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Research Article

New Oscillation Criteria for Second-Order Neutral Delay Differential Equations with Positive and Negative Coefficients

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We establish the oscillation and asymptotic criteria for the second-order neutral delay differential equations with positive and negative coefficients having the forms $[x(t) + \sum_{i\in R} c_i(t)x(\alpha_i(t))]'' + r(t)[x(t) + \sum_{i\in R} c_i(t)x(\alpha_i(t))]' + \sum_{i\in P} p_i(t)x(\beta_i(t)) - \sum_{i\in Q} q_i(t)x(\gamma_i(t)) = 0$ and $[x(t) + \sum_{i\in R} c_i(t)x(\alpha_i(t))]'' + r(t)[x(t) + \sum_{i\in R} c_i(t)x(\alpha_i(t))]' + \sum_{i\in P} p_i(t)x(\beta_i(t)) - \sum_{i\in Q} q_i(t)x(\gamma_i(t)) = f(t)$. The obtained new oscillation criteria extend and improve the recent results given in the paperof B. Karpuz et al. (2009).

1. Introduction

In this paper, we consider the oscillation of all solutions of the second-order neutral delay differential equations with positive and negative coefficients having the forms

$$\left[x(t) + \sum_{i\in R} c_i(t)x(\alpha_i(t))\right]'' + r(t)\left[x(t) + \sum_{i\in R} c_i(t)x(\alpha_i(t))\right]' + \sum_{i\in P} p_i(t)x(\beta_i(t)) - \sum_{i\in Q} q_i(t)x(\gamma_i(t)) = 0, \tag{1.1}$$

$$\left[x(t) + \sum_{i\in R} c_i(t)x(\alpha_i(t))\right]'' + r(t)\left[x(t) + \sum_{i\in R} c_i(t)x(\alpha_i(t))\right]' + \sum_{i\in P} p_i(t)x(\beta_i(t)) - \sum_{i\in Q} q_i(t)x(\gamma_i(t)) = f(t). \tag{1.2}$$

We introduce the following class of functions $D([t_0, \infty))$ equipped with the functions satisfying the following properties:

(P_1) $f \in C^1([t_0, \infty), \mathbb{R})$ is strictly increasing and $\lim_{t\to\infty} f(t) = \infty$ holds,

(P_1) $f(t) \le t$ holds for all $t \ge t_0$.

In this paper, we make the following assumptions:

(H_1) R, P, Q are bounded starting segments of positive integers; that is,

$$R = \{1, 2, \ldots, R_0\}, \quad P = \{1, 2, \ldots, P_0\}, \quad Q = \{1, 2, \ldots, Q_0\}, \quad R_0, P_0, Q_0 \in \mathbb{N}; \tag{1.3}$$

(H_2) $c_i \in C([t_0, \infty), \mathbb{R}^+)$ for all $i \in R, p_i \in C([t_0, \infty), \mathbb{R}^+)$ for all $i \in P$, and $q_i \in C([t_0, \infty), \mathbb{R}^+)$ for all $i \in Q$;

(H_3) $\alpha_i \in D([t_0, \infty))$ with $\lim\inf_{t\to\infty} \alpha_i'(t) > 0$ for all $i \in \mathrm{R}, \beta_i \in D([t_0, \infty))$ for all $i \in P$, and $\gamma_i \in D([t_0, \infty))$ for all $i \in Q$;

(H_4) $r \in C^1([t_0, \infty), \mathbb{R}^+)$, and $r'(t) \le 0$;

(H_5) $f \in C([t_0, \infty), \mathbb{R})$ and that there exists a function $F \in C^2([t_0, \infty), \mathbb{R})$ which satisfies $F'' = f$ and $\lim_{t\to\infty} F(t) = 0$.

In order to establish our main results, we will assume that there exists a mapping $\psi : Q \to P$ satisfying the following conditions:

(A_1) $\gamma_i(t) \ge \beta_{\psi(i)}(t)$ for all $t \ge t_0$ and $i \in Q$;

(A_2) $h_i \in C([t_0, \infty), \mathbb{R}^+)$ for all $i \in P$, where

$$h_i(t) := \begin{cases} p_i(t) - \displaystyle\sum_{j\in Q, \psi(j)=i} \rho_j'(t) q_j(\rho_j(t)), & i \in \psi(Q), \\ p_i(t), & i \notin \psi(Q), \end{cases} \tag{1.4}$$

and $\rho_i(t) := \gamma_i^{-1}(\beta_{\psi(i)}(t))$ for all $i \in Q$ and $t \ge t_0$;

(A_3) there exists $i_0 \in P$ such that $\lim\inf_{t\to\infty} h_{i_0}(t) > 0$ and $\lim\sup_{t\to\infty} \beta_{i_0}'(t) < \infty$.

