Research Article

Necessary and Sufficient Conditions for Schur Geometrical Convexity of the Four-Parameter Homogeneous Means

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The necessary and sufficient conditions for Schur geometrical convexity of the four-parameter means are given. This gives a unified treatment for Schur geometrical convexity of Stolarsky and Gini means.

1. Introduction and Main Result

Let $p, q \in \mathbb{R}$ and $a, b > 0$. For $a \neq b$ the Stolarsky means are defined as

\[
S_{p,q}(a,b) = \begin{cases} 
\left(\frac{q}{p} \frac{a^p - b^p}{a^q - b^q}\right)^{1/(p-q)}, & pq(p-q) \neq 0, \\
L^{1/p}(a^p, b^p), & p \neq 0, q = 0, \\
L^{1/q}(a^q, b^q), & q \neq 0, p = 0, \\
I^{1/p}(a^p, b^p), & p = q \neq 0, \\
\sqrt{ab}, & p = q = 0,
\end{cases}
\]

(1.1)
and $S_{p,q}(a,a) = a$ (see [1]), where

$$L(x, y) = \begin{cases} \frac{x - y}{\ln x - \ln y}, & x \neq y, \\ x, & x = y, \end{cases} \tag{1.2}$$

$$I(x, y) = \begin{cases} \left( \frac{x^r}{y^s} \right)^{1/(x-y)}, & x \neq y, \\ x, & x = y. \end{cases} \tag{1.3}$$

are the logarithmic mean and identric (exponential) mean of positive numbers $x$ and $y$, respectively.

Another two-parameter family of means was introduced by Gini in [2]. That are defined as

$$G_{p,q}(a,b) = \begin{cases} \left( \frac{a^p + b^q}{a^p + b^q} \right)^{1/(p-q)}, & p \neq q, \\ \exp \left( \frac{a^p \ln a + b^q \ln b}{a^p + b^q} \right), & p = q. \end{cases} \tag{1.4}$$

Stolarsky and Gini means both are contained in the so-called four-parameter means [3], which are defined as follows.

**Definition 1.1.** Let $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $a \neq b$ and $(p, q), (r, s) \in \mathbb{R} \times \mathbb{R}$. Then the four-parameter homogeneous means denoted by $F(p, q; r, s; a, b)$ are defined as follows:

$$F(p, q; r, s; a, b) = \left( \frac{L(a^{pr}, b^{qs})}{I(a^{pr}, b^{qs})} \right)^{1/(p-q)(r-s)} \quad \text{if}\ pqrs(p-q)(r-s) \neq 0, \tag{1.5}$$

or

$$F(p, q; r, s; a, b) = \left( \frac{a^{pr} - b^{qs}}{a^{pr} - b^{qs}} \right)^{1/(p-q)(r-s)} \quad \text{if}\ pqrs(p-q)(r-s) \neq 0. \tag{1.6}$$

If $pqrs(p-q)(r-s) = 0$, then $F(p, q; r, s; a, b)$ are defined as their corresponding limits, for example:

$$F(p, p; r, s; a, b) = \lim_{q \to p} F(p, q; r, s; a, b) = \left( \frac{L(a^{pr}, b^{qs})}{I(a^{pr}, b^{qs})} \right)^{1/(p-r-s)}, \quad \text{if}\ prs(r-s) \neq 0, \quad p = q,$$

$$F(p, 0; r, s; a, b) = \lim_{q \to 0} F(p, q; r, s; a, b) = \left( \frac{L(a^{pr}, b^{qs})}{I(a^{pr}, b^{qs})} \right)^{1/(p-r-s)}, \quad \text{if}\ prs(r-s) \neq 0, \quad q = 0,$$

$$F(0, 0; r, s; a, b) = \lim_{p \to 0} F(p, 0; r, s; a, b) = G(a, b), \quad \text{if}\ rs(r-s) \neq 0, \quad p = q = 0, \tag{1.7}$$
where \( L(x, y), I(x, y) \) denote logarithmic mean and identric (exponential) mean, respectively, \( G(a, b) = \sqrt{ab} \).

The Schur convexity of \( S_{p,q}(a, b) \) and \( G_{p,q}(a, b) \) on \((0, \infty) \times (0, \infty)\) with respect to \((a, b)\) was investigated by Qi et al. [4], Shi et al. [5], Li and Shi [6], and Chu and Zhang [7]. Until now, they have been perfectly solved by Chu and Zhang [7], Wang and Zhang [8], respectively. Recently, Chu and Xia also proved the same result as Wang and Zhang [9].

The Schur convexity of \( S_{p,q}(a, b) \) and \( G_{p,q}(a, b) \) on \([0, \infty) \times [0, \infty)\) and \((-\infty, 0] \times (-\infty, 0]\) with respect to \((p, q)\) was investigated by Qi [10] and Sándor [11], respectively. Now Schur convexity of a four-parameter homogeneous means family containing Stolarsky and Gini means on \((-\infty, \infty) \times (-\infty, \infty)\) with respect to \((p, q)\) has been perfectly solved by Yang [12].

