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Research Article

Mixed Approximation for Nonexpansive Mappings in Banach Spaces

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The mixed viscosity approximation is proposed for finding fixed points of nonexpansive mappings, and the strong convergence of the scheme to a fixed point of the nonexpansive mapping is proved in a real Banach space with uniformly Gâteaux differentiable norm. The theorem about Halpern type approximation for nonexpansive mappings is shown also. Our theorems extend and improve the correspondingly results shown recently.

1. Introduction and Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$, E^* denote the dual space of E, and $\langle\cdot,\cdot\rangle$ denote the generalized duality pairing. Let $J : E \to 2^{E^*}$ denote the normalized duality mapping defined by

$$J(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|f^*\|\|x\|, \|f^*\| = \|x\|\}, \quad \forall x \in E. \tag{1.1}$$

It is well known that the following results: $\forall x, y \in E$, $\forall j(x+y) \in J(x+y)$, $\forall j(x) \in J(x)$,

$$\|x\|^2 + 2\langle y, j(x)\rangle \le \|x+y\|^2 \le \|x\|^2 + \langle y, j(x+y)\rangle. \tag{1.2}$$

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of Banach space E, the norm of E is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.3}$$

exists for each $x, y \in U$. Such an E is called a smooth Banach space. The norm of Banach space E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit $\lim_{t \to 0}((\|x + ty\| - \|x\|)/t)$ is attained uniformly for $x \in U$. A Banach space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, \quad x \neq y \text{ implies } \frac{\|x + y\|}{2} < 1, \tag{1.4}$$

to be uniformly convex if for all $\varepsilon \in [0, 2]$, $\exists \delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \text{ implies } \frac{\|x + y\|}{2} < 1 - \delta_\varepsilon. \tag{1.5}$$

It is well known that (see [1, 2]): (1) if E has a uniformly Gâteaux differentiable norm, then J is norm-to-weak* continuous on bounded set of E. (2) If a Banach space E admits a sequentially continuous duality mapping J from weak topology to weak star topology, then the duality mapping J is single-valued. (3) Each uniformly convex Banach space E is reflexive and strictly convex and has fixed point property for nonexpansive self-mappings; every uniformly smooth Banach space E is a reflexive Banach space with a uniformly Gâteaux differentiable norm and has fixed point property for nonexpansive self-mappings.

Let C be a nonempty closed convex subset of a Banach space E. If D is a nonempty subset of C, then a mapping $P : C \to D$ is said to be a retraction if $Px = x$ for all $x \in D$. A mapping $P : C \to D$ is said to be a sunny if $P(Px + t(x - Px)) = Px$, $\forall x \in C$ where $t > 0$. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D. For more details, see [2]. Note that every closed convex subset of a Hilbert space is a nonexpansive retract. In the sequel, we always take P to denote the sunny nonexpansive retraction of E onto C.

Let $T : C \to C$ be a nonexpansive mapping; for a sequence $\{\alpha_n\} \subset (0, 1)$ and a fixed contractive mapping $f : C \to C$, the sequence $\{x_n\}$, iteratively defined in C by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

is said to be viscosity approximation. If $f(x_n) = y_0$ for a given $y_0 \in C$, it is called Halpern approximation which was first introduced by Halpern [3] in 1967. Under the following assumption:

$$\begin{aligned} &\text{(i)} \quad \lim_{n \to \infty} \alpha_n = 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \\ &\text{(ii)} \quad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \left(\text{or } \lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1\right), \end{aligned} \tag{1.7}$$