A function x is called a solution of (1.1) (or (1.2)) provided that x satisfies (1.1) (or (1.2)) identically on $[t_0, \infty)$, $x + \sum_{i\in R} c_i(t) x \circ \alpha_i \in C^2([t_0, \infty), \mathbb{R})$ and $x \in C([t_{-1}, \infty), \mathbb{R})$, where $t_{-1} := \min\{\alpha, \beta, \gamma\}$, $\alpha := \min\{\alpha_i(t_0) : i \in R\}$, $\beta := \min\{\beta_i(t_0) : i \in P\}$, and $\gamma := \min\{\gamma_i(t_0) : i \in Q\}$. We restrict our attention only to the nontrivial solution x, that is, to the solution x such that $\sup\{|x(t)| : t \ge t_1\} > 0$ for all $t_1 \ge t_0$. A nontrivial solution of (1.1) (or (1.2)) is called oscillatory if it has arbitrary large zeros, otherwise, it is called nonoscillatory.

The oscillation and nonoscillation of solutions of second-order neutral delay differential equations have been studied by many authors; see [1–10]. However, to the best of our knowledge,there seem to be few oscillation results for (1.1) and (1.2).

Recently,Manojlović et al. [4] andWeng and Sun [10] have studied oscillation and asymptotic behavior of all solutions of the following equations:

$$\begin{gathered} \left[x(t) + \sum_{i=1}^{l} c_i(t) x(t-\tau_i)\right]'' + \sum_{i=1}^{m} p_i(t) x(t-\delta_i) - \sum_{i=1}^{n} q_i(t) x(t-\sigma_i) = 0, \quad t \ge t_0, \\ \left[x(t) + \sum_{i=1}^{l} c_i(t) x(t-\tau_i)\right]'' + \sum_{i=1}^{m} p_i(t) x(t-\delta_i) - \sum_{i=1}^{n} q_i(t) x(t-\sigma_i) = f(t), \quad t \ge t_0, \end{gathered} \tag{1.5}$$

and several well-known results have been obtained.

By using weaker conditions than in [4, 10], Karpuz et al. [1] have established oscillation criteria for differential equation

$$\left[x(t)+\sum_{i\in R} r_i(t)x(\alpha_i(t))\right]''+\sum_{i\in P} p_i(t)x\big(\beta_i(t)\big)-\sum_{i\in Q} q_i(t)x\big(\gamma_i(t)\big)=f(t). \tag{1.6}$$

In this paper, we shall continue in the direction to study the oscillatory properties of (1.1) and (1.2). We establish new oscillation criteria for (1.1) and (1.2), which extend and improve the corresponding results in [1, 4, 10]. We also give two examples to illustrate our main results.

2. Main Results

The following properties of the set $L^1([t_0,\infty))$ in [1] are needed for our subsequent discussion.

Property 1. *If $f\in L^1([t_0,\infty))$ and $f\in C([t_0,\infty),\mathbb{R}^+)$, then* $\lim\inf_{t\to\infty} f(t)=0$.

Corollary 2.1. *Suppose that $f\in L^1([t_0,\infty))$ and $\lim_{t\to\infty} f(t)$ exists; then $\lim_{t\to\infty} f(t)=0$.*

Property 2. *If $f\in C([t_0,\infty),\mathbb{R})$ and $f\in L^1([t_1,\infty))$, where $t_1\geq t_0$, then we have $f\in L^1([t_0,\infty))$.*

Property 3. *Let t_1 be such that $g(t_1)\geq t_0$. Suppose $g\in D([t_1,\infty))$ with $\limsup_{t\to\infty} g'(t)<\infty$ and $f\in C([t_0,\infty),\mathbb{R}^+)$. If $f\circ g\in L^1([t_1,\infty))$ holds, then $f\in L^1([t_0,\infty))$.*

