Research Article

On Regularized Quasi-Semigroups and Evolution Equations

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We introduce the notion of regularized quasi-semigroup of bounded linear operators on Banach spaces and its infinitesimal generator, as a generalization of regularized semigroups of operators. After some examples of such quasi-semigroups, the properties of this family of operators will be studied. Also some applications of regularized quasi-semigroups in the abstract evolution equations will be considered. Next some elementary perturbation results on regularized quasi-semigroups will be discussed.

1. Introduction and Preliminaries

The theory of quasi-semigroups of bounded linear operators, as a generalization of strongly continuous semigroups of operators, was introduced in 1991 [1], in a preprint of Barcnas and Leiva. This notion, its elementary properties, exponential stability, and some of its applications in abstract evolution equations are studied in [2–5]. The dual quasi-semigroups and the controllability of evolution equations are also discussed in [6].

Given a Banach space X , we denote by $B(X)$ the space of all bounded linear operators on X . A biparametric commutative family $\{R(s, t)\}_{s, t \geq 0} \subseteq B(X)$ is called a quasi-semigroup of operators if for every $s, t, r \geq 0$ and $x \in X$, it satisfies

- (1) $R(t, 0) = I$, the identity operator on X ,
- (2) $R(r, s + t) = R(r + t, s)R(r, t)$,
- (3) $\lim_{(s, t) \rightarrow (s_0, t_0)} \|R(s, t)x - R(s_0, t_0)x\| = 0, x \in X$,
- (4) $\|R(s, t)\| \leq M(s + t)$, for some continuous increasing mapping $M : [0, \infty) \rightarrow [0, \infty)$.

Also regularized semigroups and their connection with abstract Cauchy problems are introduced in [7] and have been studied in [8–12] and many other papers.

We mention that if $C \in B(X)$ is an injective operator, then a one-parameter family $\{T(t)\}_{t \geq 0} \subseteq B(X)$ is called a C -semigroup if for any $s, t \geq 0$ it satisfies $T(s+t)C = T(s)T(t)$ and $T(0) = C$.

In this paper we are going to introduce regularized quasi-semigroups of operators.

In Section 2, some useful examples are discussed and elementary properties of regularized quasi-semigroups are studied.

In Section 3 regularized quasi-semigroups are applied to find solutions of the abstract evolution equations. Also perturbations of the generator of regularized quasi-semigroups are also considered in this section. Our results are mainly based on the work of Barcenas and Leiva [1].

2. Regularized Quasi-Semigroups

Suppose X is a Banach space and $\{K(s, t)\}_{s, t \geq 0}$ is a two-parameter family of operators in $B(X)$. This family is called commutative if for any $r, s, t, u \geq 0$,

$$K(r, t)K(s, u) = K(s, u)K(r, t). \quad (2.1)$$

Definition 2.1. Suppose C is an injective bounded linear operator on Banach space X . A commutative two-parameter family $\{K(s, t)\}_{s, t \geq 0}$ in $B(X)$ is called a regularized quasi-semigroups (or C -quasi-semigroups) if

- (1) $K(t, 0) = C$, for any $t \geq 0$;
- (2) $CK(r, t+s) = K(r+t, s)K(r, t)$, $r, t, s \geq 0$;
- (3) $\{K(s, t)\}_{s, t \geq 0}$ is strongly continuous, that is,

$$\lim_{(s, t) \rightarrow (s_0, t_0)} \|K(s, t)x - K(s_0, t_0)x\| = 0, \quad x \in X; \quad (2.2)$$

- (4) there exists a continuous and increasing function $M : [0, \infty) \rightarrow [0, \infty)$, such that for any $s, t > 0$, $\|K(s, t)\| \leq M(s+t)$.