The Schur geometrical convexity was introduced by Zhang [13]. In [8, 14], Wand and Zhang proved that \( G_{p,q}(a, b) \) is Schur geometrically convex (Schur geometrically concave) on \((0, \infty) \times (0, \infty)\) with respect to \((a, b)\) if \( p + q \geq (\leq)0 \). Chu et al. [15] pointed out that this conclusion is also true for \( S_{p,q}(a, b) \). Shi et al. [5, 16], Li and Shi [6], and Gu and Shi [17] also obtained similar results.

The purpose of this paper is to present the necessary and sufficient conditions for Schur geometrical convexity of the four-parameter homogeneous means. This gives a unified treatment for Schur geometrical convexity of Stolarsky and Gini means with respect to \((a, b)\).

Our main result is as follows.

**Theorem 1.2.** For fixed \((p, q), (r, s) \in \mathbb{R} \times \mathbb{R}\) the four-parameter homogeneous means \( F(p, q; r, s; a, b) \) are Schur geometrically convex (Schur geometrically concave) on \((0, \infty) \times (0, \infty)\) with respect to \((a, b)\) if and only if \((p + q)(r + s) > (<)0\).

**2. Definitions and Lemmas**

**Definition 2.1** (see [18, 19]). Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \) \( (n \geq 2) \).

(i) \( x \) is said to be majorized by \( y \) (in symbol \( x < y \)) if

\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i] \quad \text{for } 1 \leq k \leq n - 1, \quad \sum_{i=1}^{n} x[i] = \sum_{i=1}^{n} y[i],
\]

(2.1)

where \( x[1] \geq x[2] \cdots \geq x[n] \) and \( y[1] \geq y[2] \cdots \geq y[n] \) are rearrangements of \( x \) and \( y \) in a decreasing order.

(ii) \( x \geq y \) means \( x_i \geq y_i \) for all \( i = 1, 2, \ldots, n \). Let \( \Omega \subset \mathbb{R}^n \) \( (n \geq 2) \). The function \( \phi : \Omega \to \mathbb{R} \) is said to be increasing if \( x \geq y \) implies \( \phi(x) \geq \phi(y) \). \( \phi \) is said to be decreasing if and only if \( -\phi \) is increasing.

(iii) \( \Omega \subset \mathbb{R}^n \) is called a convex set if \( (ax_1 + \beta y_1, \ldots, ax_n + \beta y_n) \in \Omega \) for all \( x \) and \( y \), where \( a, \beta \in [0, 1] \) with \( a + \beta = 1 \).

(iv) Let \( \Omega \subset \mathbb{R}^n \) \( (n \geq 2) \) be a set with nonempty interior. Then \( \phi : \Omega \to \mathbb{R} \) is said to be Schur convex if \( x < y \) on \( \Omega \) implies \( \phi(x) \leq \phi(y) \). \( \phi \) is said to be Schur concave if \( -\phi \) is Schur convex.
Definition 2.2 (see [13, 20]). Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+ (n \geq 2)$. Denote
\[
\ln x = (\ln x_1, \ln x_2, \ldots, \ln x_n), \quad \ln y = (\ln y_1, \ln y_2, \ldots, \ln y_n). \quad (2.2)
\]

(i) $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $x$ and $y$, where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

(ii) Let $\Omega \subset \mathbb{R}^n_+ (n \geq 2)$ be a set with nonempty interior. Then function $\phi : \Omega \to \mathbb{R}$ is said to be Schur geometrically convex on $\Omega$ if $\ln x < \ln y$ on $\Omega$ implies $\phi(x) \leq \phi(y)$. $\phi$ is said to be Schur geometrically concave if $-\phi$ is Schur geometrically convex.

Definition 2.3 (see [18]). (i) $\Omega \subset \mathbb{R}^n (n \geq 2)$ is called symmetric set if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix $P$.

(ii) The function $\phi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix $P$, $\phi(Px) = \phi(x)$ for all $x \in \Omega$.

Lemma 2.4 (see [18, 19]). Let $\Omega \subset \mathbb{R}^n$ be a symmetric set with nonempty interior $\Omega^0$ and $\phi : \Omega \to \mathbb{R}$ be continuous on $\Omega$ and differentiable in $\Omega^0$. Then $\phi$ is Schur convex (Schur concave) on $\Omega$ if and only if $\phi$ is symmetric on $\Omega$ and
\[
(x_1 - x_2) \left( \frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0 \quad (2.3)
\]
holds for any $x = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

Lemma 2.5 (see [13, Theorem 1.4, page 108]). Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric set with nonempty interior geometrically convex set $\Omega^0$. Let $\phi : \Omega \to \mathbb{R}_+$ be continuous on $\Omega$ and differentiable in $\Omega^0$. Then $\phi$ is Schur geometrically convex (Schur geometrically concave) on $\Omega$ if and only if $\phi$ is symmetric on $\Omega$ and
\[
(\ln x_1 - \ln x_2) \left( x_1 \frac{\partial \phi}{\partial x_1} - x_2 \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0 \quad (2.4)
\]
holds for any $x = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

3. **Schur Geometrical Convexity of Two-Parameter Homogeneous Functions**

The more general form of two-parameter homogeneous means is the so-called two-parameter homogeneous functions first introduced by Yang [21]. For conveniences, we record it as follows.

