Research Article

The Optimal Upper and Lower Power Mean Bounds for a Convex Combination of the Arithmetic and Logarithmic Means

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For $p \in \mathbb{R}$, the power mean $M_p(a, b)$ of order p , logarithmic mean $L(a, b)$, and arithmetic mean $A(a, b)$ of two positive real values a and b are defined by $M_p(a, b) = ((a^p + b^p)/2)^{1/p}$, for $p \neq 0$ and $M_p(a, b) = \sqrt{ab}$, for $p = 0$, $L(a, b) = (b - a)/(\log b - \log a)$, for $a \neq b$ and $L(a, b) = a$, for $a = b$ and $A(a, b) = (a + b)/2$, respectively. In this paper, we answer the question: for $\alpha \in (0, 1)$, what are the greatest value p and the least value q , such that the double inequality $M_p(a, b) \leq \alpha A(a, b) + (1 - \alpha)L(a, b) \leq M_q(a, b)$ holds for all $a, b > 0$?

1. Introduction

For $p \in \mathbb{R}$, the power mean $M_p(a, b)$ of order p and logarithmic mean $L(a, b)$ of two positive real values a and b are defined by

$$M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \quad (1.1)$$

$$L(a, b) = \begin{cases} \frac{b - a}{\log b - \log a}, & a \neq b, \\ a, & a = b, \end{cases} \quad (1.2)$$

respectively. In the recent past, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for power mean or logarithmic mean can

be found in the literature [1–15]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [16–18]. In [16] the authors study a variant of Jensen's functional equation involving L , which appears in a heat conduction problem. A representation of L as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [11]. In [19, 20] it is shown that L can be expressed in terms of Gauss's hypergeometric function ${}_2F_1$. And, in [20] the authors prove that the reciprocal of the logarithmic mean is strictly totally positive, that is, every $n \times n$ determinant with elements $1/L(a_i, b_i)$, where $0 < a_1 < a_2 < \cdots < a_n$ and $0 < b_1 < b_2 < \cdots < b_n$, is positive for all $n \geq 1$.

Let $A(a, b) = (1/2)(a + b)$, $G(a, b) = \sqrt{ab}$, and $H(a, b) = 2ab/(a + b)$ be the arithmetic, geometric, and harmonic means of two positive numbers a and b , respectively. Then it is well known that

$$\begin{aligned} \min\{a, b\} \leq H(a, b) = M_{-1}(a, b) \leq G(a, b) = M_0(a, b) \\ \leq L(a, b) \leq A(a, b) = M_1(a, b) \leq \max\{a, b\}, \end{aligned} \quad (1.3)$$

and all inequalities are strict for $a \neq b$.

In [21], Alzer and Janous established the following best possible inequality:

$$M_{\log 2 / \log 3}(a, b) \leq \frac{2}{3}A(a, b) + \frac{1}{3}G(a, b) \leq M_{2/3}(a, b) \quad (1.4)$$

for all $a, b > 0$.

In [11, 13, 22] the authors present bounds for L in terms of G and A

$$G^{2/3}(a, b)A^{1/3}(a, b) < L(a, b) < \frac{2}{3}G(a, b) + \frac{1}{3}A(a, b) \quad (1.5)$$

for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for L in terms of power means are proved by Lin [12]

$$M_0(a, b) < L(a, b) < M_{1/3}(a, b). \quad (1.6)$$

The main purpose of this paper is to answer the question: for $\alpha \in (0, 1)$, what are the greatest value p and the least value q , such that the double inequality $M_p(a, b) \leq \alpha A(a, b) + (1 - \alpha)L(a, b) \leq M_q(a, b)$ holds for all $a, b > 0$?

2. Lemmas

In order to establish our results we need several lemmas, which we present in this section.