For a C -quasi-semigroups $\{K(s, t)\}_{s, t \geq 0}$ on Banach space X , let D be the set of all $x \in X$ for which the following limits exist in the range of C :

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{K(s, t)x - Cx}{t} &= \lim_{t \rightarrow 0^+} \frac{K(s-t, t)x - Cx}{t}, \quad s > 0 \\ \lim_{t \rightarrow 0^+} \frac{K(0, t)x - Cx}{t} &. \end{aligned} \quad (2.3)$$

Now for $x \in D$ and $s \geq 0$, define

$$A(s)x = C^{-1} \lim_{t \rightarrow 0^+} \frac{K(s, t)x - Cx}{t}. \quad (2.4)$$

$\{A(s)\}_{s \geq 0}$ is called the infinitesimal generator of the regularized quasi-semigroup $\{K(s, t)\}_{s, t \geq 0}$. Somewhere we briefly apply generator instead of infinitesimal generator.

Here are some useful examples of regularized quasi-semigroups.

Example 2.2. Let $\{T_t\}_{t \geq 0}$ be an exponentially bounded strongly continuous C -semigroup on Banach space X , with the generator A . Then

$$K(s, t) := T_t, \quad s, t \geq 0, \quad (2.5)$$

defines a C -quasi-semigroup with the generator $A(s) = A$, $s \geq 0$, and so $D = D(A)$.

Example 2.3. Let $X = \text{BUC}(\mathbb{R})$, the space of all bounded uniformly continuous functions on $\mathbb{R}$ with the supremum-norm. Define $C, K(s, t) \in B(X)$, by

$$Cf(x) = e^{-x^2}f(x), \quad K(s, t)f(x) = e^{-x^2}f(t^2 + 2st + x), \quad s, t \geq 0. \quad (2.6)$$

One can see that $\{K(s, t)\}_{s, t \geq 0}$ is a regularized C -quasi-semigroup of operators on X , with the infinitesimal generator $A(s)f = 2sf$ on D , where $D = \{f \in X : f' \in X\}$.

Example 2.4. Let $\{T_t\}_{t \geq 0}$ be a strongly continuous semigroup of operators on Banach space X , with the generator A . If $C \in B(X)$ is injective and commutes with T_t , $t \geq 0$, then

$$K(s, t) := Ce^{T_{s+t}-T_s}, \quad s, t \geq 0, \quad (2.7)$$

is a C -quasi-semigroup with the generator $A(s) = AT_s$. Thus $D = D(A)$. In fact, for $x \in D$, boundedness of C implies that

$$CA(s)x = \lim_{t \rightarrow 0^+} \frac{Ce^{T_{s+t}-T_s}x - Cx}{t} = C \lim_{t \rightarrow 0^+} \frac{e^{T_{s+t}-T_s}x - x}{t} = C \frac{d}{ds} \Big|_{t=0} (T_{s+t} - T_s)x = CAT_sx. \quad (2.8)$$

Now injectivity of C implies that $A(s)x = AT_sx$, and so $D = D(A)$.

Example 2.5. Let $\{T_t\}_{t \geq 0}$ be a strongly continuous exponentially bounded C -semigroup of operators on Banach space X , with the generator A . For $s, t \geq 0$, define

$$K(s, t) = T(g(s+t) - g(s)), \quad s, t \geq 0, \quad (2.9)$$

where $g(t) = \int_0^t a(s)ds$, and $a \in C[0, \infty)$, with $a(t) > 0$. We have $K(s, 0) = T(0) = C$ and the C -semigroup properties of $\{T(t)\}_{t \geq 0}$ imply that

$$\begin{aligned} CK(r, s+t) &= CT(g(r+t+s) - g(r)) \\ &= CT(g(r+t+s) - g(t+r) + g(t+r) - g(r)) \\ &= T(g(r+t+s) - g(t+r))T(g(t+r) - g(r)) \\ &= K(r+t, s)K(r, t). \end{aligned} \quad (2.10)$$

So $\{K(s, t)\}_{s, t \geq 0}$ is a C -quasi-semigroup (the other properties can be also verified easily). Also $D = D(A)$ and for $x \in D$, $A(s)x = a(s)Ax$.

Some elementary properties of regularized quasi-semigroups can be seen in the following theorem.

