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Research Article

Permanence and Extinction of a Generalized Gause-Type Predator-Prey System with Periodic Coefficients

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We study the permanence and extinction of a generalized Gause-type predator-prey system with periodic coefficients. We provide a sufficient and necessary condition to guarantee the predator and prey species to be permanent and a sufficient condition for the existence of a periodic solution. In addition we prove that when the predator population tends to extinction, the prey population keeps oscillating above a positive population level.

1. Introduction

Permanence of a dynamical system has always been a hot issue in the past few decades. The concept of permanence has been introduced and investigated by several authors, each using his own terminology: "cooperativity" in the earlier papers of Schuster et al. [1], and Hofbauer [2], "permanent coexistence" by Hutson and Vickers [3], "uniform persistence" in Butler et al. [4], and "ecological stability" by Svirezhev and Logofet [5–7] (for more detailed statements of the concept see [8]).

Many important results have been found in recent years [1–37]. Some authors (see [21, 28, 31, 33]) have considered the following two species periodic Lotka-Volterra predator-prey system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)(b_1(t) - a_{11}(t)x_1(t)) - a_{12}(t)x_1(t)x_2(t), \\ \dot{x}_2(t) &= x_2(t)(-b_2(t) - a_{22}(t)x_2(t)) + a_{21}(t)x_1(t)x_2(t), \end{aligned} \tag{1.1}$$

where $b_i(t)$ and $a_{ij}(t)$ $(i,j = 1,2)$ are periodic functions on R with common period $\omega > 0$ and $a_{ij}(t) \geq 0$ for all $t \in R$. They have established sufficient and necessary conditions for the existence of positive ω-periodic solutions of the system by using different methods, respectively. Teng [31] has given sufficient and necessary conditions for the uniform persistence of the system.

Cui [16] has considered the permanence of the following Lotka-Volterra predator-prey model with periodic coefficients:

$$\begin{aligned} \dot{x} &= x\left(a(t) - b(t)x - \frac{c(t)y}{p(t)+x}\right), \\ \dot{y} &= y\left(-d(t) + \frac{e(t)x}{p(t)+x} - f(t)y\right). \end{aligned} \tag{1.2}$$

He provided a sufficient and necessary condition to guarantee the predator and prey species to be permanent. In Theorems 2.2 and 2.3 he set a precondition that $f(t) \not\equiv 0$. This restricts the application of the theorems more or less, since many researchers often neglect the logistic term in the predator equation when the population level of the predator is relatively low and the competition between predators can be ignored, and it proved to be an unnecessary precondition in our paper. However, the research methods in his work inspired me, and many proofs, especially in the first half of this paper, are analogous to [16].

In this paper we consider the permanence of the following generalized Gause-type predator-prey system,

$$\begin{aligned} \dot{x} &= x\big(f(t,x) - g(t,x)y\big), \\ \dot{y} &= y\big(\gamma(t)g(t,x)x - \mu(t) - h(t)y\big), \end{aligned} \tag{1.3}$$

where $f(t+\omega,x) = f(t,x)$, $g(t+\omega,x) = g(t,x)$ for all t and $\gamma(\cdot)$, $\mu(\cdot)$, $h(\cdot)$ are all periodic continuous functions with common period $\omega > 0$; $\gamma(\cdot)$, $\mu(\cdot)$ are positive, and $h(\cdot)$ is nonnegative. We emphasize that our model includes the case when $h(t) \equiv 0$.

In the absence of predators, system (1.3) becomes

$$\dot{x} = xf(t,x), \tag{1.4}$$

where f is a real-valued function defined on

$$\mathbf{R}^2_{+0} = \left\{(t,x) \in \mathbf{R}^2 : t \geq 0, x \geq 0\right\}. \tag{1.5}$$

Vance and Coddington [35] have studied system (1.4) and proved the existence of a unique periodic solution under some assumptions. Apart from the assumption we mentioned above that $f(\cdot, x)$ is ω-periodic, the other assumptions with a mild modification are as follows.

(A1) Function f is continuous and differentiable with respect to x on $\mathbf{R}^2_{+0}$, and $\partial f/\partial x$ is continuous on $\mathbf{R}^2_{+0}$.

(A2) There are continuous functions p and λ with $p(x) > 0$ for $x > 0$ and $\lambda(t) \geq 0$ for $t \geq 0$, such that

$$\frac{\partial f}{\partial x} \leq -p(x)\lambda(t) \quad \text{for } t \geq 0,\ x \geq 0,$$
$$\int_0^\infty \lambda(s)ds = \infty. \tag{1.6}$$

(A3) There exist constants $\beta \geq 0$ and $K > 0$, such that

(i) $f(t, K) \leq \beta$ for $t \geq 0$,

(ii) $\int_t^{t+\omega} f(s, K)ds \leq 0$ for $t \geq 0$.

