Research Article

Hyers-Ulam Stability of Polynomial Equations

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Received 27 April 2010; Revised 3 August 2010; Accepted 30 September 2010

Academic Editor: John M. Rassias

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We prove the Hyers-Ulam stability of the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$. We give an affirmative answer to a problem posed by Li and Hua (2009).

1. Introduction and Preliminaries

A classical question in the theory of functional equations is that “when is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$ ”. Such a problem was formulated by Ulam [1] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [2]. It gave rise to the *stability theory* for functional equations. The result of Hyers was generalized by Rassias [3]. The topic of the Hyers-Ulam stability of functional equations and its applications has been studied by a number of mathematicians; see [3–40] and references therein.

Recently, Li and Hua [41] discussed and proved the Hyers-Ulam stability of the polynomial equation

$$x^n + \alpha x + \beta = 0, \quad (1.1)$$

where $x \in [-1, 1]$ and proved the following.

Theorem 1.1. *If $|\alpha| > n$, $|\beta| < |\alpha| - 1$, and $y \in [-1, 1]$ satisfy the inequality*

$$|y^n + \alpha y + \beta| \leq \varepsilon, \quad (1.2)$$

then there exists a solution $v \in [-1, 1]$ of (1.1) such that

$$|y - v| \leq K\varepsilon, \quad (1.3)$$

where $K > 0$ is constant.

They also asked an open problem whether the real polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad (1.4)$$

has the Hyers-Ulam stability for the case that this real polynomial equation has some solutions in $[a, b]$. The aim of this paper is to give a positive answer to this problem. First of all, we give the definition of the Hyers-Ulam stability.

Definition 1.2. One says that (1.4) has the Hyers-Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\varepsilon > 0$, $y \in [-1, 1]$, if

$$\left| a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 \right| \leq \varepsilon \quad (1.5)$$

then there exists some $z \in [-1, 1]$ satisfying

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 \quad (1.6)$$

such that $|y - z| \leq K\varepsilon$. One calls such K a Hyers-Ulam stability constant for (1.4). For the complex polynomial equation, $[-1, 1]$ is replaced by closed unit disc

$$D = \{z \in \mathbb{C}; |z| \leq 1\}. \quad (1.7)$$

2. Main Results

The aim of this work is to investigate the Hyers-Ulam stability for (1.4).

Theorem 2.1. *If*

$$|a_0| < |a_1| - (|a_2| + |a_3| + \cdots + |a_n|), \quad (2.1)$$

$$|a_1| > 2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| + n|a_n|, \quad (2.2)$$

then there exists an exact solution $v \in [-1, 1]$ of (1.4).

Proof. If we set

$$g(x) = \frac{1}{a_1} \left(-a_0 - a_2 x^2 - a_3 x^3 - \cdots - a_{n-1} x^{n-1} - a_n x^n \right), \quad (2.3)$$

for $x \in [-1, 1]$, then we have

$$\begin{aligned} |g(x)| &= \frac{1}{|a_1|} \left| -a_0 - a_2x^2 - \cdots - a_{n-1}x^{n-1} - a_nx^n \right| \\ &\leq \frac{1}{|a_1|} (|a_0| + |a_2| + \cdots + |a_{n-1}| + |a_n|) \\ &\leq 1 \end{aligned} \quad (2.4)$$

by (2.1).

Let $X = [-1, 1]$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space and g maps X to X . Now, we will show that g is a contraction from X to X . For any $x, y \in X$, we have

$$\begin{aligned} d(g(x), g(y)) &= \left| \frac{1}{a_1} (-a_0 - a_2x^2 - \cdots - a_nx^n) - \frac{1}{a_1} (-a_0 - \cdots - a_ny^n) \right| \\ &\leq \frac{1}{|a_1|} |x - y| \left\{ |a_2| |x + y| + \cdots + |a_n| |x^{n-1} + \cdots + y^{n-1}| \right\} \\ &\leq \frac{1}{|a_1|} |x - y| \{ 2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| + n|a_n| \}. \end{aligned} \quad (2.5)$$

For $x, y \in [-1, 1]$, $x \neq y$, from (2.2), we obtain

$$d(g(x), g(y)) \leq \lambda d(x, y). \quad (2.6)$$

Here

$$\lambda = \frac{2|a_2| + 3|a_3| + \cdots + (n-1)|a_{n-1}| + n|a_n|}{|a_1|} < 1. \quad (2.7)$$

Thus g is a contraction from X to X . By the Banach contraction mapping theorem, there exists a unique $v \in X$ such that

$$g(v) = v. \quad (2.8)$$

Hence (1.4) has a solution on $[-1, 1]$. $\square$

As an application of Rouché's theorem, we prove the following theorem for complex polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \quad (2.9)$$

which is much better than the above result. In fact, we prove the following theorem.