Theorem 2.6. *Suppose $\{K(s, t)\}_{s, t \geq 0}$ is a C -quasi-semigroup with the generator $\{A(s)\}_{s \geq 0}$ on Banach space X . Then*

(i) *for any $x \in D$ and $s_0, t_0 \geq 0$, $K(s_0, t_0)x \in D$ and*

$$K(s_0, t_0)A(s)x = A(s)K(s_0, t_0)x; \quad (2.11)$$

(ii) *for each $x_0 \in D$,*

$$\frac{\partial}{\partial t} K(r, t)Cx_0 = A(r+t)K(r, t)Cx_0 = K(r, t)A(r+t)Cx_0; \quad (2.12)$$

(iii) *if $A(s)$ is locally integrable, then for each $x_0 \in D$ and $r \geq 0$,*

$$K(r, t)x_0 = Cx_0 + \int_0^t A(r+s)K(r, s)x_0 ds, \quad t \geq 0; \quad (2.13)$$

(iv) *let $f : [0, \infty) \rightarrow X$ be a continuous function; then for every $t \in [0, \infty)$,*

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} K(s, u)f(u)du = K(s, t)f(t); \quad (2.14)$$

(v) *Let $C' \in B(X)$ be injective and for any $s, t \geq 0$, $C'K(s, t) = K(s, t)C'$. Then $R(s, t) := C'K(s, t)$ is a CC' -quasi-semigroup with the generator $\{A(s)\}_{s \geq 0}$,*

(vi) *Suppose $\{R(s, t)\}_{s, t \geq 0}$ is a quasi-semigroup of operators on Banach space X with the generator $\{A(s)\}_{s \geq 0}$, and $C \in B(X)$ commutes with every $R(s, t)$, $s, t \geq 0$. Then $K(s, t) := CR(s, t)$ is a C -quasi-semigroup of operators on X with the generator $\{A(s)\}_{s \geq 0}$.*

Proof. First we note that from the commutativity of $\{K(s, t)\}_{s, t \geq 0}$:

$$CK(s, t) = K(s, t)C \quad s, t \geq 0. \quad (2.15)$$

Also $x \in D$ implies that

$$\lim_{t \rightarrow 0^+} \frac{K(s, t)x - Cx}{t} = CA(s)x \quad s \geq 0. \quad (2.16)$$

Thus from continuity of $K(s_0, t_0)$, we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{K(s, t)K(s_0, t_0)x - CK(s_0, t_0)x}{t} &= K(s_0, t_0) \lim_{t \rightarrow 0^+} \frac{K(s, t)x - Cx}{t} \\ &= K(s_0, t_0)CA(s)x \\ &= CK(s_0, t_0)A(s)x. \end{aligned} \quad (2.17)$$

Thus $K(s_0, t_0)x \in D$ and $A(s)K(s_0, t_0) = K(s_0, t_0)A(s)x$.

To prove (ii), consider the quotient

$$\begin{aligned} \frac{K(r, t+s)Cx_0 - K(r, t)Cx_0}{s} &= \frac{K(r+t, s)K(r, t)x_0 - K(r, t)Cx_0}{s} \\ &= K(r, t) \frac{K(r+t, s)x_0 - Cx_0}{s}, \end{aligned} \quad (2.18)$$

which tends to $K(r, t)CA(r+t)x_0$ as $s \rightarrow 0^+$.

Also for $s < 0$,

$$\begin{aligned} \frac{K(r, t+s)Cx_0 - K(r, t)Cx_0}{s} &= \frac{K(r, t)Cx_0 - K(r, t+s)Cx_0}{-s} \\ &= \frac{K(r+t+s, -s)K(r, t+s)x_0 - K(r, t+s)Cx_0}{-s} \\ &= K(r, t+s) \frac{K(r+t+s, -s)x_0 - Cx_0}{-s} \\ &= K(r, t+s) \frac{1}{-s} (K(r+t+s, -s)x_0 \\ &\quad - K(r+t, -s)x_0 + K(r+t, -s)x_0 - Cx_0). \end{aligned} \quad (2.19)$$