Definition 3.1. Assume that $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{0\}$ is $n$-order homogeneous, continuous and exists first partial derivatives and $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+, (p, q) \in \mathbb{R} \times \mathbb{R}$. 
Abstract and Applied Analysis

If \( f(x, y) > 0 \) for \( (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+ \} \) and \( f(x, x) = 0 \) for all \( x \in \mathbb{R}_+ \), then define

\[
\mathcal{K}_f(p, q; a, b) = \left( \frac{f(a^p, b^q)}{f(1, 1)} \right)^{1/(p-q)} \quad \text{if } p \neq q, \ p q \neq 0,
\]

\[
\mathcal{K}_f(p, p; a, b) = \lim_{q \to p} \mathcal{K}_f(p, q; a, b) = G_{f,p}(a, b) \quad \text{if } p = q \neq 0,
\]

where

\[
G_{f,p}(a, b) = G_f^{1/p}(a^p, b^p), \quad G_f(x, y) = \exp \left( \frac{x f_x(x, y) \ln x + y f_y(x, y) \ln y}{f(x, y)} \right),
\]

\( f_x(x, y) \) and \( f_y(x, y) \) denote first-order partial derivatives with respect to first and second component of \( f(x, y) \), respectively.

If \( f(x, y) > 0 \) for all \( (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \), then define further

\[
\mathcal{K}_f(p, 0; a, b) = \left( \frac{f(a^p, b^0)}{f(1, 1)} \right)^{1/p} \quad \text{if } p \neq 0, \ q = 0;
\]

\[
\mathcal{K}_f(0, q; a, b) = \left( \frac{f(a^0, b^q)}{f(1, 1)} \right)^{1/q} \quad \text{if } p = 0, \ q \neq 0;
\]

\[
\mathcal{K}_f(0, 0; a, b) = a^{f_x(1,1)/f(1,1)} b^{f_y(1,1)/f(1,1)} \quad \text{if } p = q = 0.
\]

Since \( f(x, y) \) is a homogeneous function, \( \mathcal{K}_f(p, q; a, b) \) is also one and called a homogeneous function with parameters \( p \) and \( q \) and simply denoted by \( \mathcal{K}_f(p, q) \) or \( \mathcal{K}_f \) sometimes.

Concerning the monotonicity and log-convexity of two-parameter homogeneous functions, there have been some literatures such as [3, 21, 22], which yield some new and interesting inequalities for means.

The two-parameter homogeneous functions \( \mathcal{K}_f(p, q; a, b) \) have some well properties (see [21–23]) such as the following lemma.

**Lemma 3.2 (see [23]).** Let \( f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) be a homogenous and differentiable function and

\[
T(t) = T(t; a, b) := \ln f(a^t, b^t), \quad (t; a, b) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+.
\]

Then we have

\[
\frac{\partial T(t; a, b)}{\partial t} = \frac{a^t f_x(a^t, b^t) \ln a + b^t f_y(a^t, b^t) \ln b}{f(a^t, b^t)},
\]

\[
\ln \mathcal{K}_f(p, q; a, b) = \int_0^1 \frac{\partial T(tp + (1-t)q; a, b)}{\partial t} \, dt.
\]

Next we give another property.
Lemma 3.3. Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a homogenous and $m$-time differentiable function. Then $\mathcal{H}_f(p, q; a, b) \in C^{m-1}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+)$. 

Proof. Since $f(x, y)$ has continuous partial derivatives of $m$ order with respect to $x, y$ on $\mathbb{R}_+ \times \mathbb{R}_+$, the integrand in (3.6) has continuous partial derivatives of $m - 1$ order with respect to $p, q, a, b$ on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$, that is $\mathcal{H}_f(p, q; a, b) \in C^{m-1}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+)$. \hfill \Box

For the Schur geometrical convexity, we have the following result.

Theorem 3.4. Assume that $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a symmetric, $n$-order homogeneous, continuous, and three-time differentiable function. If for any $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $x \neq y$

\[
\mathcal{N}(x, y) = (x - y) \left( x(\ln f)_x - y(\ln f)_y - 2xy\mathcal{C} \ln \left( \frac{x}{y} \right) \right) > (<>0), \quad \text{where } \mathcal{C} = (\ln f)_{xy},
\]

(3.7)

then $\mathcal{H}_f(p, q; a, b)$ is Schur geometrically convex on $(0, \infty) \times (0, \infty)$ with respect to $(a, b)$ if and only if $p + q > (<>0)$ and Schur geometrically concave if and only if $p + q < (>)0$.