(A4) There exist constants $\alpha \geq 0$ and $0 < \delta \leq K$, such that

(i) $f(t, \delta) \geq -\alpha$ for $t \geq 0$,

(ii) $\int_t^{t+\omega} f(s, \delta)ds > 0$ for $t \geq 0$.

In addition, we assume that the ω-periodic function $g(\cdot, x)$ satisfies the following.

(A5) Function $g(t, x)$ is continuous with respect to t, and $G(t, x) := xg(t, x)$ is strictly monotonely increasing with respect to x,

(A6) $g(t, x)$ is nonnegative and is positive when $x > 0$,

(A7) $g(t, x)$ is bounded on $\mathbf{R}^2_{+0}$, that is there is a constant M_g such that $g(t, x) \leq M_g$ for $(t, x) \in \mathbf{R}^2_{+0}$.

Traditionally $G(t, x)$ represents a grazing rate. Usually when the amount of prey increases, the grazing rate increases and eventually tends to a maximal value as the prey population tends to infinity. But here we do not emphasize that $G(t, x)$ is bounded because its unnecessary theoretically. Assumption (A5) means that there is a higher capture rate when there is a larger amount of prey. $g(t, x)$ is directly proportional to the mean possibility density for each individual prey being captured, and (A6), (A7) suggest that there is a possibility, but its not definitely for each individual prey being captured. In ecology $G(t, x)$ is called the functional response or grazing function, for example, the Holling-type grazing function:

$$\frac{R_m x^n}{\alpha + x^n}, \quad n \geq 1, \tag{1.7}$$

where $R_m,\ \alpha > 0$ or the generalized form

$$\frac{R_m(t) x^n}{\alpha(t) + x^n}, \quad n \geq 1, \tag{1.8}$$

where $R_m(t)$, $\alpha(t) > 0$; the Ivlev grazing function

$$R_m\left(1 - e^{-\lambda x}\right), \tag{1.9}$$

where R_m, $\lambda > 0$. Obviously all of these functions satisfy (A5)–(A7).

In this paper we will establish sufficient and necessary conditions for the permanence of system (1.3). In the next section we state our main results. These results are proved in Section 3. Two applications are given in Section 4.

2. Main Results

Throughout this paper, we will assume that all the functions $f(\cdot, x)$, $g(\cdot, x)$, $\gamma(\cdot)$, $\mu(\cdot)$, and $h(\cdot)$ are continuous and periodic with common period $\omega > 0$. For any continuous ω-periodic function $u(t)$ defined on R, we denote

$$\overline{u(t)} = \frac{1}{\omega}\int_0^{\omega} u(t)dt, \qquad u^M = \max_{t\in[0,\omega]} u(t), \qquad u^L = \min_{t\in[0,\omega]} u(t). \tag{2.1}$$

In order to describe our main results, we first introduce a lemma.

Lemma 2.1 (see [35]). *Suppose that f satisfies (A1)–(A4). Then system* (1.4) *possesses a unique ω-periodic positive solution $x^*(t)$ which is globally asymptotically stable with respect to the positive x-axis.*

Theorem 2.2. *Suppose that f satisfies (A1)–(A4), g satisfies (A5)–(A7). Then system* (1.3) *is permanent provided that*

$$\overline{\gamma(t)g(t, x^*(t))x^*(t) - \mu(t)} > 0, \tag{2.2}$$

where $x^*(t)$ is the unique periodic solution of (1.4) *given by Lemma 2.1.*

Theorem 2.3. *Suppose that f satisfies (A1)–(A4), g satisfies (A5)–(A7), and*

$$\overline{\gamma(t)g(t, x^*(t))x^*(t) - \mu(t)} \le 0, \tag{2.3}$$

then

(i) $\lim_{t\to\infty} y(t) = 0$,

(ii) $\lim_{t\to\infty} \inf x(t) > 0$,

for any solution $(x(t), y(t))$ of system (1.3) *with positive initial conditions, where $x^*(t)$ is the unique periodic solution of* (1.4) *given by Lemma 2.1.*

By Theorems 2.2 and 2.3, we have the following corollary.