Property 4. *Let t_1 be such that $g(t_1)\geq t_0$. Suppose $g\in D([t_1,\infty))$ with $\lim\inf_{t\to\infty} g'(t)>0$, $f\circ g\in C([t_1,\infty),\mathbb{R}^+)$, and $f\in C([t_0,\infty),\mathbb{R}^+)$. If $f\in L^1([t_0,\infty))$ holds, then $f\circ g\in L^1([t_1,\infty))$ holds.*

For simplicity, we denote the set of bounded functions by

$$B([t_0,\infty)):=\{f\in C([t_0,\infty),\mathbb{R}):\|f\|<\infty\}, \tag{2.1}$$

where

$$\|f\|:=\sup\{|f(t)|,\ t\geq t_0\}. \tag{2.2}$$

For an arbitrary function $\psi: Q\to P$, which satisfies (A_1)–(A_3), we denote the function $\varphi:[t_0,\infty)\to\mathbb{R}^+$ by

$$\varphi(t):=\sum_{i\in Q}\int_{\rho_i(t)}^{t} q_i(v)dv,\quad t\geq t_0. \tag{2.3}$$

In this section, for convenience, we suppose that $q_i\equiv 0$ holds for all $i\in Q$ on $[t_{-1},t_0)$. We start with the following Theorem.

Theorem 2.2. *Assume that* (H_1)–(H_4) *hold and there exists a mapping* $\psi : Q \to P$ *which satisfies* (A_1)–(A_3) *and that* $\sum_{i\in R} c_i \in B([t_0,\infty))$. *If* $\varphi \in L^1([t_0,\infty))$, *then every solution* x *of* (1.1) *is oscillatory.*

Proof. Suppose that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0$. Therefore, we may assume existence of $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} \varphi(u)du < 1, \quad \forall t \geq t_1, \quad x(\alpha_i(t)) > 0, \quad \forall i \in R, \quad x(\beta_i(t)) > 0, \quad \forall i \in P. \tag{2.4}$$

Now, we set

$$w(t) := x(t) + \sum_{i\in R} c_i(t)x(\alpha_i(t)), \quad t \geq t_1, \tag{2.5}$$

$$z(t,s) := w(t) + \int_s^t r(u)w(u)du - \sum_{i\in Q}\int_s^t\int_{\rho_i(u)}^u q_i(v)x(\gamma_i(v))dv\,du, \quad t \geq s \geq t_1. \tag{2.6}$$

By $z'(t,s)$, we denote differential of functions with respect to the first component. Considering (2.5), we rewrite (1.1) in the form

$$w''(t) + r(t)w'(t) + \sum_{i\in P} p_i(t)x(\beta_i(t)) - \sum_{i\in Q} q_i(t)x(\gamma_i(t)) = 0 \tag{2.7}$$

on $[t_1,\infty)$. By Leibnitz's rule, (2.7) and (H_4), we have

$$\begin{aligned}
z''(t,t_1) &= w''(t) + r'(t)w(t) + r(t)w'(t) - \sum_{i\in Q} q_i(t)x(\gamma_i(t)) + \sum_{i\in Q}\rho_i'(t)q_i(\rho_i(t))x(\beta_{\psi(i)}(t)) \\
&\leq w''(t) + r(t)w'(t) - \sum_{i\in Q} q_i(t)x(\gamma_i(t)) + \sum_{i\in Q}\rho_i'(t)q_i(\rho_i(t))x(\beta_{\psi(i)}(t)) \\
&= -\sum_{i\in P} p_i(t)x(\beta_i(t)) + \sum_{i\in Q}\rho_i'(t)q_i(\rho_i(t))x(\beta_{\psi(i)}(t)) \\
&= -\sum_{i\in\psi(Q)} p_i(t)x(\beta_i(t)) - \sum_{i\notin\psi(Q)} p_i(t)x(\beta_i(t)) + \sum_{i\in\psi(Q)}\sum_{j\in Q,\psi(j)=i}\rho_j'(t)q_j(\rho_j(t))x(\beta_i(t)) \\
&= -\sum_{i\in\psi(Q)}\left[p_i(t) - \sum_{j\in Q,\psi(j)=i}\rho_j'(t)q_j(\rho_j(t))\right]x(\beta_i(t)) - \sum_{i\notin\psi(Q)} p_i(t)x(\beta_i(t)). \\
&= -\sum_{i\in P} h_i(t)x(\beta_i(t)) \leq 0, \quad \forall t \geq t_1,
\end{aligned} \tag{2.8}$$