Xu [4] proved the strong convergence of $\{x_n\}$ to a fixed point of T in Hilbert spaces and in uniformly smooth Banach spaces in 2004. In [3], Halpern pointed out that the condition (i) is necessary for the convergence of the Halpern approximation to a fixed point of T. At the same time, he put forth the following open problem: *is the condition (i) a sufficient condition for the convergence of the Halpern approximation to a fixed point of T?* which was put forward by Reich in [5] also. In order to answer the open question, many authors have done extensively some works; see [6–11] and the references therein. In [7–9], the strong convergence of the Halpern approximation depends on the convergence of the path $x_t = ty_0+(1-t)Tx_t (t \in (0,1))$. In [6], Song got rid of the dependence on the convergence of the path x_t, and proved the convergence of the Halpern approximation under the assumptions for $\{\alpha_n\}$ and T as follows:

$$\begin{aligned} &\text{(iii)} \quad \sum_{n=0}^{\infty}\alpha_n = \infty, \quad \sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty \quad \left(\text{or } \lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = 1\right), \\ &\text{(iv)} \quad K_{\min}\bigcap F(T) \neq \emptyset, \end{aligned} \tag{1.8}$$

where $K_{\min} = \{z \in C : \mathcal{U}_n\|x_n - z\|^2 = \inf_{y\in C}\mathcal{U}_n\|x_n - y\|^2\}$ and $\mathcal{U}_n$ is a Banach limit. Recently, many authors have studied extensively the problem of approximating a fixed point of nonexpansive nonself-mappings $T : C \to E$ in a Banach space, by using the Halpern type iteration (see [12–16]) and the viscosity type iteration (see [17–23]).

In this work, on one hand, we will prove the strong convergence of the mixed viscosity iterative scheme, which is introduced as follows: for any chosen $x_0 \in C$,

$$\begin{aligned} y_n &= (1-\theta_n)x_n + \theta_n T x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)\left[(1-\beta_n)x_n + \beta_n T y_n\right]), \quad \forall n \geq 0, \end{aligned} \tag{1.9}$$

where $f : C \to C$ is a fixed contractive mapping, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ are the real number sequences in $(0,1)$, to a fixed point of the nonexpansive mapping $T : C \to C$ in a real Banach space with uniformly Gâteaux differentiable norm under the following conditions:

$$\begin{aligned} &\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=0}^{\infty}\alpha_n = \infty, \\ &0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1, \\ &\lim_{n\to\infty}\theta_n = 0, \end{aligned} \tag{C1}$$

$$\text{(iv)}' \quad \widetilde{K}_{\min}\bigcap F(T) \neq \emptyset, \tag{1.10}$$

where $\widetilde{K}_{\min} = \{z \in K : \mathcal{U}_n\|x_n - z\|^2 = \inf_{y\in C}\mathcal{U}_n\|x_n - y\|^2\}$, $\mathcal{U}_n$ is a Banach limit, and $\{x_n\}$ is defined by (1.9). On the other hand, we will show that the condition (iv)$'$ is not necessary for proving the strong convergence of the mixed viscosity iterative scheme. As the applications, we will show some results about mixed viscosity type approximation and Halpern type approximation for nonexpansive nonself-mappings also. Our theorems complement and generalize the corresponding results in [8, 9, 16, 24–26].

Now, we recall the following lemmas for proving our theorems firstly.

Lemma 1.1 (see [27]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E, $\{t_n\} \subset [0,1]$ a sequence satisfying*

$$0 < \liminf_{n\to\infty} t_n \le \limsup_{n\to\infty} t_n < 1. \tag{1.11}$$

Suppose that $x_{n+1} = t_n y_n + (1-t_n)x_n$, for all $n = 0,1,2,\ldots$, and

$$\limsup_{n\to\infty}\left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{1.12}$$

Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 1.2 (see [28]). *Let $\{a_n\}$, $\{b_n\}$ and, $\{c_n\}$ be sequences of nonnegative real numbers such that $a_{n+1} \le (1-\alpha_n)a_n + \alpha_n b_n + c_n, \forall n \ge 0$. If $\{\alpha_n\} \subset [0,1]$,*

$$\sum_{n=1}^{\infty}\alpha_n = \infty, \qquad \limsup_{n\to\infty} b_n \le 0, \qquad \sum_{n=1}^{\infty} c_n < \infty, \tag{1.13}$$

then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

In this section, the mixed viscosity iterations for a contractive self-mapping f for approximating to a fixed point of nonexpansive mapping $T : C \to C$ are studied in a real Banach space.