Proof. (1) In the case of $p \neq q$. We have

\[
\ln \mathcal{H}_f(p, q; a, b) = \frac{\ln f(a^p, b^p) - \ln f(a^q, b^q)}{p - q}.
\]

(3.8)

Some simple partial derivative computations yield

\[
\frac{\partial \ln \mathcal{H}_f}{\partial a} = \frac{1}{\mathcal{H}_f} \frac{\partial \mathcal{H}_f}{\partial a} = \frac{1}{p - q} \left( \frac{pa^{p-1}f_x(a^p, b^p)}{f(a^p, b^p)} - \frac{qa^{q-1}f_x(a^q, b^q)}{f(a^q, b^q)} \right),
\]

\[
\frac{\partial \ln \mathcal{H}_f}{\partial b} = \frac{1}{\mathcal{H}_f} \frac{\partial \mathcal{H}_f}{\partial b} = \frac{1}{p - q} \left( \frac{pb^{p-1}f_y(a^p, b^p)}{f(a^p, b^p)} - \frac{qb^{q-1}f_y(a^q, b^q)}{f(a^q, b^q)} \right),
\]

(3.9)

hence,

\[
\frac{1}{\mathcal{H}_f} \left( a \frac{\partial \mathcal{H}_f}{\partial a} - b \frac{\partial \mathcal{H}_f}{\partial b} \right) = \frac{g(p) - g(q)}{p - q},
\]

(3.10)

where

\[
g(t) = \frac{ta^pf_x(a^p, b^p)}{f(a^p, b^p)} - \frac{tb^pf_y(a^p, b^p)}{f(a^p, b^p)}.
\]

(3.11)

It is easy to verify that $g(t)$ is even on $(-\infty, \infty)$. In fact, since $f(x, y)$ is $n$-order homogeneous and symmetric, for arbitrary $\lambda > 0$, we have

\[
f(\lambda x, \lambda y) = \lambda^n f(x, y), \quad f_x(\lambda x, \lambda y) = \lambda^{n-1} f_x(x, y), \quad f_y(\lambda x, \lambda y) = \lambda^{n-1} f_y(x, y),
\]

\[
f(x, y) = f(y, x), \quad f_x(x, y) = f_y(y, x), \quad f_y(x, y) = f_x(y, x).
\]

(3.12)
Thus,

\[
g(-t) = \frac{-ta^{-t}f_x(a^{-t}, b^{-t})}{f(a^{-t}, b^{-t})} - \frac{-tb^{-t}f_y(a^{-t}, b^{-t})}{f(a^{-t}, b^{-t})} \\
= \frac{-ta^{-t}(a'b')^{-(n-1)}f_x(b', a')}{(a'b')^{-n}f(b', a')} - \frac{-tb^{-(n-1)}f_y(b', a')}{(a'b')^{-n}f(b', a')} \\
= -\frac{tb^{t}f_y(a', b')}{f(a', b')} + \frac{ta^{t}f_x(a', b')}{f(a', b')} = g(t). \tag{3.13}
\]

Let \(a^t = x, b^t = y\). Then

\[
g'(t) = x(\ln f)_x + t \left( \left( \frac{x f_x(x, y)}{f(x, y)} \right)_x \frac{dx}{dt} + \left( \frac{x f_x(x, y)}{f(x, y)} \right)_y \frac{dy}{dt} \right) \\
- y(\ln f)_y - t \left( \left( \frac{y f_y(x, y)}{f(x, y)} \right)_x \frac{dx}{dt} + \left( \frac{y f_y(x, y)}{f(x, y)} \right)_y \frac{dy}{dt} \right) \\
= x(\ln f)_x + t \left( x \left( \frac{x f_x(x, y)}{f(x, y)} \right)_x \ln a + y \left( \frac{x f_x(x, y)}{f(x, y)} \right)_y \ln b \right) \\
- y(\ln f)_y - t \left( x \left( \frac{y f_y(x, y)}{f(x, y)} \right)_x \ln a + y \left( \frac{y f_y(x, y)}{f(x, y)} \right)_y \ln b \right). \tag{3.14}
\]

Note \(xf_x(x, y)/f(x, y)\) and \(yf_y(x, y)/f(x, y)\) both are 0-order homogeneous with respect to \(x\) and \(y\), then

\[
x \left( \frac{x f_x(x, y)}{f(x, y)} \right)_x + y \left( \frac{x f_x(x, y)}{f(x, y)} \right)_y = 0, \\
x \left( \frac{y f_y(x, y)}{f(x, y)} \right)_x + y \left( \frac{y f_y(x, y)}{f(x, y)} \right)_y = 0. \tag{3.15}
\]

and then

\[
x \left( \frac{x f_x(x, y)}{f(x, y)} \right)_x = -y \left( \frac{x f_x(x, y)}{f(x, y)} \right)_y = -xy\mathcal{O}, \\
y \left( \frac{y f_y(x, y)}{f(x, y)} \right)_y = -x \left( \frac{y f_y(x, y)}{f(x, y)} \right)_x = -xy\mathcal{O}. \tag{3.16}
\]
Therefore,

\[
g'(t) = x (\ln f)_x + t x y \partial (\ln b - \ln a) - y (\ln f)_y - t x y \partial (\ln a - \ln b) \\
= x (\ln f)_x - y (\ln f)_y - 2 t x y \partial (\ln a - \ln b) \\
= x (\ln f)_x - y (\ln f)_y - 2 t x y \ln \left( \frac{x}{y} \right) = \frac{\mathcal{N}(x, y)}{x - y} \quad \text{for } x \neq y. \tag{3.17}
\]