Lemma 2.1. *If $\alpha \in (0, 1)$, then $(1 + 2\alpha)(\log 2 - \log \alpha) > 3 \log 2$.*

Proof. For $\alpha \in (0, 1)$, let $f(\alpha) = (1 + 2\alpha)(\log 2 - \log \alpha)$, then simple computations lead to

$$f'(\alpha) = 2(\log 2 - 1) - 2 \log \alpha - \frac{1}{\alpha}, \quad (2.1)$$

$$f''(\alpha) = \frac{1}{\alpha^2}(1 - 2\alpha). \quad (2.2)$$

From (2.2) we clearly see that $f''(\alpha) > 0$ for $\alpha \in (0, 1/2)$, and $f''(\alpha) < 0$ for $\alpha \in (1/2, 1)$. Then from (2.1) we get

$$f'(\alpha) \leq f'\left(\frac{1}{2}\right) = 4(\log 2 - 1) < 0 \quad (2.3)$$

for $\alpha \in (0, 1)$.

Therefore $f(\alpha) > f(1) = 3 \log 2$ for $\alpha \in (0, 1)$ follows from (2.3). $\square$

Lemma 2.2. *Let $\alpha \in (0, 1)$, if $p = \log 2 / (\log 2 - \log \alpha)$, then*

$$-p^3 + (4\alpha - 1)p^2 - 3\alpha p + \alpha < 0. \quad (2.4)$$

Proof. For $\alpha \in (0, 1)$, let $t = -\log \alpha$, then $t \in (0, +\infty)$ and

$$-p^3 + (4\alpha - 1)p^2 - 3\alpha p + \alpha = \frac{f(t)}{(t + \log 2)^3 e^t}, \quad (2.5)$$

where $f(t) = (t + \log 2)^3 - 3 \log 2(t + \log 2)^2 + (\log 2)^2(t + \log 2)(4 - e^t) - (\log 2)^3 e^t$.

To prove Lemma 2.2 we need only to prove that $f(t) < 0$ for $t \in (0, +\infty)$. Elementary calculations yield that

$$f(0) = 0, \quad (2.6)$$

$$f'(t) = 3(t + \log 2)^2 - 6 \log 2(t + \log 2) - (\log 2)^2 t e^t - (1 + 2 \log 2)(\log 2)^2 e^t + 4(\log 2)^2, \quad (2.7)$$

$$f'(0) = -2(\log 2)^3 < 0, \quad (2.8)$$

$$\lim_{t \rightarrow +\infty} f'(t) = -\infty, \quad (2.9)$$

$$f''(t) = 6t - (\log 2)^2 t e^t - 2(\log 2)^2 (1 + \log 2) e^t, \quad (2.10)$$

$$f''(0) = -2(1 + \log 2)(\log 2)^2 < 0, \quad (2.11)$$

$$\lim_{t \rightarrow +\infty} f''(t) = -\infty, \quad (2.12)$$

$$f'''(t) = 6 - (\log 2)^2 t e^t - (\log 2)^2 (3 + 2 \log 2) e^t, \quad (2.13)$$

$$f'''(0) = 6 - 3(\log 2)^2 - 2(\log 2)^3 > 0, \quad (2.14)$$

$$\lim_{t \rightarrow +\infty} f'''(t) = -\infty, \quad (2.15)$$

$$f^{(4)}(t) = -(\log 2)^2 t e^t - 2(\log 2)^2 (2 + \log 2) e^t < 0 \quad (2.16)$$

for $t \in (0, +\infty)$.

Making use of a computer and the mathematica software, from (2.10) we get

$$f''(1.15) = 0.01679 \dots, \quad (2.17)$$

$$f''(1.16) = -0.0077 \dots. \quad (2.18)$$

From (2.14)–(2.16) we clearly see that there exists a unique $t_0 \in (0, +\infty)$, such that $f'''(t) > 0$ for $t \in [0, t_0]$ and $f'''(t) < 0$ for $t \in (t_0, +\infty)$. Hence we know that $f''(t)$ is strictly increasing in $[0, t_0]$ and strictly decreasing in $[t_0, +\infty)$.