which implies $z'(\cdot,t_1)$ and $z(\cdot,t_1)$ is eventually strictly monotonic on $[t_1,\infty)$. Hence there exists $t_2 \geq t_1$ such that either $z'(t,t_1) < 0$ or $z'(t,t_1) > 0$ holds for all $t \geq t_2$.

We consider the following two possible cases:

Case 1 ($z'(t,t_1) > 0$ for all $t \geq t_2$). Integrating (2.8) from t_2 to ∞, we have

$$\infty > z'(t_2,t_1) \geq z'(t_2,t_1) - \lim_{t\to\infty} z'(t,t_1) = -\int_{t_2}^{\infty} z''(u,t_1)du = \sum_{i\in P}\int_{t_2}^{\infty} h_i(u)x(\beta_i(u))du, \tag{2.9}$$

which implies that $\sum_{i\in P} h_i(t)\cdot(x\circ\beta_i)(t) \in L^1[t_2,\infty)$. Therefore, for $i_0 \in P$ for which (A_3) holds, we have $x\circ\beta_{i_0} \in L^1[t_2,\infty)$. Then we conclude that $x \in L^1[t_0,\infty)$ by Property 3. Hence $\sum_{i\in R} c_i \in B([t_0,\infty))$. (H_3) and Property 4 imply that $w \in L^1[t_1,\infty)$. Since $r(t)$ is bounded, $b(t) = r(t)w(t)$ is also integrable in $[t_1,\infty)$. So we obtain that there exists a constant $M > 0$ such that

$$\int_{t_1}^{t} r(u)w(u)du \leq M, \quad \forall t \geq t_1. \tag{2.10}$$

Let

$$u(t) = w(t) + \int_{t_1}^{t} r(u)w(u)du, \quad \forall t \geq t_2. \tag{2.11}$$

From (2.6), we have

$$u'(t) = z'(t,t_1) + \sum_{i\in Q}\int_{\rho_i(t)}^{t} q_i(v)x(\gamma_i(v))dv > 0. \tag{2.12}$$

Then $u(t)$ is bounded and monotonous and $\lim_{t\to\infty}u(t)$ exists. We can suppose that $\lim_{t\to\infty}u(t) = \mu > 0$ since $u(t) > x(t) > 0$ and $u'(t) > 0$.

So there exists $t_3 \geq t_2$ such that $u(t) > \mu - \varepsilon/2,\ t \geq t_3$, for arbitrary $\varepsilon \in (0,\mu - M)$; by

$$u(t) = w(t) + \int_{t_3}^{t} r(u)w(u)du > \mu - \frac{\varepsilon}{2}, \tag{2.13}$$

we have

$$\begin{aligned} w(t) &> -\int_{t_3}^{t} r(u)w(u)du + \mu - \frac{\varepsilon}{2} \\ &> \mu - \frac{1}{2}(\mu - M) - M \\ &= \frac{1}{2}(\mu - M). \end{aligned} \tag{2.14}$$

This implies that $w(t) \notin L^1([t_1,\infty)$, which is a contradiction.

Case 2 ($z'(t,t_1) < 0$ for all $t \geq t_2$). Since $z'(\cdot,t_1)$ is nonincreasing by (2.8), the inequality $z'(t,t_1) \leq z'(t_2,t_1)$ implies that $\lim_{t\to\infty} z(t,t_1) = -\infty$. Hence $z(\cdot,t_1) \notin B([t_1,\infty))$. We claim that $x \in B([t_0,\infty))$. On contrary, there exists $t_3 \geq t_2$ such that $z(t_3,t_1) < 0$ and $x(t_3) = \sup\{x(t) : t \in [t_0,t_3)\}$. We get the following contradiction:

$$\begin{aligned} 0 > z(t_3,t_1) &= w(t_3) + \int_{t_1}^{t_3} r(u)w(u)du - \sum_{i\in Q}\int_{t_1}^{t_3}\int_{\rho_i(u)}^{u} q_i(v)x\big(\gamma_i(v)\big)dv\,du \\ &\geq x(t_3) + x(t_3)\int_{t_1}^{t_3} r(u)du - \sum_{i\in Q}\int_{t_1}^{t_3}\int_{\rho_i(u)}^{u} q_i(v)x\big(\gamma_i(v)\big)dv\,du \\ &\geq x(t_3)\left[1 + \int_{t_1}^{t_3} r(u)du - \int_{t_1}^{t_3}\varphi(u)du\right] > 0, \end{aligned} \tag{2.15}$$

since

$$\int_{t_1}^{t_3}\varphi(u)du < \int_{t_1}^{\infty}\varphi(u)du < 1. \tag{2.16}$$

Thus $\|x\| < \infty$. Accordingly, by (2.4) and (2.6), it follows that

$$z(t,t_1) \geq w(t) + \int_{t_1}^{t} r(u)w(u)du - \|x\|\int_{t_1}^{t}\varphi(u)du \geq -\|x\|\int_{t_1}^{t}\varphi(u)du > -\|x\|, \quad \forall t \geq t_2. \tag{2.17}$$

Therefore, $\|x\| \geq \|z(\cdot,t_1)\|$ holds and we see that $z(\cdot,t_1) \in B([t_1,\infty))$, which is a contradiction.

Therefore, we completed the proof by considering both possible cases.

□

Remark 2.3. When $r(t) = 0$, and Theorem 2.2 reduces to Theorem 3.1 in [1]. So Theorem 2.2 extends and improves the corresponding results in [1, 4, 10].

Theorem 2.4. *Assume that* (H_1)–(H_5) *hold and there exists a mapping* $\psi : Q \to P$ *which satisfies* (A_1)–(A_3). *Furthermore, assume that* $\sum_{i\in R} c_i \in B([t_0,\infty))$. *If* $\varphi \in L^1([t_0,\infty))$, *then every solution* x *of* (1.2) *is oscillatory or tends to zero asymptotically.*

Proof. Suppose that x is a nonoscillatory solution of (1.2). Without loss of generality, we assume that $x(t) > 0$ for $t \geq t_0$. Therefore, we may assume existence of a constant $\varepsilon > 0$ and $t_1 \geq t_0$ such that (2.4) and $F(t) \leq \varepsilon$ hold for all $t \geq t_1$. Now, for $t \geq s \geq t_1$, set

$$W(t) := w(t) - F(t), \tag{2.18}$$

$$Z(t,s) := W(t) + \int_{s}^{t} r(u)w(u)du - \sum_{i\in Q}\int_{s}^{t}\int_{\rho_i(u)}^{u} q_i(v)x\big(\gamma_i(v)\big)dv\,du + \varepsilon, \tag{2.19}$$

where $w(t)$ is defined on the interval $[t_1, \infty)$ as in (2.5). Then as in (2.8), we have

$$Z''(t, t_1) \leq -\sum_{i \in P} h_i(t) x(\beta_i(t)) \leq 0, \quad \forall t \geq t_1. \tag{2.20}$$

Thus there exists $t_2 \geq t_1$ satisfying either $Z'(t, t_1) > 0$ or $Z'(t, t_1) < 0$ for all $t \geq t_2$.

We consider the following two possible cases.

Case 1 ($Z'(t, t_1) > 0$ for all $t \geq t_2$). In this case, one can show that $w \in L^1([t_1, \infty))$ as shown in above proofs. Since $r(t)$ is bounded, $r(t)w(t)$ is also integrable in $[t_1, \infty)$. Let

$$v(t) = W(t) + \int_{t_1}^{t} r(u)w(u)du, \quad t \geq t_2. \tag{2.21}$$

By (2.19), we have

$$v'(t) = Z'(t, s) + \sum_{i \in Q} \int_{\rho_i(t)}^{t} q_i(v) x(\gamma_i(v)) dv > 0, \quad t \geq t_2; \tag{2.22}$$

then $v(t)$ is bounded and monotonous and $\lim_{t \to \infty} v(t)$ exists. By (2.21), we can obtain that $\lim_{t \to \infty} w(t)$ exists. Letting