By the mean values theorem, there is a \( \xi \) between \( |p| \) and \( |q| \) such that

\[
\frac{g(p) - g(q)}{p - q} = \frac{g(|p|) - g(|q|)}{|p| - |q|} = \frac{|p| - |q|}{p - q} g'(\xi) = \frac{p + q}{|p| + |q|} g'(\xi) = \frac{p + q}{|p| + |q|} \frac{\mathcal{N}(x, y)}{x - y}, \quad \text{for } x \neq y,
\]

where \( x = a^\xi, \ y = b^\xi \). Thus we have

\[
(\ln a - \ln b) \left( a \frac{\partial \mathcal{H}_f}{\partial a} - b \frac{\partial \mathcal{H}_f}{\partial b} \right) = \mathcal{H}_f \left( \frac{p + q}{|p| + |q|} \ln \left( \frac{a}{b} \right) \frac{\mathcal{N}(x, y)}{x - y} \right)
\]

\[
= \mathcal{H}_f \left( \frac{p + q}{|p| + |q|} \frac{\mathcal{N}(x, y)}{x - y} \right) \left( \frac{x}{y} \right) \ln \frac{x}{y}
\]

\[
= \begin{cases} 
> 0 & \text{if } p + q > (>)0, \\
< 0 & \text{if } p + q < (>)0.
\end{cases}
\]

By Lemma 2.5, our required result is derived immediately.

(2) In the case of \( p = q \neq 0 \). By Lemma 3.3 together with (3.10) and (3.17), we have

\[
\frac{1}{\mathcal{H}_f(p, p)} \left( a \frac{\partial \mathcal{H}_f(p, p)}{\partial a} - b \frac{\partial \mathcal{H}_f(p, p)}{\partial b} \right) = \lim_{q \to p} \frac{1}{\mathcal{H}_f(p, q)} \left( a \frac{\partial \mathcal{H}_f(p, q)}{\partial a} - b \frac{\partial \mathcal{H}_f(p, q)}{\partial b} \right)
\]

\[
= \lim_{q \to p} \frac{g(p) - g(q)}{p - q} = g'(p) = \frac{\mathcal{N}(x, y)}{x - y},
\]

(3.20)
where \( x = a^p, y = b^p \). Hence we have

\[
\begin{aligned}
\ln a - \ln b \left( a \frac{\partial H_f(p,p)}{\partial a} - b \frac{\partial H_f(p,p)}{\partial b} \right) &= H_f(p,p) \left( \ln a - \ln b \right) \frac{A(x,y)}{x-y} \\
&= p^{-1} H_f(p,p) A(x,y) \frac{\ln x - \ln y}{x-y} \\
&= \begin{cases} 
> 0 & \text{if } p > (>)0 , \\
< 0 & \text{if } p < (>)0 .
\end{cases}
\end{aligned}
\]

By Lemma 2.5, the required result holds.

(3) In the case of \( p = q = 0 \). By Lemma 3.3 and (3.20), we have

\[
\begin{aligned}
\frac{1}{H_f(0,0)} \left( a \frac{\partial H_f(0,0)}{\partial a} - b \frac{\partial H_f(0,0)}{\partial b} \right) &= \lim_{p \to 0} \left( a \frac{\partial H_f(p,p)}{\partial a} - b \frac{\partial H_f(p,p)}{\partial b} \right) \\
&= \lim_{p \to 0} g'(p) .
\end{aligned}
\]

However,

\[
g'(0) = \left. \left( x \ln f \right)_x - \left. y \ln f \right)_y - 2xy \mathcal{O} \ln \left( \frac{x}{y} \right) \right|_{x=1,y=1} \\
= 1 \cdot \frac{f_x(1,1)}{f(1,1)} - 1 \cdot \frac{f_y(1,1)}{f(1,1)} - 2 \cdot 1 \cdot \mathcal{O}(1,1) \cdot \ln \left( \frac{1}{1} \right) = 0 ,
\]

where \( f_x(1,1) = f_y(1,1) \) due to the symmetry of \( f(x, y) \). Thus

\[
\ln a - \ln b \left( a \frac{\partial H_f(p,p)}{\partial a} - b \frac{\partial H_f(p,p)}{\partial b} \right) = 0 .
\]

Summarizing the above three cases, this proof of Theorem 3.4 is complete. \( \Box \)
4. Proof of Main Result

Establishing the Theorem 3.4, we are in a position to prove main result.