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Banach space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) = \{x \in C : Tx = x\} \neq \emptyset$ and $f : C \to C$ a fixed contractive mapping with the contractive coefficient $k \in (0,1)$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ in $(0,1)$ satisfy (C1) and for any $x_0 \in C$, the sequence $\{x_n\}$ is defined as follows:*

$$\begin{gathered} y_n = (1-\theta_n)x_n + \theta_n T x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\left[(1-\beta_n)x_n + \beta_n T y_n\right]), \quad \forall n \ge 0; \end{gathered} \tag{2.1}$$

then we obtain the following:

(1) *$\{x_n\}$ is bounded;*

(2) *$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Proof. First we show that $\{x_n\}$ is bounded. Now let $z \in F(T)$, then

$$\begin{aligned}
\|y_n - z\| &= \|(1-\theta_n)(x_n - z) + \theta_n(Tx_n - Tz)\| \le \|x_n - z\|; \\
\|x_{n+1} - z\| &= \left\|\alpha_n f(x_n) + (1-\alpha_n)\left[(1-\beta_n)x_n + \beta_n T y_n\right] - z\right\|, \\
&\le \alpha_n \left\|\left(f(x_n) - f(z)\right) + \left(f(z) - z\right)\right\| \\
&\quad + (1-\alpha_n)\left\|(1-\beta_n)(x_n - z) + \beta_n(Ty_n - z)\right\| \\
&\le \alpha_n k\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\beta_n\|Ty_n - Tz\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|x_n - z\| \\
&\le \alpha_n k\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\beta_n\|y_n - z\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|x_n - z\| \\
&\le \alpha_n\|f(z) - z\| + (1-\alpha_n(1-k))\|x_n - z\| \\
&\le \max\left\{\frac{1}{1-k}\|f(z) - z\|, \|x_n - z\|\right\} \\
&\le \max\left\{\frac{1}{1-k}\|f(z) - z\|, \|x_0 - z\|\right\}.
\end{aligned} \tag{2.2}$$

Therefore, $\{x_n\}$ is bounded, so are $\{Tx_n\}$, $\{y_n\}$, $\{Ty_n\}$, and $\{f(x_n)\}$. Next, we show that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.3}$$

For all $n \ge 0$, let

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)u_n, \tag{2.4}$$

where $\gamma_n = (1-\alpha_n)(1-\beta_n) \in (0,1)$, then

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1. \tag{2.5}$$

Note that

$$\begin{aligned} u_{n+1}-u_n &= \frac{x_{n+2}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{x_{n+1}-\gamma_n x_n}{1-\gamma_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})\left[(1-\beta_{n+1})x_{n+1}+\beta_{n+1}Ty_{n+1}\right]-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n)+(1-\alpha_n)\left[(1-\beta_n)x_n+\beta_n Ty_n\right]-\gamma_n x_n}{1-\gamma_n} \\ &= \frac{\alpha_{n+1}}{1-\gamma_{n+1}}\left(f(x_{n+1})+Ty_{n+1}\right) - \frac{\alpha_n}{1-\gamma_n}\left(f(x_n)+Ty_n\right) - \left(Ty_{n+1}-Ty_n\right), \end{aligned} \tag{2.6}$$

$$\left\|Ty_{n+1}-Ty_n\right\| \le \left\|y_{n+1}-y_n\right\| \le \|x_{n+1}-x_n\| + \theta_{n+1}\|x_{n+1}-Tx_{n+1}\| + \theta_n\|x_n-Tx_n\|. \tag{2.7}$$

Hence, we have that

$$\begin{aligned} \|u_{n+1}-u_n\| \le{}& \frac{\alpha_{n+1}}{1-\gamma_{n+1}}\left\|f(x_{n+1})+Ty_{n+1}\right\| + \frac{\alpha_n}{1-\gamma_n}\left\|f(x_n)+Ty_n\right\| \\ &+ \|x_{n+1}-x_n\| + \theta_{n+1}\|x_{n+1}-Tx_{n+1}\| + \theta_n\|x_n-Tx_n\|. \end{aligned} \tag{2.8}$$