Now the strongly continuity of $\{K(s, t)\}_{s, t \geq 0}$ implies that

$$\lim_{s \rightarrow 0^-} K(r+t+s, -s)x_0 - K(r+t, -s)x_0 = 0. \quad (2.20)$$

Thus

$$\lim_{s \rightarrow 0^-} \frac{K(r+t+s, -s)x_0 - Cx_0}{-s} = CA(r+t)x_0. \quad (2.21)$$

Hence by the strongly continuity of $K(s, t)$,

$$\lim_{s \rightarrow 0^-} \frac{K(r, t+s)Cx_0 - K(r, t)Cx_0}{s} = K(r, t)CA(r+t)x_0. \quad (2.22)$$

Thus $(\partial/\partial t)K(r, t)Cx_0 = K(r, t)CA(r+t)x_0$. The second equality holds by (i).

Now integrating of this equation, we have

$$K(r, t)Cx_0 - Cx_0 = C \int_0^t K(r, s)A(r + s)x_0 ds. \quad (2.23)$$

Hence injectivity of C implies (iii).

(iv) is trivial from continuity of f and strongly continuity of $\{K(s, t)\}_{s, t \geq 0}$. In (v), obviously $\{R(s, t)\}_{s, t \geq 0}$ is a $C'C$ -quasi-semigroup. For $x \in D$, we have

$$\frac{R(s, t)x - CC'x}{t} = C' \frac{K(s, t)x - Cx}{t}, \quad (2.24)$$

which tends to $C'CA(s)$, as $t \rightarrow 0^+$. This proves (v).

(vi) can be seen easily. □

3. Evolution Equations and Regularized Quasi-Semigroups

Suppose C is an injective bounded linear operator on Banach space X and $r > 0$. In this section, we study the solutions of the following abstract evolution equation using the regularized quasi-semigroups:

$$\begin{aligned} \dot{x}(t) &= A(t + r)x(t), \quad t > 0, \\ x(0) &= C^2x_0, \quad x_0 \in X. \end{aligned} \quad (3.1)$$

One can see [13, 14] for a comprehensive studying of abstract evolution equations.

Theorem 3.1. *Let $\{A(s)\}_{s \geq 0}$ be the infinitesimal generator of a C -quasi-semigroups $\{K(s, t)\}_{s, t \geq 0}$ on Banach space X , with domain D . Then for each $x_0 \in D$ and $r \geq 0$, the initial value problem (3.1) admits a unique solution.*

Proof. Let $x(t) = K(r, t)Cx_0$. By Theorem 2.6(ii), $x(t)$ is a solution of (3.1).

Now we show that this solution is unique. Suppose $y(s)$ is another solution of (3.1). Trivially $y(s) \in D$. Let $t > 0$. For $s \in [0, t]$ and $x \in X$, define

$$F(s)x = K(r + s, t - s)Cx, \quad G(s) = F(s)Cy(s). \quad (3.2)$$

From C -quasi-semigroup properties, for small enough $h > 0$, we have

$$\begin{aligned} K(r + s, t - s)C &= K(r + s + t - s - (t - s - h), t - s - h)K(r + s, t - s - (t - s - h)) \\ &= K(r + s + h, t - s - h)K(r + s, h). \end{aligned} \quad (3.3)$$

So

$$\begin{aligned}
 \frac{F(s+h)x - F(s)x}{h} &= \frac{K(r+s+h, t-s-h)Cx - K(r+s+h, t-s-h)K(r+s, h)x}{h} \\
 &= -K(r+s+h, t-s-h) \left[\frac{K(r+s, h)x - Cx}{h} \right] \\
 &\longrightarrow -K(r+s, t-s)CA(r+s)x, \quad \text{as } h \longrightarrow 0.
 \end{aligned} \tag{3.4}$$