From (2.11), (2.12), (2.17), (2.18) and the monotonicity of $f''(t)$ in $[0, t_0]$ and in $[t_0, +\infty)$ we know that there exist exactly two numbers $t_1, t_2 \in (0, +\infty)$ with $t_1 < t_2$, such that $f''(t) < 0$ for $t \in [0, t_1) \cup (t_2, +\infty)$ and $f''(t) > 0$ for $t \in (t_1, t_2)$, and t_2 satisfies

$$1.15 < t_2 < 1.16. \quad (2.19)$$

Hence, we know that $f'(t)$ is strictly decreasing in $[0, t_1] \cup [t_2, +\infty)$ and strictly increasing in $[t_1, t_2]$.

Making use of a computer and the mathematica software, from (2.7) and (2.19), we get

$$\begin{aligned} f'(t_2) &< 3(1.16 + \log 2)^2 - 6 \log 2 (1.15 + \log 2) - 1.15 \times e^{1.15} \times (\log 2)^2 \\ &\quad - (1 + 2 \log 2) \times (\log 2)^2 \times e^{1.15} + 4(\log 2)^2 \\ &= -0.807 \dots < 0. \end{aligned} \quad (2.20)$$

Now, (2.8), (2.9), (2.20) and the monotonicity of $f'(t)$ in $[0, t_1] \cup [t_2, +\infty)$ and in $[t_1, t_2]$ imply that

$$f'(t) < 0 \quad (2.21)$$

for $t \in (0, +\infty)$.

Therefore, $f(t) < 0$ for $t \in (0, +\infty)$ follows from (2.6) and (2.21). $\square$

Lemma 2.3. For $\alpha \in (0, 1)$ and $g(t) = \alpha(t - t^p)(\log t)^2 + 2(1 - \alpha)(t + t^p) \log t - 2(1 - \alpha)(t - 1)(1 + t^p)$, one has the following.

(1) If $p = \log 2 / (\log 2 - \log \alpha)$, then there exists $\lambda \in (1, +\infty)$ such that $g(t) > 0$ for $t \in (1, \lambda)$ and $g(t) < 0$ for $t \in (\lambda, +\infty)$.

(2) If $p = (1 + 2\alpha)/3$, then $g(t) < 0$ for $t \in (1, +\infty)$.

Proof. Let $g_1(t) = t^{1-p}g'(t)$, $g_2(t) = t^p g_1'(t)$, $g_3(t) = t g_2'(t)$, $g_4(t) = t^{2-p} g_3'(t)$, $g_5(t) = t g_4'(t)$, and $p \in \{\log 2 / (\log 2 - \log \alpha), (1 + 2\alpha)/3\}$, then simple computations lead to

$$g(1) = 0, \quad (2.22)$$

$$\lim_{t \rightarrow +\infty} g(t) = -\infty, \quad (2.23)$$

$$g_1(t) = \alpha(t^{1-p} - p)(\log t)^2 + 2(t^{1-p} + p - \alpha p - \alpha) \log t + 2(1 - \alpha)(1 + p)(1 - t), \quad (2.24)$$

$$g_1(1) = 0, \quad (2.25)$$

$$\lim_{t \rightarrow +\infty} g_1(t) = -\infty, \quad (2.26)$$

$$\begin{aligned} g_2(t) &= \alpha(1 - p)(\log t)^2 + 2(1 + \alpha - p - \alpha p t^{p-1}) \log t + 2(p - \alpha p - \alpha)t^{p-1} \\ &\quad - 2(1 - \alpha)(1 + p)t^p + 2, \end{aligned} \quad (2.27)$$

$$g_2(1) = 0, \quad (2.28)$$

$$\lim_{t \rightarrow +\infty} g_2(t) = -\infty, \quad (2.29)$$

$$\begin{aligned} g_3(t) &= 2\alpha(1 - p)(1 + p t^{p-1}) \log t + 2[(1 - \alpha)p^2 - (1 + \alpha)p + \alpha]t^{p-1} \\ &\quad - 2p(1 - \alpha)(1 + p)t^p + 2(1 + \alpha - p), \end{aligned} \quad (2.30)$$