$$a(t) = w(t) + \int_{t_1}^{t} r(u)w(u)du, \tag{2.23}$$

we can obtain that $\lim_{t \to \infty} a(t)$ exists. Suppose that $\lim_{t \to \infty} a(t) = \theta$, where $\theta \in [0, \infty)$. We claim $\theta = 0$. Suppose that $\theta \in (0, \infty)$. By $r(t)w(t) \in L^1([t_1, \infty))$, we see that there exists $N > 0$, such that

$$\int_{t_1}^{t} r(u)w(u)du \leq N. \tag{2.24}$$

Because $\lim_{t \to \infty} a(t) = \theta > 0$, there exists $t_3 \geq t_2$, such that $a(t) > \theta - (1/2)\epsilon$, $t \geq t_3$, for arbitrary $\epsilon \in (0, \theta - N)$. But by

$$w(t) + \int_{t_3}^{t} r(u)w(u)du > \theta - \frac{\epsilon}{2}, \tag{2.25}$$

we have

$$\begin{aligned} w(t) &> -\int_{t_3}^{t} r(u)w(u)du + \theta - \frac{\epsilon}{2} \\ &> \theta - \frac{1}{2}(\theta - N) - N \\ &= \frac{1}{2}(\theta - N); \end{aligned} \tag{2.26}$$

this implies that $w(t) \notin L^1([t_2, \infty)$, which is a contradiction. Therefore, $\lim_{t\to\infty} a(t) = 0$. Since $0 < x(t) < a(t)$ for $t \geq t_2$, we have that $\lim_{t\to\infty} x(t) = 0$.

Case 2 ($Z'(t, t_1) < 0$ for all $t \geq t_2$). Then we have that $Z(\cdot, t_1) \notin B([t_1, \infty))$ by (2.20). We claim that $x \in B([t_0, \infty))$. On contrary, there exists $t_3 \geq t_2$ such that $z(t_3, t_1) < 0$ and $x(t_3) = \sup\{x(t) : t \in [t_0, t_3)\}$ hold and $\lim_{t\to\infty} Z(t, t_1) < 0$.

Taking (2.4), (2.5), (2.18), and (2.19) into account, we get the following contradiction:

$$\begin{aligned} 0 > Z(t_3, t_1) &= W(t_3) + \int_{t_1}^{t_3} r(u)w(u)du - \sum_{i\in Q} \int_{t_1}^{t_3} \int_{\rho_i(u)}^{u} q_i(v)x(\gamma_i(v))dv\,du + \epsilon \\ &\geq x(t_3)\left[1 + \int_{t_1}^{t_3} r(u)du - \int_{t_1}^{t_3} \varphi(u)du\right] > 0. \end{aligned} \tag{2.27}$$

Hence $\|x\| < \infty$. Accordingly, using (2.18), (2.19), and the fact that $w(t) > 0$ on $[t_2, \infty)$, we have that

$$\begin{aligned} Z(t, t_2) &:= W(t) + \int_{t_2}^{t} r(u)w(u)du - \sum_{i\in Q} \int_{s}^{t} \int_{\rho_i(u)}^{u} q_i(v)x(\gamma_i(v))dv\,du + \epsilon \\ &\geq w(t) + \int_{t_2}^{t} r(u)w(u)du - \|x\| \int_{t_2}^{t} \varphi(u)du \\ &\geq -\|x\|, \quad \forall t \geq t_2. \end{aligned} \tag{2.28}$$

Thus $\|x\| \geq \|Z(\cdot, t_1)\|$ and $Z(\cdot, t_1) \in B([t_1, \infty))$, which is a contradiction.

Therefore, we completed the proof by considering both possible cases. □

Remark 2.5. If there exists $F \in C^2([t_0, \infty), \mathbb{R})$ such that $F'' = f$ on $[t_0, \infty)$ and $l_0 := \lim_{t\to\infty} F(t)$ exists and is finite, then $G := F - l_0$ on $[t_0, \infty)$ satisfies (H_5).

Corollary 2.6. *If all conditions of Theorem 2.4 hold, then every nonoscillatory solution of* (1.2) *converges to zero at infinity.*

Remark 2.7. When $r(t) = 0$, Theorem 2.4 reduces to Theorem 3.3 in [1]. So Theorem 2.4 extends and improves the corresponding results in [1, 4, 10].