This means that

$$\dot{F}(s)x = -K(r+s, t-s)CA(r+s)x. \tag{3.5}$$

Therefore, from this, the fact that $y(s)$ satisfies (3.1), and $CF(s) = F(s)C$, we obtain that

$$\begin{aligned}
 \dot{G}(s) &= \dot{F}(s)Cy(s) + F(s)C\dot{y}(s) = -K(r+s, t-s)CA(r+s)Cy(s) + K(r+s, t-s)C^2\dot{y}(s) \\
 &= -K(r+s, t-s)CA(r+s)Cy(s) + K(r+s, t-s)C^2A(r+s)y(s) = 0.
 \end{aligned} \tag{3.6}$$

Hence for every $s \in (0, t)$, $\dot{G}(s) = 0$. Consequently, $G(s)$ is a constant function on $[0, t]$. In particular, $G(0) = G(t)$. So from $y(0) = Cx_0$, we have

$$G(0) = F(0)Cy(0) = K(r, t)C^2x_0 = G(t) = F(t)Cy(t) = K(r+t, 0)C^2y(t) = C^3y(t). \tag{3.7}$$

Hence $C^2K(r, t)x_0 = C^3y(t)$. Now injectivity of C implies that $y(t) = K(r, t)Cx_0$, which proves the uniqueness of the solution. $\square$

Now with the above notation, we consider the inhomogeneous evolution equation

$$\begin{aligned}
 \dot{x}(t) &= A(r+t)x(t) + C^2f(t), \quad 0 < t \leq T, \\
 x(0) &= C^2x_0, \quad x_0 \in D.
 \end{aligned} \tag{3.8}$$

The following theorem guarantees the existence and uniqueness of solutions of (3.8) with some sufficient conditions on f .

Theorem 3.2. *Let $K(s, t)$ be a C -quasi-semigroup on Banach space X , with the generator $\{A(s)\}_{s \geq 0}$ whose domain is D . If $f : [0, T] \rightarrow D$ is a continuous function, each operator $A(s)$ is closed, and*

$$C \int_0^t K(r+s, t-s)f(s)ds \in D, \quad 0 < t \leq T, \tag{3.9}$$

then the initial value equation (3.8) admits a unique solution

$$x(t) = K(r, t)Cx_0 + \int_0^t K(r+s, t-s)Cf(s)ds. \tag{3.10}$$

Proof. For the existence of the solution, it is enough to show that $x(t)$ in (3.10) is continuously differentiable and satisfies (3.8).

Trivially $x(0) = Cx_0$. We know that $y(t) = K(r, t)Cx_0$ is a solution of (3.1) by Theorem 3.1. Define

$$g(t) = \int_0^t K(r+s, t-s)Cf(s)ds, \quad (3.11)$$

which is in D by our hypothesis. We have

$$\begin{aligned} \frac{g(t+h) - g(t)}{h} &= \frac{1}{h} \left[\int_0^{t+h} K(r+s, t+h-s)Cf(s)ds - \int_0^t K(r+s, t-s)Cf(s)ds \right] \\ &= \frac{1}{h} \left[\int_0^t K(r+s, t+h-s)Cf(s)ds - \int_0^t K(r+s, t-s)Cf(s)ds \right. \\ &\quad \left. + \int_t^{t+h} K(r+s, t+h-s)Cf(s)ds \right]. \end{aligned} \quad (3.12)$$

On the other hand, the C -quasi-semigroup properties imply that

$$\begin{aligned} K(r+s, t+h-s)Cf(s) &= K(r+s+t+h-s-h, h)K(r+s, t+h-s-h)f(s) \\ &= K(r+t, h)K(r+s, t-s)f(s). \end{aligned} \quad (3.13)$$

So

$$\begin{aligned} \frac{g(t+h) - g(t)}{h} &= \frac{1}{h} \left[\int_0^t K(r+t, h)K(r+t, t-s)f(s)ds \right. \\ &\quad \left. - \int_0^t K(r+s, t-s)Cf(s)ds + \int_t^{t+h} K(r+s, t+h-s)Cf(s)ds \right] \\ &= \int_0^t K(r+t, t-s) \left(\frac{K(r+t, h)f(s) - Cf(s)}{h} \right) ds \\ &\quad + \frac{1}{h} \int_t^{t+h} K(r+s, t+h-s)Cf(s)ds. \end{aligned} \quad (3.14)$$