Then it follows from the boundedness of $\{x_n\}$, $\{Tx_n\}$, $\{Ty_n\}$, and $\{f(x_n)\}$, and (C1) that

$$\limsup_{n\to\infty}\left(\|u_{n+1}-u_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{2.9}$$

It follow from (2.9) and Lemma 1.1 that

$$\lim_{n\to\infty}\|x_n-u_n\| = 0, \tag{2.10}$$

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{2.11}$$

Since

$$\begin{gathered} x_{n+1}-x_n = \alpha_n\left(f(x_n)-x_n\right) + (1-\alpha_n)\beta_n\left(Ty_n-x_n\right), \\ x_n-y_n = \theta_n(x_n-Tx_n), \end{gathered} \tag{2.12}$$

$$(1-\alpha_n)\beta_n\left\|Ty_n-x_n\right\| \le \|x_{n+1}-x_n\| + \alpha_n\left\|f(x_n)-x_n\right\|, \tag{2.13}$$

from (C1), (2.11) and, the boundedness of $\{x_n\}$, $\{Tx_n\}$, and $\{f(x_n)\}$, we have

$$\lim_{n\to\infty}\left\|x_n-Ty_n\right\| = 0, \qquad \lim_{n\to\infty}\left\|x_n-y_n\right\| = 0. \tag{2.14}$$

From

$$\begin{aligned}\|x_{n+1}-Tx_n\| &\le \|Ty_n-Tx_n\| + \|x_n-Ty_n\| + \|x_{n+1}-x_n\| \\ &\le \|y_n-x_n\| + \|x_n-Ty_n\| + \|x_{n+1}-x_n\|,\end{aligned} \tag{2.15}$$

we obtain

$$\lim_{n\to\infty}\|x_{n+1}-Tx_n\| = 0. \tag{2.16}$$

Therefore, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. This completes the proof. □

Proposition 2.2 (see [18]). *Let C be a nonempty closed convex subset of a real Banach space E which has uniformly Gâteaux differentiable norm. Suppose that $\{x_n\}$ is a bounded sequence of E such that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\mathcal{U}_n$ is a Banach limit. If $z \in C$ such that $\mathcal{U}_n\|x_n - z\|^2 = \inf_{y\in C}\mathcal{U}_n\|x_n - y\|^2$, then*

$$\limsup_{n\to\infty}\langle y-z, j(x_n-z)\rangle \le 0, \quad \forall y \in C. \tag{2.17}$$

Let $\{x_n\}$ be defined by (2.1) and $\alpha_n \in (0,1)$, it follows from Theorem 2.1 that $\{x_n\}$ is bounded. Let

$$\varphi(y) = \mathcal{U}_n\|x_n - y\|^2 \quad \forall y \in C, \tag{2.18}$$

then $\varphi(y)$ is convex and continuous. If E is reflexive, there exists $z \in C$ such that $\varphi(z) = \inf_{y\in C}\varphi(y)$ (see [2], Theorem 1.3.11). Let

$$\widetilde{K}_{\min} = \{z \in C : \varphi(z) = \inf_{y\in C}\varphi(y)\}, \tag{2.19}$$

then $\widetilde{K}_{\min} \neq \emptyset$ is a closed convex subset in a reflexive Banach space E.

Theorem 2.3. *Let C be a nonempty closed convex subset of a real Banach space E which has uniformly Gâteaux differentiable norm. Suppose that $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, f is a fixed contractive mapping with the contractive coefficient $k \in (0,1)$, and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ in $(0,1)$ satisfy (C1). If $\widetilde{K}_{\min} \bigcap F(T) \neq \emptyset$, then the sequence $\{x_n\}$ defined by (2.1) converges strongly to a fixed point of T as $n \to \infty$.*