3. Examples

In this section, we provide two examples to illustrate our main results.

Example 3.1. Consider the following equation:

$$[x(t) + 3x(t-\pi)]'' + \frac{1}{1+t^2}[x(t) + 3x(t-\pi)]' + (t+1)x(t-2\pi) - e^{-t-2\pi}x(t-\pi) = 0, \qquad t \geq 1. \tag{3.1}$$

Here, we have

$$\begin{gathered} c_1(t) = 3, \qquad \alpha_1 = t-\pi, \qquad \beta_1 = t-2\pi, \qquad \gamma_1 = t-\pi, \\ p_1(t) = t+1, \qquad q_1(t) = e^{-t-2\pi}, \qquad r(t) = \frac{1}{1+t^2}. \end{gathered} \tag{3.2}$$

Set the function ψ with $\psi(1) = 1$, then $\rho_1(t) = t - \pi$.

By simple calculation, we have

$$\begin{gathered} h_1(t) = t + 1 - e^{-t-\pi}, \\ \varphi(t) = e^{-t-\pi} - e^{-t-2\pi}, \\ \lim_{t\to\infty} \inf h_1(t) = \infty, \\ \int_1^\infty \varphi(u)du = e^{-1-\pi} - e^{-1-2\pi}. \end{gathered} \tag{3.3}$$

Therefore, according to Theorem 2.2, every solution x of (3.1) is oscillatory. Clearly, the known results in [1–10] cannot be applied to (3.1).

Example 3.2. Consider the following equation:

$$\begin{gathered} \left[x(t) + e^{-t}x\left(\frac{t}{2}\right)\right]'' + \frac{1}{1+t^2}\left[x(t) + e^{-t}x\left(\frac{t}{2}\right)\right]' + x\left(\frac{t}{2} - 2\pi\right) + x\left(\frac{t}{2} - \pi\right) \\ - e^{-t}x\left(\frac{t}{2} - \frac{3\pi}{2}\right) - \frac{3}{4}e^{-t}x\left(\frac{t}{2} - \pi\right) = e^{-t} - \frac{1}{t^3}\sin\frac{1}{t}, \quad t \geq 0. \end{gathered} \tag{3.4}$$

Here, we have

$$
\begin{gathered}
R = \{1\}, \qquad P = \{1,2\}, \qquad Q = \{1,2\}, \\
c_1(t) = e^{-t}, \qquad p_1(t) = p_2(t) = 1, \qquad q_1(t) = e^{-t}, \qquad q_2(t) = -\frac{3}{4}e^{-t}, \\
\alpha_1(t) = \frac{t}{2}, \qquad \beta_1(t) = \frac{t}{2} - 2\pi, \qquad \beta_2(t) = \frac{t}{2} - \pi, \\
\gamma_1(t) = \frac{t}{2} - \frac{3\pi}{2}, \qquad \gamma_2(t) = \frac{t}{2} - \pi, \\
f(t) = e^{-t} + \frac{1}{t^3}\sin\frac{1}{t}, \qquad r(t) = \frac{1}{1+t^2}.
\end{gathered}
\tag{3.5}
$$

Set the function $\psi : Q \to P$ with $\psi(i) = 1$ for $i = 1,2$; then $\rho_1(t) = t - \pi$ and $\rho_2(t) = t - 2\pi$.

By simple calculation, we have

$$
\begin{gathered}
F(t) = e^{-t} + t\sin\frac{1}{t}, \\
h_1(t) = 1 - e^{-t+\pi} - \frac{3}{4}e^{-t+2\pi}, \quad h_2(t) = 1, \\
\varphi(t) = \int_{t-\pi}^{t} e^{-v}du + \frac{3}{4}\int_{t-2\pi}^{t} e^{-v}dv, \\
\liminf_{t\to\infty} h_2(t) = 1, \quad \int_0^\infty \varphi(u)du = \frac{3}{4}\left(e^{2\pi} - 1\right) + e^{\pi} - 1, \qquad \lim_{t\to\infty} F(t) = 1.
\end{gathered}
\tag{3.6}
$$

Therefore, according to Theorem 2.4 and Remark 2.5, every solution x of (3.4) is oscillatory or tends to zero asymptotically. Clearly, the known results in [1–10] cannot be applied to (3.4).

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