Since the range of f is in D , passing to the limit when $h \rightarrow 0$, and using Theorem 2.6(v), we have

$$\begin{aligned} \dot{g}(t) &= \int_0^t K(r+s, t-s)CA(r+t)f(s)ds + K(r+t, t-t)Cf(t) \\ &= \int_0^t K(r+s, t-s)CA(r+t)f(s)ds + C^2f(t). \end{aligned} \quad (3.15)$$

Therefore, $\dot{g}(t)$ exists. Also by our hypothesis $A(r+t)$ is closed, and $\int_0^t K(r+s, t-s)Cf(s)ds \in D$, thus

$$\int_0^t K(r+s, t-s)CA(r+t)f(s)ds = A(r+t) \int_0^t K(r+s, t-s)Cf(s)ds. \quad (3.16)$$

Consequently,

$$\dot{g}(t) = A(r+t)g(t) + C^2f(t), \quad t \geq 0. \quad (3.17)$$

Hence

$$\begin{aligned} \dot{x}(t) &= \frac{\partial}{\partial t} K(r, t)Cx_0 + A(r+t) \int_0^t K(r+s, t-s)Cf(s)ds + C^2f(t) \\ &= A(r+t) \left(K(r, t)Cx_0 + \int_0^t K(r+s, t-s)Cf(s)ds \right) + C^2f(t) \\ &= A(r+t)x(t) + C^2f(t). \end{aligned} \quad (3.18)$$

This completes the proof. $\square$

We conclude this section with two simple perturbation theorems and some examples, as applications of our discussion.

Theorem 3.3. (a) Suppose B is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ and $\{A(s)\}_{s \geq 0}$ with domain D is the generator of a regularized C -quasi-semigroup $\{K(s, t)\}_{s, t \geq 0}$, which commutes with $\{T(t)\}_{t \geq 0}$. Then $\{A(s) + B\}_{s \geq 0}$ with domain $D \cap D(B)$ is the infinitesimal generator of a regularized C -quasi-semigroup.

(b) Suppose B is the infinitesimal generator of an exponentially bounded C -semigroup $\{T(t)\}_{t \geq 0}$ and $\{A(s)\}_{s \geq 0}$ with domain D is the generator of a quasi-semigroup (resp., regularized C' -quasi-semigroup) $\{K(s, t)\}_{s, t \geq 0}$, which commutes with $\{T(t)\}_{t \geq 0}$. Then $\{A(s) + B\}_{s \geq 0}$ with domain $D \cap D(B)$ is the infinitesimal generator of a C -regularized quasi-semigroup (resp., regularized CC' -quasi-semigroup).

Proof. In (a) and (b), define

$$R(s, t) = T(t)K(s, t). \quad (3.19)$$

One can see that $\{R(s, t)\}_{s, t \geq 0}$ is a C -regularized quasi-semigroup (in (b), resp., regularized CC' -quasi-semigroup). We just prove that $\{A(s) + B\}_{s \geq 0}$ is its generator. In (a), let $\{B(s)\}_{s \geq 0}$ be the infinitesimal generator of $\{R(s, t)\}_{s, t \geq 0}$ and $x \in D \cap D(B)$. Hence

$$\lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}, \quad \lim_{t \rightarrow 0^+} \frac{K(s, t)x - Cx}{t} \quad (3.20)$$

exist in X and the range of C , respectively. Now the fact that C commutes with $T(t)$ and strongly continuity of $T(t)$ implies that

$$\lim_{t \rightarrow 0^+} T(t) \frac{K(s, t)x - Cx}{t} \quad (3.21)$$

exists in the range of C . So

$$\lim_{t \rightarrow 0^+} \frac{R(s, t)x - Cx}{t} = \lim_{t \rightarrow 0^+} \frac{T(t)K(s, t)x - Cx}{t} = \lim_{t \rightarrow 0^+} T(t) \frac{K(s, t)x - Cx}{t} + C \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \quad (3.22)$$

exists in the range of C and

$$CB(s)x = \lim_{t \rightarrow 0^+} \frac{R(s, t)x - Cx}{t} = CA(s)x + CBx. \quad (3.23)$$

By injectivity of C , $B(s)x = A(s)x + Bx$.