Theorem 2.2. *If*

$$|a_0| < |a_1| - (|a_2| + |a_3| + \cdots + |a_n|), \quad (2.10)$$

then there exists an exact solution in open unit disc for (2.9).

Proof. If we set

$$g(z) = \frac{1}{a_1} \left(-a_0 - a_2 z^2 - a_3 z^3 - \cdots - a_{n-1} z^{n-1} - a_n z^n \right), \quad (2.11)$$

then we have

$$\begin{aligned} |g(z)| &= \frac{1}{|a_1|} \left| -a_0 - a_2 z^2 - \cdots - a_{n-1} z^{n-1} - a_n z^n \right| \\ &\leq \frac{1}{|a_1|} (|a_0| + |a_2| + \cdots + |a_{n-1}| + |a_n|), \quad \text{for } |z| \leq 1 \\ &< 1 \end{aligned} \quad (2.12)$$

by (2.10).

Since $|g(z)| < 1$ for $|z| = 1$, then $|g(z)| < |z| = 1$ and by Rouché's theorem, we observe that $g(z) - z$ has exactly one zero in $|z| < 1$ which implies that g has a unique fixed point in $|z| < 1$. $\square$

Theorem 2.3. *If the conditions of Theorem 2.1 hold and $y \in [-1, 1]$ satisfies the inequality*

$$\left| a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 \right| \leq \varepsilon, \quad (2.13)$$

then (1.4) has the Hyers-Ulam stability.

Proof. Let $\varepsilon > 0$ and $y \in [-1, 1]$ such that

$$\left| a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 \right| \leq \varepsilon. \quad (2.14)$$

We will show that there exists a constant K independent of ε and v such that

$$|y - v| \leq K\varepsilon \quad (2.15)$$

for some $v \in [-1, 1]$ satisfying (1.4).

Let us introduce the abbreviation $K = 1/|a_1|(1 - \lambda)$. Then

$$\begin{aligned} |y - v| &= |y - g(y) + g(y) - g(v)| \leq |y - g(y)| + |g(y) - g(v)| \\ &\leq \left| y - \frac{1}{a_1}(-a_0 - a_2 y^2 - \cdots - a_n y^n) \right| + \lambda |y - v| \\ &= \frac{1}{|a_1|} |a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0| + \lambda |y - v|. \end{aligned} \quad (2.16)$$

Thus, we have

$$\begin{aligned} |y - v| &\leq \frac{1}{|a_1|(1 - \lambda)} |a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0| \\ &\leq K\varepsilon \end{aligned} \quad (2.17)$$

by (2.13) and so the result follows. $\square$

Corollary 2.4. *In Theorem 2.2, if there exists $y \in D$ satisfying the inequality (2.13), then (2.9) has the Hyers-Ulam stability.*

Remark 2.5. For $a_n = 1$, $a_i = 0$, for $2 \leq i \leq n - 1$, combining Theorems 2.1 and 2.3 gives Theorem 1.1.

Remark 2.6. By the similar way, one can easily prove the Hyers-Ulam stability of (1.4) on any finite interval $[a, b]$.

Remark 2.7. Let f be any complex function such that f is analytic in

$$\Delta = \{z \in \mathbb{C} : |z| < R, R > 0\}. \quad (2.18)$$

It is an interesting open problem whether f has the Hyers-Ulam stability for the case that f has some zeros in Δ .

We note that there is an error in the proof of Theorem 2.2 of [41], when Li and Hua stated that if (X, d) is a complete metric linear space then metric d is invariant, more precisely

$$d(x, y) = d(x - y, 0) \quad (2.19)$$

for all $x, y \in X$. We give a counterexample for this case. Suppose that $X = \mathbb{R}$, and we define metric d on X as follows:

$$d(x, y) = |x + [x] - (y + [y])|, \quad (2.20)$$

for all $x, y \in X$, (X, d) is a complete metric linear space, and d is not an invariant metric on X , that is, there are $x, y \in X$ such that

$$d(x, y) \neq d(x - y, 0). \quad (2.21)$$

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