Proof of Theorem 1.2. It follows from [3, Section 1], that $F(p, q; r, s; a, b) = \mathcal{K}(p, q; a, b)$, where $\mathcal{K}_L = \mathcal{K}_L(r, s) = \mathcal{K}_L(r, s; x, y) = S_{r,s}(x, y)$ is symmetric with respect to $x$ and $y$. From Lemma 3.3, it follows that $\mathcal{K}_L = \mathcal{K}_L(r, s; x, y) \in C^\infty(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$. Thus we have

$$
\begin{align*}
\lim_{s \to r} (\ln \mathcal{K}_L(r, s))_{x} &= (\ln \mathcal{K}_L(r, r))_{x}, \\
\lim_{s \to r} (\ln \mathcal{K}_L(r, s))_{y} &= (\ln \mathcal{K}_L(r, r))_{y}, \\
\lim_{s \to r} (\ln \mathcal{K}_L(r, s))_{xy} &= (\ln \mathcal{K}_L(r, r))_{xy}, \\
\lim_{s \to 0} (\ln \mathcal{K}_L(r, s))_{x} &= (\ln \mathcal{K}_L(r, 0))_{x}, \\
\lim_{s \to 0} (\ln \mathcal{K}_L(r, s))_{y} &= (\ln \mathcal{K}_L(r, 0))_{y}, \\
\lim_{s \to r} (\ln \mathcal{K}_L(r, s))_{xy} &= (\ln \mathcal{K}_L(r, r))_{xy}, \\
\lim_{r \to 0} (\ln \mathcal{K}_L(r, r))_{x} &= (\ln \mathcal{K}_L(0, 0))_{x}, \\
\lim_{r \to 0} (\ln \mathcal{K}_L(r, r))_{y} &= (\ln \mathcal{K}_L(0, 0))_{y}, \\
\lim_{r \to 0} (\ln \mathcal{K}_L(r, r))_{xy} &= (\ln \mathcal{K}_L(0, 0))_{xy}.
\end{align*}
$$

(1) In the case of $rs(r - s) \neq 0$.

Simple partial derivative calculations yield

$$
\begin{align*}
\ln \mathcal{K}_L &= \frac{1}{r - s} \left( |s| + \ln |x^r - y^r| - \ln |r| - \ln |x^s - y^s| \right), \\
(\ln \mathcal{K}_L)_x &= \frac{1}{r - s} \left( \frac{rx^{r-1}}{x^r - y^r} - \frac{sxs^{-1}}{x^s - y^s} \right), \\
(\ln \mathcal{K}_L)_y &= \frac{1}{r - s} \left( \frac{-ry^{r-1}}{x^r - y^r} + \frac{ysy^{-1}}{x^s - y^s} \right), \\
(\ln \mathcal{K}_L)_{xy} &= \frac{1}{xy(r - s)} \left( \frac{r^2x^{r'}y'}{(x^r - y^r)^2} - \frac{s^2x^{s'}y'}{(x^s - y^s)^2} \right).
\end{align*}
$$
Hence,

\[
\mathcal{N}(x, y) = (x - y) \left( x(\ln x)_x - y(\ln y)_y - 2xy \ln \left( \frac{x}{y} \right) \right)
\]

\[
= \frac{x - y}{r - s} \left( \frac{r(x^r + y^r)}{x^r - y^r} - \frac{2r^2x^r y^r \ln(x/y)}{(x^r - y^r)^2} \right)
\]

\[
- \frac{x - y}{r - s} \left( \frac{s(x^s + y^s)}{x^s - y^s} - \frac{2s^2x^s y^s \ln(x/y)}{(x^s - y^s)^2} \right)
\]

\[
= (x - y) \frac{P(r) - P(s)}{r - s},
\]

where

\[
P(t) = t \left( \frac{x^t + y^t}{x^t - y^t} - \frac{2x^t y^t \ln(x^t / y^t)}{(x^t - y^t)^2} \right).
\]

It is easy to check that \(P(t)\) is even and increasing (decreasing) on \((0, \infty)\) if \(x > (<) y\). Indeed,

\[
P(-t) = -t \left( \frac{x^{-t} + y^{-t}}{x^{-t} - y^{-t}} - \frac{2x^{-t} y^{-t} \ln(x^{-t} / y^{-t})}{(x^{-t} - y^{-t})^2} \right) = P(t).
\]

With \((x/y)^t = u\), then \(t = \ln u / \ln(x/y)\), and then \(P(t)\) can be written as

\[
P(t) = \frac{1}{\ln(x/y)} \left( \frac{u + 1}{u - 1} \ln u - \frac{2u \ln^2 u}{(u - 1)^2} \right).
\]