The proof the other parts is similar. $\square$

Theorem 3.4. Let $K(s, t)$ be a C -quasi-semigroup of operator on Banach space X with the generator $\{A(s)\}$ on domain D . If $B \in B(X)$ commutes with $K(s, t)$, $s, t \geq 0$, and $B^2 = B$, then $\{BA(s)\}_{s \geq 0}$ is the infinitesimal generator of C -regularized quasi-semigroup

$$R(s, t) = B(K(s, t) - C) + C. \quad (3.24)$$

Proof. The C -quasi-semigroup properties of $\{R(s, t)\}_{s, t \geq 0}$ can be easily verified. We just prove that its generator is $\{BA(s)\}_{s \geq 0}$. Let $x \in D$; we have

$$\frac{R(s, t)x - Cx}{t} = \frac{B(K(s, t) - C)x + Cx - Cx}{t} = B \frac{K(s, t)x - Cx}{t} \quad (3.25)$$

which tends to $BA(s)x$, as $t \rightarrow 0$. This completes the proof. $\square$

Example 3.5. Let $r > 0$. Consider the following initial value problem:

$$\begin{aligned} \frac{\partial}{\partial t} x(t, \varepsilon) &= 2(r + t) \frac{\partial}{\partial \varepsilon} x(t, \varepsilon) + \varepsilon x(t, \varepsilon), \\ x(0, \varepsilon) &= e^{-4\varepsilon^2} x_0(\varepsilon), \quad \varepsilon, t \geq 0. \end{aligned} \quad (3.26)$$

Let $X = BUC(\mathbb{R})$, with the supremum-norm. Define $C \in B(X)$ by $Cx(\varepsilon) = e^{-\varepsilon^2} x(\varepsilon)$, $x(\cdot) \in X$. Also define $B : D(B) \rightarrow X$ by $Bx(\varepsilon) = \varepsilon x(\varepsilon)$, where $D(B) = \{x \in X : Bx \in X\}$. It is well known that B is the infinitesimal generator of C -regularized semigroup $T(t)$, defined by $T(t)x(\varepsilon) = e^{-\varepsilon^2 + \varepsilon t} x(\varepsilon)$. Now with $D = \{x \in X : \dot{x} \in X\}$, if $A(s) : D \rightarrow X$ is defined by $A(s)x = 2s\dot{x}$, then by Example 2.3, $\{A(s)\}_{s \geq 0}$ is the infinitesimal generator of the regularized

C^2 -quasi-semigroup $K(s, t)x(\varepsilon) = e^{-\varepsilon^2}x(t^2 + 2st + \varepsilon)$. Using Theorem 3.3 and the fact that $T(t)K(s, r) = K(s, r)T(t)$, $s, t, r \geq 0$, we obtain that $\{A(s) + B\}$ is the infinitesimal generator of regularized C^2 -quasi-semigroup $R(s, t) = T(t)K(s, t)$. Also using these operators, (3.26) can be written as

$$\begin{aligned}\dot{x}(t) &= (A(r + t) + B)x(t), \\ x(0) &= C^4x_0.\end{aligned}\tag{3.27}$$

Thus by Theorem 3.1 for any $x_0 \in D \cap D(B)$, (3.26) has the unique solution

$$x(t, \varepsilon) = R(r, t)C^2x_0(\varepsilon) = e^{-4\varepsilon^2 + \varepsilon t}x_0(t^2 + 2rt + \varepsilon).\tag{3.28}$$