$$g_3(1) = 2(1 + 2\alpha - 3p), \quad (2.31)$$

$$\lim_{t \rightarrow +\infty} g_3(t) = -\infty, \quad (2.32)$$

$$g_4(t) = 2\alpha(1-p)t^{1-p} - 2\alpha p(1-p)^2 \log t - 2p^2(1-\alpha)(1+p)t + 2(p-1)\left[(1-\alpha)p^2 - (1+2\alpha)p + \alpha\right], \quad (2.33)$$

$$g_4(1) = 2p(1+2\alpha-3p), \quad (2.34)$$

$$\lim_{t \rightarrow +\infty} g_4(t) = -\infty, \quad (2.35)$$

$$g_5(t) = 2\alpha(1-p)^2 t^{1-p} - 2p^2(1-\alpha)(1+p)t - 2\alpha p(1-p)^2, \quad (2.36)$$

$$g_5(1) = -2\left[p^3 - (4\alpha-1)p^2 + 3\alpha p - \alpha\right], \quad (2.37)$$

$$g'_5(t) = 2\alpha(1-p)^3 t^{-p} - 2p^2(1-\alpha)(1+p), \quad (2.38)$$

$$g'_5(1) = -2\left[p^3 - (4\alpha-1)p^2 + 3\alpha p - \alpha\right]. \quad (2.39)$$

(1) If $p = \log 2 / (\log 2 - \log \alpha)$, then from (2.31), (2.34), (2.37)–(2.39), and Lemmas 2.1–2.2 we clearly see that

$$g_3(1) > 0, \quad (2.40)$$

$$g_4(1) > 0, \quad (2.41)$$

$$g_5(1) < 0, \quad (2.42)$$

$$g'_5(1) < 0, \quad (2.43)$$

and $g'_5(t)$ is strictly decreasing in $[1, +\infty)$.

From (2.43) and the monotonicity of $g'_5(t)$ we know that $g_5(t)$ is strictly decreasing in $[1, +\infty)$.

The monotonicity of $g_5(t)$ and (2.42) implies that $g_5(t) < 0$ for $t \in [1, +\infty)$, then we conclude that $g_4(t)$ is strictly decreasing in $[1, +\infty)$.

From the monotonicity of $g_4(t)$ and (2.35) together with (2.41) we clearly see that there exists $t_1 \in (1, +\infty)$, such that $g_4(t) > 0$ for $t \in [1, t_1)$ and $g_4(t) < 0$ for $t \in (t_1, +\infty)$. Hence we know that $g_3(t)$ is strictly increasing in $[1, t_1]$ and strictly decreasing in $[t_1, +\infty)$.

The monotonicity of $g_3(t)$ in $[1, t_1]$ and in $[t_1, +\infty)$ together with (2.32) and (2.40) imply that there exists $t_2 \in (1, +\infty)$, such that $g_3(t) > 0$ for $t \in [1, t_2)$ and $g_3(t) < 0$ for $t \in (t_2, +\infty)$. Then we know that $g_2(t)$ is strictly increasing in $[1, t_2]$ and strictly decreasing in $[t_2, +\infty)$.

From (2.28) and (2.29) together with the monotonicity of $g_2(t)$ in $[1, t_2]$ and in $[t_2, +\infty)$ we clearly see that there exists $t_3 \in (1, +\infty)$, such that $g_2(t) > 0$ for $t \in [1, t_3)$ and $g_2(t) < 0$ for $t \in (t_3, +\infty)$. Hence we know that $g_1(t)$ is strictly increasing in $[1, t_3]$ and strictly decreasing in $[t_3, +\infty)$.

Equations (2.25) and (2.26) together with the monotonicity of $g_1(t)$ in $[1, t_3]$ and in $[t_3, +\infty)$ imply that there exists $t_4 \in (1, +\infty)$, such that $g_1(t) > 0$ for $t \in [1, t_4)$ and $g_1(t) < 0$ for $t \in (t_4, +\infty)$. Then we conclude that $g(t)$ is strictly increasing in $[1, t_4]$ and strictly decreasing in $[t_4, +\infty)$.