Corollary 2.4. *Suppose that f satisfies (A1)–(A4), g satisfies (A5)–(A7). Then system* (1.3) *is permanent if and only if* (2.2) *holds.*

Lemma 2.5 (see Theorem 15.5 in [30, 37]). *Consider a periodic system*

$$\dot{x} = F(t,x), \quad F(t+\omega,x) = F(t,x), \quad \omega > 0, \tag{2.4}$$

where $F(t,x) \in C(R \times R^2, R^2)$ satisfies a local Lipschitz condition. If all solutions of the above system exist in the future and one of them is bounded, then there exists a periodic solution of period ω.

By Theorem 2.2 and Lemma 2.5, we have the following corollary (see proof in Section 3).

Corollary 2.6. *Suppose f satisfies (A1)–(A4), g satisfies (A5)–(A7), and condition* (2.2) *holds. If in addition $g(t,x)$ satisfies a local Lipschitz condition with respect to x, then system* (1.3) *has a positive ω-periodic solution.*

3. Proof of Our Main Results

Lemma 3.1 (see [31]). *If $a(t)$, $b(t)$ are ω-periodic functions, $b(t) \geq 0$, for all $t \in R$ and $\overline{b(t)} > 0$, then*

$$\dot{u} = u(a(t) - b(t)u) \tag{3.1}$$

has a unique nonnegative ω-periodic solution u^* which is globally asymptotically stable with respect to the positive u-axis. Moreover, if $\overline{a(t)} > 0$, then $u^*(t) > 0$, for all $t \in R$ and if $\overline{a(t)} \leq 0$, then $u^*(t) = 0$.

Lemma 3.2. *Suppose that f satisfies (A1)–(A4), then there exists an $\varepsilon_0 > 0$, such that for any $0 < \varepsilon \leq \varepsilon_0$, system*

$$\dot{x} = x\big(f(t,\ x) - \varepsilon\big) \tag{3.2}$$

possesses an ω-periodic positive solution $x_\varepsilon^*(t)$ which is globally asymptotically stable with respect to the positive x-axis.

Proof. It suffices to show that if f satisfies (A1)–(A4), then there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, $\widetilde{f} = f - \varepsilon$ satisfies (A1)–(A4). Obviously $\widetilde{f}$ satisfies (A1)–(A3). Taking $\varepsilon_0 = (1/2\omega)\int_0^\omega f(s,\delta)ds$, its easy to see $\widetilde{f}$ satisfies (A4). □

From a comparison theorem (see Theorem 1.1 [37]) and Lemma 2.1, we have the following lemma.

Lemma 3.3. *Suppose that f satisfies (A1)–(A4), $x(t)$ is a solution of system* (1.4), *and $x^*(t)$ is the periodic solution of system* (1.4) *given by Lemma 2.1. Let $v(t)$ be a right maximal (right minimal)*

solution of a scalar differential equation $\dot{v} = w(t,v)$ *on* $[t_0,+\infty) \subset [0,+\infty)$, *where* $w(t,v)$ *is continuous and satisfies* $w(t,x) \le (\ge)\ xf(t,x)$ *for all* $(t,x) \in R^2_{+0}$. *Then the following conclusions hold.*

(i) *If* $v(t_0) \le (\ge)\ x(t_0)$, *then* $v(t) \le (\ge) x(t)$ *for* $t \ge t_0$.

(ii) *For all* $\varepsilon > 0$ *there exist a* $\tau \ge t_0$ *such that* $v(t) < x^*(t) + \varepsilon (v(t) > x^*(t) - \varepsilon)$ *for* $t \ge \tau$.

Proposition 3.4. *Under assumptions (A1)–(A7), there exist* M_x *and* M_y, *such that*

$$\lim_{t\to\infty} \sup\ x(t) \le M_x, \qquad \lim_{t\to\infty} \sup\ y(t) \le M_y \tag{3.3}$$

for all solutions $(x(t), y(t))$ *of system* (1.3) *with positive initial values.*

Proof. Obviously, $R^2_+ = \{(x,y) : x \ge 0,\ y \ge 0\}$ is a positively invariant set of system (1.3). Given any solution $(x(t), y(t))$ of (1.3) with positive initial values, from system (1.3) we have

$$\dot{x} \le xf(t,x). \tag{3.4}$$

The following equation

$$\dot{v} = vf(t,v), \tag{3.5}$$

has a globally asymptotically stable positive ω-periodic solution $v^*(t)$ by Lemma 2.1. By Lemma 3.3, there exists $T_1 > 0$, such that

$$x(t) < v^*(t) + 1, \quad \text{for } t > T_1. \tag{3.6}$$

Let $M_x = \max_{0\le t\le\omega}\{v^*(t) + 1\}$. We have

$$\lim_{t\to\infty} \sup\ x(t) \le M_x. \tag{3.7}$$

By (1.3),

$$\begin{aligned}
\gamma^M \dot{x} + \dot{y} &= \gamma^M xf(t,x) - \left(\gamma^M - \gamma(t)\right) g(t,x)xy - \mu(t)y - h(t)y^2 \\
&\le \gamma^M xf(t,x) - \mu(t)y \\
&\le \gamma^M xf(t,x) - \mu^L y \\
&= \gamma^M x\left(f(t,x) + \mu^L\right) - \mu^L\left(\gamma^M x + y\right).
\end{aligned} \tag{3.8}$$

