \phi_1^{-2\gamma/(1+\gamma)}(x) \, dx < +\infty. \tag{2.19}
\]

So we obtain \( u_1^{-\gamma} \in L^1(\Omega) \).

Let \( \Omega_j = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > r/2j \} \), \( j = 1, 2, 3, \ldots \), and let \( u_j \) be the unique solution of

\[
-\Delta u + k(x)u_j^{-\gamma} = \lambda u_j^p, \quad \text{in } \Omega_j,
\]

\[
 u = u_{j-1}, \quad \text{on } \overline{\Omega} \setminus \Omega_j,
\]

for \( j = 1, 2, 3, \ldots \), and with \( u_0 = \bar{u} = \lambda^{1/(1-p)}u^* \), where

\[
r = \max_{x \in \Omega} \min_{y \in \partial \Omega} |x - y|. \tag{2.21}
\]

We claim that \( u_j \) is nonincreasing with respect to \( j \) in \( \overline{\Omega} \) for all \( j \in \mathbb{N} \). Indeed, since \( \bar{u} \) is a supersolution of problem (1.1) for all \( \lambda > 0 \), then we have

\[
-\Delta (u_0 - u_1) = -\Delta u_0 + \Delta u_1
\]

\[
= -\Delta u_0 + k(x)u_0^{-\gamma} - \lambda u_0^p
\]

\[
= -\Delta \bar{u} + k(x)\bar{u}^{-\gamma} - \lambda \bar{u}^p
\]

\[
> 0,
\]

for all \( x \in \Omega_1 \). Since \( u_1 = u_0 \) in \( \overline{\Omega} \setminus \Omega_1 \), so by the maximum principle, one has \( u_0 \geq u_1 \) in \( \overline{\Omega} \). So when \( j = 0 \) our claim is true. We assume that our claim is true when \( j = n \); that is, \( u_n \leq u_{n-1} \) in \( \overline{\Omega} \). Then we obtain

\[
-\Delta (u_n - u_{n+1}) = -\Delta u_n + \Delta u_{n+1}
\]

\[
= \lambda \left( u_{n-1}^p - u_n^p \right) + k(x) \left( u_n^{-\gamma} - u_{n-1}^{-\gamma} \right)
\]

\[
> 0,
\]

for all \( x \in \Omega_{n+1} \). Since \( u_n = u_{n+1} \) in \( \overline{\Omega} \setminus \Omega_{n+1} \), so by the maximum principle, one has \( u_n \geq u_{n+1} \) in \( \overline{\Omega} \). Thus by the induction, one obtains

\[
 u_{j+1} \leq u_j, \quad \text{in } \overline{\Omega}, \tag{2.24}
\]
for all \( j \in N \). Then by the monotonicity of \( u_j \), we have

\[
-\Delta u_j = \lambda u_j^p - k(x)u_j^q,
\]

\[
\geq \lambda u_j^p - k(x)u_j^q,
\]

for all \( x \in \Omega_j \) and \( j \in N^+ \). According to the definitions of \( u_j \) and \( u_0 \), we obtain that \( u_j \) is a supersolution of problem (1.1) for all \( j \in N^+ \). Let \( u_1 \) be a classical solution of problem (1.1), thus one has

\[
u_1(x) \leq u_{j+1}(x) \leq u_j(x) \leq u_0(x), \quad \text{in } \Omega.
\] (2.26)

Assume that \( v_1(x) = \lim_{j \to \infty} u_j(x) \) for all \( x \in \Omega \), then by standard elliptic arguments (see [17]) it follows that \( v_1 \) is a solution of problem (1.1), and \( v_1 \geq u_1 \) in \( \Omega \) for any \( u_1 \). Therefore, \( v_1 \) is the maximal solution of problem (1.1). According to the above arguments, problem (1.1) has a maximal solution for \( \lambda \geq \lambda^* \).

To complete the proof of Theorem 1.1, setting

\[
\sigma = \{ \lambda > 0 \mid \text{problem (1.1) has at least one solution } u_1 \},
\]

\[
\bar{\lambda} = \inf \sigma,
\] (2.27)

then \([\lambda^*, +\infty) \subset \sigma, \bar{\lambda} \leq \lambda^* \). It suffices to prove that if \( \lambda_0 \in \sigma \), then \([\lambda_0, +\infty) \subset \sigma \); that is, assume that \( \lambda > \lambda_0 \), then problem (1.1) has at least one solution. Let \( u_{\lambda_0} \) be a solution of problem (1.1) corresponding to \( \lambda_0 \), then \( u_{\lambda_0} \) is a subsolution of problem (1.1) with every fixed \( \lambda > \lambda_0 \). Since \( \bar{u} = \lambda^{1/(1-p)}u^* \) is a supersolution of problem (1.1) for any \( \lambda > 0 \), then one has

\[
\lambda^{1/(1-p)}u^* \geq \lambda_0^{1/(1-p)}u^* \geq u_{\lambda_0}, \quad \text{in } \Omega,
\] (2.28)

for all \( \lambda > \lambda_0 \). According to Lemma 2.1, problem (1.1) has at least one solution \( u_1 \in C^{2+\alpha}(\Omega) \cap C(\Omega) \) for all \( \lambda > \lambda_0 \). Moreover,

\[
u_{\lambda_0}(x) \leq u_1(x) \leq \bar{u}(x), \quad \text{in } \Omega.
\] (2.29)