Now (2.22), (2.23) and the monotonicity of $g(t)$ in $[1, t_4]$ and in $[t_4, +\infty)$ imply that there exists $\lambda \in (1, +\infty)$, such that $g(t) > 0$ for $t \in [1, \lambda)$ and $g(t) < 0$ for $t \in (\lambda, +\infty)$.

(2) If $p = (1 + 2\alpha)/3$, then (2.31), (2.34), and (2.37)–(2.39) lead to

$$g_4(1) = g_3(1) = 0, \quad (2.44)$$

$$g'_5(1) = g_5(1) = -\frac{1}{27} \left[8(1 - \alpha^3) + 12\alpha(1 - \alpha^2) + 60\alpha^2(1 - \alpha) \right] < 0, \quad (2.45)$$

and $g'_5(t)$ is strictly decreasing in $[1, +\infty)$.

Therefore, Lemma 2.3(2) follows from (2.22), (2.25), (2.28), (2.44), (2.45), and the monotonicity of $g'_5(t)$. $\square$

3. Main Result

Theorem 3.1. *For $\alpha \in (0, 1)$, the double inequality $M_{\log 2/(\log 2 - \log \alpha)}(a, b) \leq \alpha A(a, b) + (1 - \alpha)L(a, b) \leq M_{(1+2\alpha)/3}(a, b)$ holds for all $a, b > 0$, each inequality becomes an equality if and only if $a = b$, and the given parameters $\log 2/(\log 2 - \log \alpha)$ and $(1 + 2\alpha)/3$ in each inequality are best possible.*

Proof. If $a = b$, then from (1.1) and (1.2) we clearly see that $M_{\log 2/(\log 2 - \log \alpha)}(a, b) = \alpha A(a, b) + (1 - \alpha)L(a, b) = M_{(1+2\alpha)/3}(a, b) = a$ for $\alpha \in (0, 1)$. Next, we assume that $a \neq b$.

Firstly, we prove that $M_{\log 2/(\log 2 - \log \alpha)}(a, b) < \alpha A(a, b) + (1 - \alpha)L(a, b) < M_{(1+2\alpha)/3}(a, b)$ for $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $p \in \{\log 2/(\log 2 - \log \alpha), (1 + 2\alpha)/3\}$, then (1.1) and (1.2) leads to

$$\alpha A(a, b) + (1 - \alpha)L(a, b) - M_p(a, b) = b \left[\frac{\alpha(t+1) \log t + 2(1 - \alpha)(t-1)}{2 \log t} - \left(\frac{t^p + 1}{2} \right)^{1/p} \right]. \quad (3.1)$$

Let

$$f(t) = \log \left[\frac{\alpha(t+1) \log t + 2(1 - \alpha)(t-1)}{2 \log t} \right] - \frac{1}{p} \log \left(\frac{t^p + 1}{2} \right), \quad (3.2)$$

then

$$\lim_{t \rightarrow 1} f(t) = 0, \quad (3.3)$$

$$f'(t) = \frac{g(t)}{t [\alpha(t+1) \log t + 2(1 - \alpha)(t-1)] (1 + t^p) \log t}, \quad (3.4)$$

where $g(t) = \alpha(t - t^p)(\log t)^2 + 2(1 - \alpha)(t + t^p) \log t - 2(1 - \alpha)(t - 1)(1 + t^p)$.

If $p = \log 2 / (\log 2 - \log \alpha)$, then it is not difficult to verify that

$$\lim_{t \rightarrow +\infty} f(t) = 0. \quad (3.5)$$

From (3.4) and Lemma 2.3(1) we know that there exists $\lambda \in (1, +\infty)$, such that $f(t)$ is strictly increasing in $[1, \lambda]$ and strictly decreasing in $[\lambda, +\infty)$. Then (3.3) and (3.5) together with the monotonicity of $f(t)$ in $[1, \lambda]$ and in $[\lambda, +\infty)$ imply that $f(t) > 0$ for $t \in (1, +\infty)$, and from (3.1) and (3.2) we know that $\alpha A(a, b) + (1 - \alpha)L(a, b) > M_{\log 2 / (\log 2 - \log \alpha)}(a, b)$ for all $a, b > 0$ with $a \neq b$.