Proof. Take $z \in \widetilde{K}_{\min} \bigcap F(T)$. It follows from Theorem 2.1 that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. For $f(z) \in C$, by Proposition 2.2, we have that

$$\limsup_{n\to\infty}\langle f(z) - z, j(x_n - z)\rangle \le 0. \tag{2.20}$$

Next, we show that $\lim_{n\to\infty} x_n = z$. Since

$$\begin{aligned} x_{n+1} - (\alpha_n f(x_n) + (1-\alpha_n)z) &= (x_{n+1} - z) - \alpha_n(f(x_n) - z), \\ \|x_{n+1} - (\alpha_n f(x_n) + (1-\alpha_n)z)\| &\le \|(1-\alpha_n)[(1-\beta_n)x_n + \beta_n T y_n] - (1-\alpha_n)z\| \\ &= \|(1-\alpha_n)[(1-\beta_n)(x_n - z) + \beta_n(Ty_n - Tz)]\| \\ &\le (1-\alpha_n)\|x_n - z\|, \end{aligned} \tag{2.21}$$

then

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \langle x_{n+1} - (\alpha_n f(x_n) + (1-\alpha_n)z), j(x_{n+1} - z)\rangle \\ &\quad + \alpha_n\langle f(x_n) - z, j(x_{n+1} - z)\rangle \\ &\le \|x_{n+1} - (\alpha_n f(x_n) + (1-\alpha_n)z)\|\|x_{n+1} - z\| \\ &\quad + \alpha_n(\langle f(x_n) - f(z), j(x_{n+1} - z)\rangle + \langle f(z) - z, j(x_{n+1} - z)\rangle) \\ &\le \frac{1}{2}(1-\alpha_n)\left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) + \frac{1}{2}\alpha_n\left(\|f(x_n) - f(z)\|^2 + \|x_{n+1} - z\|^2\right) \\ &\quad + \alpha_n\langle f(z) - z, j(x_{n+1} - z)\rangle \\ &\le \frac{1}{2}\left(1 - \alpha_n\left(1 - k^2\right)\right)\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n\langle f(z) - z, j(x_{n+1} - z)\rangle. \end{aligned} \tag{2.22}$$

Hence,

$$\|x_{n+1} - z\|^2 \le \left(1 - \alpha_n\left(1 - k^2\right)\right)\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, j(x_{n+1} - z)\rangle. \tag{2.23}$$

Let $a_n = \|x_n - z\|^2$ and $b_n = (2/(1-k^2))\langle y - z, j(x_{n+1} - z)\rangle$; it follows from Lemma 1.2 that $\{x_n\}$ converges strongly to $z \in C$. This completes the proof. □

Proposition 2.4. *Let E be a real reflexive Banach space with uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $\{x_n\}$ defined by* (2.1). *Then $\widetilde{K}_{\min} \bigcap F(T) \neq \emptyset$.*

Proof. From the reflexivity of E and the definition of $\widetilde{K}_{\min}$, it follows that $\widetilde{K}_{\min}$ is a nonempty closed convex subset of E. By Theorem 2.1, we know that $\lim_{n\to\infty}\|x_{n+1} - Tx_n\| = 0$.

Claim that $T(\widetilde{K}_{\min}) \subset \widetilde{K}_{\min}$. Indeed, for any $x \in \widetilde{K}_{\min}$, we have

$$\begin{aligned}\varphi(Tx) &= \mathcal{U}_n\|x_n - Tx\|^2 = \mathcal{U}_n\|x_{n+1} - Tx\|^2 \\ &\le \mathcal{U}_n(\|x_{n+1} - Tx_n\| + \|Tx_n - Tx\|)^2 \le \mathcal{U}_n\|x_n - x\|^2 = \varphi(x).\end{aligned} \tag{2.24}$$

Therefore, $Tx \in \widetilde{K}_{\min}$ and $T(\widetilde{K}_{\min}) \subset \widetilde{K}_{\min}$.