Example 3.6. For a given sequence $(p_n)_{n \in \mathbb{N}}$ of complex numbers with nonzero elements and $(y_n)_{n \in \mathbb{N}}$, consider the following equation:

$$\begin{aligned}\frac{d}{dt}x_n(t) &= e^{in(t+1)}x_n(t) + p_nx_n(t), \\ x_n(0) &= p_n^2y_n, \quad n \in \mathbb{N}.\end{aligned}\tag{3.29}$$

Let X be the space c_0 , the set of all complex sequence with zero limit at infinity. For a bounded sequence $p = (p_n)_{n \in \mathbb{N}}$, define $A : D(A) \rightarrow X$ and M_p on X by

$$A(x_n)_{n \in \mathbb{N}} = (e^{in}x_n)_{n \in \mathbb{N}}, \quad M_p(x_n)_{n \in \mathbb{N}} = (p_nx_n).\tag{3.30}$$

One can easily see that $D(A) = \{(x_n)_{n \in \mathbb{N}} \in c_0 : (e^{in}x_n)_{n \in \mathbb{N}} \in c_0\}$ and M_p is a bounded linear operator which is injective. It is well known that A is the infinitesimal generator of strongly continuous semigroup

$$T(t)(x_n)_{n \in \mathbb{N}} = (e^{in(1+t)}x_n)_{n \in \mathbb{N}}.\tag{3.31}$$

Thus by Example 2.4, $\{A(t)\}_{t \geq 0}$, defined by

$$A(t)(x_n)_{n \in \mathbb{N}} := AT(t)(x_n)_{n \in \mathbb{N}} = (e^{in(1+t)}x_n)_{n \in \mathbb{N}},\tag{3.32}$$

is the infinitesimal generator of the M_p -quasi-semigroup

$$K(s, t) = M_p(e^{T(s+t)-T(s)}).\tag{3.33}$$

Using these operators, one can rewrite (3.29) as

$$\begin{aligned}\dot{x}(t) &= (A(t) + M_p)x(t), \\ x(0) &= M_p^2 y_0,\end{aligned}\tag{3.34}$$

where $x_0 = (y_n)_{n \in \mathbb{N}}$. Trivially $T(t)$ commutes with $K(r, s)$, for any $r, s, t \geq 0$. Now using Theorem 3.3 we obtain that $\{A(t) + M_p\}_{t \geq 0}$ is the infinitesimal generator of M_p -quasi-semigroup

$$R(s, t) = T(t)K(s, t).\tag{3.35}$$

Also from Theorem 3.1, with $r = 0$, for any $y \in D(A)$, (3.34) has a unique solution

$$x(t) = R(0, t)M_p y = T(t)K(0, t)M_p^2 x_0.\tag{3.36}$$

But from definition of $K(s, t)$, for a given $(x_n)_{n \in \mathbb{N}} \in c_0$,

$$K(0, t)(x_n)_{n \in \mathbb{N}} = e^{T(t)-I} = e^{-1} \sum_{k=0}^{\infty} \frac{T^k(t)}{k!} (x_n)_{n \in \mathbb{N}} = e^{-1} \sum_{k=0}^{\infty} \left(\frac{e^{iknt} x_n}{k!} \right)_{n \in \mathbb{N}} = e^{-1} \left(\sum_{k=0}^{\infty} \frac{e^{iknt} x_n}{k!} \right)_{n \in \mathbb{N}}.\tag{3.37}$$

So the solution of (3.34) is

$$x(t) = R(0, t)M_p y = \left(\sum_{k=0}^{\infty} \frac{e^{ikt(n+1)-1} p_n^4 y_n}{k!} \right)_{n \in \mathbb{N}},\tag{3.38}$$

or equivalently the solution of (3.29) is

$$x_n(t) = \sum_{k=0}^{\infty} \frac{e^{ikt(n+1)-1} p_n^4 y_n}{k!}, \quad n \in \mathbb{N}.\tag{3.39}$$

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