If $p = (1 + 2\alpha)/3$, then from Lemma 2.3(2) and (3.1)–(3.4) we clearly see that $\alpha A(a, b) + (1 - \alpha)L(a, b) < M_{(1+2\alpha)/3}(a, b)$ for all $a, b > 0$ with $a \neq b$.

Secondly, we prove that the parameters $\log 2 / (\log 2 - \log \alpha)$ and $(1 + 2\alpha)/3$ cannot be improved in each inequality.

For any $\varepsilon > 0$ and $x > 1$, from (1.1) and (1.2) we get

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{M_{\log 2 / (\log 2 - \log \alpha) + \varepsilon}(1, x)}{\alpha A(1, x) + (1 - \alpha)L(1, x)} &= \frac{2}{\alpha} \times \left(\frac{1}{2}\right)^{(\log 2 - \log \alpha) / (\log 2 + \varepsilon(\log 2 - \log \alpha))} \\ &> \frac{2}{\alpha} \times \left(\frac{1}{2}\right)^{(\log 2 - \log \alpha) / \log 2} \\ &= 1. \end{aligned} \quad (3.6)$$

Inequality (3.6) implies that for any $\varepsilon > 0$ there exists $X = X(\varepsilon) > 1$, such that $M_{\log 2 / (\log 2 - \log \alpha) + \varepsilon}(1, x) > \alpha A(1, x) + (1 - \alpha)L(1, x)$ for $x \in (X, +\infty)$. Hence the parameter $\log 2 / (\log 2 - \log \alpha)$ cannot be improved in the left-side inequality.

Next for $0 < \varepsilon < (1 + 2\alpha)/3$, let $0 < x < 1$, then (1.1) and (1.2) leads to

$$\begin{aligned} &[\alpha A(1, 1+x) + (1 - \alpha)L(1, 1+x)]^{(1+2\alpha-3\varepsilon)/3} - [M_{(1+2\alpha)/3-\varepsilon}(1, 1+x)]^{(1+2\alpha-3\varepsilon)/3} \\ &= \left[\frac{(1 - \alpha)x + \alpha(1+x/2) \log(1+x)}{\log(1+x)} \right]^{(1+2\alpha-3\varepsilon)/3} - \frac{1 + (1+x)^{(1+2\alpha-3\varepsilon)/3}}{2} \\ &= \frac{f(x)}{[\log(1+x)]^{(1+2\alpha-3\varepsilon)/3}}, \end{aligned} \quad (3.7)$$

where $f(x) = [(1 - \alpha)x + \alpha(1+x/2) \log(1+x)]^{(1+2\alpha-3\varepsilon)/3} - ((1 + (1+x)^{(1+2\alpha-3\varepsilon)/3})/2)[\log(1+x)]^{(1+2\alpha-3\varepsilon)/3}$.

Let $x \rightarrow 0$, making use of the Taylor expansion we get

$$f(x) = \frac{1}{24} \varepsilon (1 + 2\alpha - 3\varepsilon) x^{(1+2\alpha-3\varepsilon)/3} [x^2 + o(x^2)]. \quad (3.8)$$

Equations (3.7) and (3.8) imply that for any $0 < \varepsilon < (1 + 2\alpha)/3$ there exists $0 < \delta = \delta(\varepsilon, \alpha) < 1$, such that $\alpha A(1, 1+x) + (1 - \alpha)L(1, 1+x) > M_{(1+2\alpha)/3-\varepsilon}(1, 1+x)$ for $x \in (0, \delta)$. Hence the parameter $(1 + 2\alpha)/3$ cannot be improved in the right-side inequality. $\square$

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