Consequently, the maximal solution \( v_1 \) of problem (1.1) is increasing with respect to \( \lambda \) for all \( \lambda > \bar{\lambda} \). So the proof of Theorem 1.1 is completed.

\[ \square \]

**Proof of Theorem 1.3.** Suppose to the contrary that there exists \( \lambda > 0 \) such that problem (1.1) has one solution \( u_1 \in C^2(\Omega) \cap C(\Omega) \). Let \( e \) be the unique solution of

\[
-\Delta u = 1, \quad \text{in } \Omega,
\]

\[
u = 0, \quad \text{on } \partial \Omega,
\] (2.30)
By the maximum principle, $e > 0$ in $\Omega$. We claim that for any solution $u_\lambda$ of problem (1.1), there exists a constant $M = M(\lambda) > 0$ such that

$$Me(x) > u_\lambda(x), \quad \text{in } \Omega.$$  \hfill (2.31)

Indeed, let $M = \lambda \|u_\lambda\|_{\infty}^p + 1$, then one obtains

$$-\Delta (Me - u_\lambda) = -M\Delta e + \Delta u_\lambda$$

$$= \lambda\|u_\lambda\|_{\infty}^p + 1 - \lambda u_\lambda^p(x) + k(x)u_\lambda^{-r}$$

$$> 0,$$

for all $x \in \Omega$. Since $(Me - u_\lambda)|_{\partial \Omega} = 0$, by the maximum principle we have

$$Me(x) > u_\lambda(x), \quad \text{in } \Omega.$$  \hfill (2.33)

According to Lemma 4 in [14], there exists a positive constant $C$ such that

$$e(x) \leq C\varphi_1(x), \quad \text{in } \Omega.$$  \hfill (2.34)

Since $\gamma \geq 1$, from Lemma 2.2, it follows that

$$\int_{\Omega} u_\lambda^{-r}(x)dx \geq \frac{1}{(CM)^{\gamma}} \int_{\Omega} \varphi_1^{-r}(x)dx = +\infty.$$  \hfill (2.35)

Thus we obtain

$$\int_{\Omega} u_\lambda^{-r}dx = +\infty.$$  \hfill (2.36)

Set

$$\Omega_i = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \frac{r}{2i}, \ i \in \mathbb{N}^+ \},$$

and $\bar{\Omega} = \bigcup_{i=1}^{\infty} \Omega_i$, then $\Omega_i \subset \Omega$ and $u_\lambda \in C^2(\bar{\Omega}_i)$, satisfying

$$-\Delta u_\lambda + k(x)u_\lambda^{-r} = \lambda u_\lambda^p,$$

for all $x \in \bar{\Omega}_i$ and $i \in \mathbb{N}^+$. Consequently, integrating (2.38) we have

$$-\int_{\Omega_i} \Delta u_\lambda dx + \int_{\Omega_i} k(x)u_\lambda^{-r}dx = \lambda \int_{\Omega_i} u_\lambda^p dx \leq \lambda \int_{\Omega_i} u_\lambda^p dx,$$  \hfill (2.39)
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noting that

$$\int_{\Omega} \Delta u_{\lambda} dx = \int_{\partial\Omega} \frac{\partial u_{\lambda}}{\partial n} ds,$$

(2.40)

where \( n \) denotes the outward normal to \( \partial\Omega \). From (2.39) and (2.40), letting \( i \to \infty \), one has

$$\int_{\Omega} k(x)u_{\lambda}^T dx - \int_{\partial\Omega} \frac{\partial u_{\lambda}}{\partial n} ds \leq \lambda \|u_{\lambda}\|_p \|\Omega\|,$$

(2.41)

where \( |\Omega| \) denotes the Lebesgue measure of \( \Omega \). According to (2.36) and \( k(x) > 0 \) in \( \overline{\Omega} \), one obtains

$$\int_{\partial\Omega} \frac{\partial u_{\lambda}}{\partial n} ds = +\infty.$$

(2.42)

But this is impossible, by Hopf’s maximum principle, we have

$$\frac{\partial u_{\lambda}}{\partial n} < 0,$$

(2.43)

for all \( x \in \partial\Omega \), where \( n \) denotes the outward normal to \( \partial\Omega \) at \( x \). Therefore Theorem 1.3 is true. \( \square \)

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References