Denote $w(t) = \gamma^M x(t) + y(t)$. Then we have

$$\dot{w} + \mu^L w \le \gamma^M x\left(f(t,x) + \mu^L\right). \tag{3.9}$$

Since $x(t) \le \max_{0\le t\le\omega}\{v^*(t)+1\}$, for $t \ge T_1$, there is a constant $B > 0$, such that

$$\gamma^M x\left(f(t,x)+\mu^L\right) \le B \quad \text{for } t \ge T_1, \tag{3.10}$$

by the continuity of f. It follows that

$$\dot{w} + \mu^L w \le B, \quad \text{for } t \ge T_1. \tag{3.11}$$

Notice that $\mu^L > 0$. It's easy to show that there is an $A > 0$, such that

$$\lim_{t\to\infty} \sup\, w(t) \le A. \tag{3.12}$$

By the notation $w = \gamma^M x + y$, taking $M_x = A/\gamma^M$ and $M_y = A$, Proposition 3.4 is proved. □

Proposition 3.5. *Under assumption (A1)–(A7), there exists a positive constant η_x, such that*

$$\lim_{t\to\infty} \sup\, x(t) \ge \eta_x, \tag{3.13}$$

for all solutions $(x(t), y(t))$ of (1.3) *with positive initial values.*

Proof. Suppose that (3.13) is not true. Then there is a sequence $\{z_m\} \subset R_+^2$, such that

$$\lim_{t\to\infty} \sup\, x(t,z_m) < \frac{1}{m}, \quad m = 1,2,\ldots, \tag{3.14}$$

where $(x(t,z_m), y(t,z_m))$ is the solution of (1.3) with $(x(0,z_m), y(0,z_m)) = z_m$. By assumption (A5), we can choose sufficiently small positive numbers $\varepsilon_x < 1$ and $\varepsilon_y < 1$, such that

$$\overline{\left(\gamma(t)g(t,\varepsilon_x)\varepsilon_x - \mu(t)\right)} < 0, \tag{3.15}$$

$$\overline{\phi_\varepsilon(t)} > 0, \tag{3.16}$$

where

$$\phi_\varepsilon(t) = f(t,\varepsilon_x) - M_g\varepsilon_y \exp(\alpha\omega), \qquad \alpha = \max_{0\le t\le\omega}\left(\mu(t) + \gamma(t)g(t,\varepsilon_x)\varepsilon_x + h(t)\varepsilon_y\right). \tag{3.17}$$

By (3.14), for the given $\varepsilon_x > 0$, there exists a positive integer N_0, such that

$$\lim_{t\to\infty} \sup\, x(t,z_m) < \frac{1}{m} < \varepsilon_x, \quad m > N_0. \tag{3.18}$$

For the rest of this proof, we assume that $m > N_0$. Equation (3.18) implies there exists $\tau_1^{(m)} > 0$, such that

$$x(t, z_m) < \varepsilon_x, \quad t \geq \tau_1^{(m)}, \tag{3.19}$$

and further

$$\dot{y}(t, z_m) \leq y(t, z_m)\big(\gamma(t)g(t, \varepsilon_x)\varepsilon_x - \mu(t) - h(t)y(t, z_m)\big), \quad \text{for } t \geq \tau_1^{(m)}. \tag{3.20}$$

By (3.15), and Lemma 3.1, any solution $v(t)$ of the following equation,

$$\dot{v} = v\big(\gamma(t)g(t, \varepsilon_x)\varepsilon_x - \mu(t) - h(t)v\big) \tag{3.21}$$

with positive initial conditions satisfies

$$\lim_{t\to\infty} v(t) = 0. \tag{3.22}$$

Hence,

$$\lim_{t\to\infty} y(t, z_m) = 0 \tag{3.23}$$

by Lemma 3.3. So there is a $\tau_2^{(m)} > \tau_1^{(m)}$, such that

$$y(t, z_m) < \varepsilon_y, \quad \text{for } t \geq \tau_2^{(m)}. \tag{3.24}$$

It follows that

$$\dot{x}(t, z_m) \geq x(t, z_m)\big(f(t, x(t, z_m)) - g(t, \varepsilon_x)\varepsilon_y\big), \quad t \geq \tau_2^{(m)}. \tag{3.25}$$