Direct computation yields

\[
P'(t) = \frac{1}{\ln(x/y)} \left( \frac{u + 1}{u - 1} \ln u - \frac{2u \ln^2 u}{(u - 1)^2} \right) \frac{du}{dt}
\]

\[
= u \left( \frac{(u + 1)(u - 1)/u - \ln u}{(u - 1)^2} + \frac{\ln u}{u - 1} - \frac{2u \ln^2 u}{(u - 1)^2} - 4u \frac{\ln u (u - 1)/u - \ln u}{(u - 1)^2} \right)
\]

\[
\frac{(u - 1)/\ln u = L (u + 1)L^2 - 6uL + 2u(u + 1)}{(u - 1)L^2}
\]

\[
= 2L((u + 1)/2)L - u) + 4u((u + 1)/2 - L) / (u - 1)L^2.
\]
From

\[ \frac{u^2 + 1}{2} L - u = \frac{u^2 - 1}{\ln u^2} - \sqrt{u^2} > 0, \]
\[
L - \frac{u^2 + 1}{2} < 0,
\]

it follows that \( P'(t) > 0 \) if \( u - 1 > 0 \), that is, \( x > y \) and \( P'(t) < 0 \) if \( x < y \). Namely,

\[ (x - y)P'(t) > 0 \quad \text{for} \ t > 0 \ \text{with} \ x \neq y. \]  \( (4.17) \)

By the mean values theorem, there is a \( \eta \) between \( |r| \) and \( |s| \) such that

\[ P(|r|) - P(|s|) = (|r| - |s|)P'(\eta), \]  \( (4.18) \)

and then

\[
\mathcal{N}(x, y) = (x - y) \frac{P(r) - P(s)}{r - s} = (x - y) \frac{r + s}{|r| + |s|} \frac{P(|r|) - P(|s|)}{|r| - |s|}
\]
\[
= \frac{r + s}{|r| + |s|} \cdot (x - y)P'(|\eta|)\quad \text{(4.19)}
\]

Using Theorem 3.4, for fixed \((p, q), (r, s) \in \mathbb{R} \times \mathbb{R} \) with \( rs(r - s) \neq 0 \), the four-parameter homogeneous means \( F(p, q; r, s; a, b) \) are Schur geometrically convex on \((0, \infty) \times (0, \infty)\) with respect to \((a, b)\) if and only if \( (p + q)(r + s) > 0 \) and Schur geometrically concave if and only if \( (p + q)(r + s) < 0 \).

(2) In the case of \( s = 0, r \neq 0 \).

From (4.11) together with (4.4)–(4.6) and (4.19), there is a \( \eta_1 \) between \( 0 \) and \( |r| \) such that

\[
\mathcal{N}(x, y) = (x - y) \left( x \ln \mathcal{H}_L(r, 0)_x - y \ln \mathcal{H}_L(r, 0)_y - 2xy \ln \mathcal{H}_L(r, 0)^{x_y} \ln \left( \frac{x}{y} \right) \right)
\]
\[
= \lim_{s \to 0} \left( (x - y) \left( x \ln \mathcal{H}_L(r, s)_x - y \ln \mathcal{H}_L(r, s)_y - 2xy \ln \mathcal{H}_L(r, s)^{x_y} \ln \left( \frac{x}{y} \right) \right) \right)
\]
\[
= \lim_{s \to 0} \frac{P(r) - P(s)}{r - s} = \lim_{s \to 0} \frac{r + s}{|r| + |s|} \cdot \lim_{s \to 0} (x - y)P'(|\eta_1|)
\]
\[
= \begin{cases} 
> 0 & \text{if } r > 0, \\
< 0 & \text{if } r < 0,
\end{cases} \quad \text{(by (4.17))}.
\]  \( (4.20) \)
Abstract and Applied Analysis

(3) In the case of \( r = 0, s \neq 0 \).

Since \( \mathcal{H}_L(r, s; x, y) \) is symmetric with respect to \( r \) and \( s \), it follows from case 2 that

\[
\mathcal{N}(x, y) = (x - y) \left( x \left( \ln \mathcal{H}_L(0, s) \right)_x - y \left( \ln \mathcal{H}_L(0, s) \right)_y - 2xy \left( \ln \mathcal{H}_L(r, s) \right)_{xy} \ln \left( \frac{x}{y} \right) \right)
\]

\[= \begin{cases} 
> 0 & \text{if } s > 0, \\
< 0 & \text{if } s < 0. 
\end{cases} \quad (4.21)
\]

(4) In the case of \( r = s \neq 0 \).