Since $F(T) \neq \emptyset$, there exists unique $u \in \widetilde{K}_{\min}$ such that $\|z - u\| = \inf_{x\in\widetilde{K}_{\min}}\|z - x\|$, for all $z \in F(T)$. By $Tz = z$ and $Tu \in \widetilde{K}_{\min}$, we have

$$\|z - Tu\| = \|Tz - Tu\| \le \|z - u\|. \tag{2.25}$$

Hence $u = Tu$ by the uniqueness of $u \in \widetilde{K}_{\min}$. Thus $\widetilde{K}_{\min} \bigcap F(T) \neq \emptyset$. □

By the above results, we can obtain the following theorem.

Theorem 2.5. *Let E be a real reflexive Banach space with uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and f a fixed contractive mapping with the contractive coefficient $k \in (0,1)$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ in $(0,1)$ satisfy (C1), then the sequence $\{x_n\}$ defined by (2.1) converges strongly to a fixed point of T.*

Remark 2.6. Theorem 2.5 shakes off the assumption $\lim_{n\to\infty}\|Ty_n - y_n\| = 0$ in [26] and extends Theorem 1 in [24] shown in uniformly smooth Banach spaces.

Theorem 2.7. *Let E be a real strictly convex Banach space E with uniformly Gâteaux differentiable norm and C a nonempty closed convex subset of E which is a sunny nonexpansive retract of E. Let $T : C \to E$ be a nonexpansive nonself-mapping with $F(T) = \{x \in C : Tx = x\} \neq \emptyset$ and $f : C \to C$ a fixed contractive mapping with the contractive coefficient $k \in (0,1)$. Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ in $(0,1)$ satisfy (C1), C is a sunny nonexpansive retract of E, and the sequence $\{x_n\}$ is defined as follows:*

$$\begin{gathered}\text{for any } x_0 \in C, \\ y_n = (1-\theta_n)x_n + \theta_n PTx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\left[(1-\beta_n)x_n + \beta_n PTy_n\right]), \quad \forall n \ge 0;\end{gathered} \tag{2.26}$$

If $\{z \in C : \mathcal{U}_n\|x_n - z\|^2 = \inf_{y\in C}\mathcal{U}_n\|x_n - y\|^2\} \bigcap F(T) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T as $n \to \infty$.

Proof. It follows from [12, Lemmas 3.1 and 3.3] that $F(T) = F(PT)$, and then $F(PT) \neq \emptyset$. By replacing T by PT in Theorem 2.3, we can show that the conclusion holds. □

Theorem 2.8. *Let E be a real reflexive and strictly convex Banach space with uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of a real Banach space E which is a sunny nonexpansive retract of E, $T : C \to E$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$, and f a fixed contractive mapping with the contractive coefficient $k \in (0,1)$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ in $(0,1)$ satisfy (C1), then the sequence $\{x_n\}$ defined by (2.26) converges strongly to a fixed point of T.*

Proof. It follows from [12, Lemmas 3.1 and 3.3] that $F(T) = F(PT)$, and then $F(PT) \neq \emptyset$. By replacing T by PT in Theorem 2.5, we can show that the conclusion holds. □

Corollary 2.9. *Let E be a real uniformly convex Banach space with uniformly Gâteaux differentiable norm, C, T, f, $\{\alpha_n\}$, and $\{\beta_n\}$ as Theorem 2.8. Then the sequence $\{x_n\}$ defined by (2.1) converges strongly to a fixed point of T.*

3. Some Applications

In this section, we introduce the following Halpern type approximation: for the given $x_0, u_0 \in C$, the sequence $\{x_n\}$ is defined by

$$\begin{aligned} y_n &= (1-\theta_n)x_n + \theta_n T x_n, \\ x_{n+1} &= \alpha_n u_0 + (1-\alpha_n)\left[(1-\beta_n)x_n + \beta_n T y_n\right]), \quad \forall n \geq 0, \end{aligned} \tag{3.1}$$

and show some results about Halpern type approximation for nonexpansive mappings, which generalize and improve some known conclusions.

Define $C_{\min} = \{z \in C : \varphi(z) = \inf_{y\in C}\varphi(y)\}$, where $\varphi(y) = \mathcal{U}_n\|x_n - y\|^2$ for all $y \in C$ and $\{x_n\}$ is defined by (3.1).

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Banach space E which has uniformly Gâteaux differentiable norm. Suppose that $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, f is a fixed contractive mapping with the contractive coefficient $k \in (0,1)$, and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ in $(0,1)$ satisfy (C1). If $C_{\min} \bigcap F(T) \neq \emptyset$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a fixed point of T as $n \to \infty$.*

Proof. First we show that $\{x_n\}$ is bounded. Now let $z \in F(T)$, then

$$\begin{aligned} \|y_n - z\| &= \|(1-\theta_n)(x_n - z) + \theta_n(Tx_n - Tz)\| \leq \|x_n - z\|, \\ \|x_{n+1} - z\| &= \left\|\left(\alpha_n u_0 + (1-\alpha_n)\left[(1-\beta_n)x_n + \beta_n T y_n\right]\right) - z\right\| \\ &\leq \alpha_n\|u_0 - z\| + (1-\alpha_n)\left\|(1-\beta_n)(x_n - z) + \beta_n(Ty_n - z)\right\| \\ &\leq \alpha_n\|u_0 - z\| + (1-\alpha_n)\beta_n\|Ty_n - Tz\| + (1-\alpha_n)(1-\beta_n)\|x_n - z\| \\ &\leq \alpha_n\|u_0 - z\| + (1-\alpha_n)\|x_n - z\| \\ &\leq \max\{\|u_0 - z\|, \|x_0 - z\|\}. \end{aligned} \tag{3.2}$$

Therefore, $\{x_n\}$ is bounded. Then $C_{\min}$ is well defined.

In the proofs of Theorems 2.1 and 2.3, take $f(x) = u_0$ for all $x \in C$; we can show similarly that the conclusion holds. □

Remark 3.2. If $\theta_n = 0$ for all $n = 0, 1, 2, \ldots$, Theorem 3.1 weakens the condition of $\{\beta_n\}$ of Theorem 4.1 in [6].

Theorem 3.3. *Let C be a nonempty closed convex subset of a real reflexive Banach space E which has uniformly Gâteaux differentiable norm. Suppose that $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ in $(0, 1)$ satisfy $(C1)$. Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a fixed point of T as $n \to \infty$.*

Proof. Take $f(x) = u_0$ for all $x \in C$ in Theorem 2.5; it is easy to show that the conclusion holds. □

Remark 3.4. If $\theta_n = 0$ for all $n = 0, 1, 2, \ldots$, Theorem 3.3 gets rid of the dependence on the implicit anchor-like continuous path $z_t = ty + (1 - t)Tz_t$ in Suzuki's Theorem 3 in [9] and Theorem 3.1 of C. E. Chidume and C. O. Chidume [8]. It also complements and generalizes [25, Theorem 1], which is proved in uniformly smooth Banach spaces.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real reflexive and strictly convex Banach space E which has uniformly Gâteaux differentiable norm. Suppose that C is a sunny nonexpansive retract of E, $T : C \to E$ is a nonexpansive nonself-mapping with $F(T) \neq \emptyset$, f is a fixed contractive mapping with the contractive coefficient $k \in (0, 1)$, the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ in $(0, 1)$ satisfy $(C1)$, and the sequence $\{x_n\}$ is defined as follows,*

$$\begin{gathered} \text{for the given } x_0, u_0 \in C, \\ y_n = (1 - \theta_n)x_n + \theta_n PTx_n, \\ x_{n+1} = \alpha_n u_0 + (1 - \alpha_n)\left[(1 - \beta_n)x_n + \beta_n PTy_n\right]), \quad \forall n \geq 0. \end{gathered} \tag{3.3}$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T as $n \to \infty$.

Remark 3.6. If $\theta_n = 0$ for all $n = 0, 1, 2, \ldots$, Theorem 3.3 improves and generalizes Theorem 3.2 in [16]: it gets rid of the restriction of $\delta \in (0, 1)$ and dependence on the implicit anchor-like continuous path $z_t = ty + (1 - t)PTz_t$.

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