From (4.11) together with (4.1)–(4.3), we have

\[
\mathcal{N}(x, y) = (x - y) \left( x \left( \ln \mathcal{H}_L(r, r) \right)_x - y \left( \ln \mathcal{H}_L(r, r) \right)_y - 2xy \left( \ln \mathcal{H}_L(r, s) \right)_{xy} \ln \left( \frac{x}{y} \right) \right)
\]

\[= \lim_{s \to r} \left( (x - y) \left( x \left( \ln \mathcal{H}_L(r, s) \right)_x - y \left( \ln \mathcal{H}_L(r, s) \right)_y - 2xy \left( \ln \mathcal{H}_L(r, s) \right)_{xy} \ln \left( \frac{x}{y} \right) \right) \right)
\]

\[= (x - y) \lim_{s \to r} \frac{P(r) - P(s)}{r - s} = (x - y) P'(r) \]

\[= \begin{cases} 
> 0 & \text{if } r > 0, \\
< 0 & \text{if } r < 0. \quad \text{(by (4.17))} 
\end{cases} \quad (4.22)
\]

(5) In the case of \( r = s = 0 \).

From (4.22) together with (4.7)–(4.9), we have

\[
\mathcal{N}(x, y) = (x - y) \left( x \left( \ln \mathcal{H}_L(0, 0) \right)_x - y \left( \ln \mathcal{H}_L(0, 0) \right)_y - 2xy \left( \ln \mathcal{H}_L(r, 0) \right)_{xy} \ln \left( \frac{x}{y} \right) \right)
\]

\[= \lim_{r \to 0} \left( (x - y) \left( x \left( \ln \mathcal{H}_L(r, 0) \right)_x - y \left( \ln \mathcal{H}_L(r, 0) \right)_y - 2xy \left( \ln \mathcal{H}_L(r, 0) \right)_{xy} \ln \left( \frac{x}{y} \right) \right) \right)
\]

\[= (x - y) \lim_{r \to 0} P'(r). \quad (4.23)
\]

But by (4.15) and some limit computations, we obtain

\[
\lim_{t \to 0} P'(t) (x/y)' = u \lim_{u \to 1} \left( (u + 1) \frac{(u - 1)/u - \ln u}{(u - 1)^2} + \frac{1}{u - 1} - \frac{2 \ln^2 u}{(u - 1)^2} - 4 \frac{\ln u}{u - 1} \frac{(u - 1)/u - \ln u}{(u - 1)^2} \right) = 0,
\]

which implies \( \mathcal{N}(x, y) = 0 \).
Summarizing the above five cases, our required results are derived. This proof ends.

5. Other Corollaries

The four-parameter homogeneous means \( F(p, q; r, s; a, b) \) also contain many other two-parameter means, for instance, for the identric (exponential) mean defined by (1.3), its two-parameter means are defined as follows [21, Example 2.3]:

\[
\mathcal{L}_I(p, q; a, b) = \begin{cases} 
\left( \frac{I(a^p, b^p)}{I(a^q, b^q)} \right)^{1/(p-q)}, & p \neq q, \ pq \neq 0, \\
G_{I,p}(a, b), & p = q \neq 0, \\
I^{1/p}(a^p, b^p), & p \neq 0, q = 0, \\
I^{1/q}(a^q, b^q), & p = 0, q \neq 0, \\
G(a, b), & p = q = 0,
\end{cases}
\] (5.1)

where \( G_{I,p}(a, b) = Y^{1/p}(a^p, b^p) := Y_p(a, b), \ Y(a, b) = Ie^{1-G^{1/2}}. \)

By [3], we see that

\[
\mathcal{L}_I(p, q; a, b) = F(p, q; 1, 1; a, b).
\] (5.2)

And then according to Theorem 1.2, we have the following corollary.

**Corollary 5.1.** For fixed \((p, q) \in \mathbb{R} \times \mathbb{R},\) the two-parameter identric (exponential) means \( \mathcal{L}_I(p, q; a, b) \) are Schur geometrically convex on \((0, \infty) \times (0, \infty)\) with respect to \((a, b)\) if and only if \(p + q > 0\) and Schur geometrically concave if and only if \(p + q < 0.\)

As another example, for Heronian mean defined by

\[
\text{He} = \frac{a + \sqrt{ab} + b}{3},
\] (5.3)

its two-parameter means are defined as follows:

\[
\mathcal{L}_{\text{He}}(p, q; a, b) = \begin{cases} 
\left( \frac{a^p + (\sqrt{ab})^p + b^p}{a^q + (\sqrt{ab})^q + b^q} \right)^{1/(p-q)}, & p \neq q, \ pq \neq 0, \\
\text{He}^{1/p}(a^p, b^p), & p \neq 0, q = 0, \\
\text{He}^{1/q}(a^q, b^q), & p = 0, q \neq 0, \\
G(a, b), & p = q = 0.
\end{cases}
\] (5.4)
By [3], we see that

$$A_{He}(p, q; a, b) = F(p, q; 3/2, 1/2; a, b). \quad (5.5)$$

And then according to Theorem 1.2, we have the following corollary.

**Corollary 5.2.** For fixed \((p, q) \in \mathbb{R} \times \mathbb{R}\), the two-parameter Heronian means \(A_{He}(p, q; a, b)\) are Schur geometrically convex on \((0, \infty) \times (0, \infty)\) with respect to \((a, b)\) if and only if \(p + q > 0\) and Schur geometrically concave if and only if \(p + q < 0\).

**References**