By (3.16), the equation

$$\dot{x} = x\big(f(t, x(t, z_m)) - g(t, \varepsilon_x)\varepsilon_y\big) \tag{3.26}$$

has an ω-periodic positive solution $x^*(t)$ which is globally asymptotically stable. Hence,

$$x(t, z_m) > \frac{x^*(t)}{2}, \tag{3.27}$$

for sufficiently large $t > 0$ and $m > N_0$, which is a contradiction with (3.14). This completes the proof of Proposition 3.5. □

Proposition 3.6. *Under assumption (A1)–(A7), there exists a positive constant γ_x, such that*

$$\lim_{t\to\infty} \inf x(t) \geq \gamma_x \tag{3.28}$$

for all solutions $(x(t), y(t))$ of system (1.3) with positive initial values.

Proof. Suppose that (3.28) is not true, then there exists a sequence $\{z_m\} \subset R_+^2$ such that

$$\lim_{t\to\infty} \inf x(t, z_m) < \frac{\eta_x}{2m^2}, \quad m = 1, 2, \ldots. \tag{3.29}$$

On the other hand, by Proposition 3.5, we have

$$\lim_{t\to\infty} \sup x(t, z_m) > \eta_x, \quad m = 1, 2, \ldots. \tag{3.30}$$

Hence, there are time sequences $\{s_q^{(m)}\}$ and $\{t_q^{(m)}\}$ satisfying

$$\begin{gathered} 0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_2^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \cdots, \\ s_q^{(m)} \longrightarrow \infty, \quad t_q^{(m)} \longrightarrow \infty, \quad \text{as } q \longrightarrow \infty, \\ x\left(t_q^{(m)}, z_m\right) = \frac{\eta_x}{m^2} < x(t, z_m) < \frac{\eta_x}{m} = x\left(s_q^{(m)}, z_m\right), \quad t \in \left(s_q^{(m)}, t_q^{(m)}\right). \end{gathered} \tag{3.31}$$

By Proposition 3.4, for a given positive integer m, there is a $T^{(m)} > 0$, such that

$$x(t, z_m) \leq M_x, \quad y(t, z_m) \leq M_y, \quad \text{for } t \geq T_1^{(m)}. \tag{3.32}$$

Because of $s_q^{(M)} \to \infty$ as $q \to \infty$, there is a positive integer $K^{(m)}$ such that $s_q^{(m)} > T_1^{(m)}$ since $q \geq K(m)$, and hence

$$\dot{x} \geq x(t, z_m)\left(f(t, M_x) - M_g M_y\right) := \zeta(t) x(t, z_m) \tag{3.33}$$

for $q \geq K^{(m)}$. Integrating (3.33) from $s_q^{(m)}$ to $t_q^{(m)}$ yields

$$x\left(t_q^{(m)}, z_m\right) \geq x\left(s_q^{(m)}, z_m\right) \exp \int_{s_q^{(m)}}^{t_q^{(m)}} \zeta(t) dt \tag{3.34}$$

or

$$-\int_{s_q^{(m)}}^{t_q^{(m)}} \zeta(t) dt \geq \ln m, \quad \text{for } q \geq K^{(m)}. \tag{3.35}$$

If $\overline{\zeta(t)} \geq 0$, it leads to a contradiction. Otherwise, $\overline{\zeta(t)} < 0$, we have

$$t_q^{(m)} - s_q^{(m)} \longrightarrow \infty, \quad \text{as } m \longrightarrow \infty,\ q \geq K^{(m)}, \tag{3.36}$$

since $\zeta(t)$ is bounded. By (3.15) and (3.16), there are constants $P > 0$ and $N_0 > 0$, such that

$$\frac{\eta_x}{m} < \varepsilon_x, \quad t_q^{(m)} - s_q^{(m)} > 2P, \tag{3.37}$$

$$M_y \exp \int_0^P \left(\gamma(t) g(t, \varepsilon_x)\varepsilon_x - \mu(t) - h(t)\varepsilon_y\right) dt < \varepsilon_y, \quad \int_0^P \phi_\varepsilon(t) dt > 0 \tag{3.38}$$

for $m \geq N_0$, $q \geq K^{(m)}$, and $a \geq P$. Equation (3.37) implies

$$x(t, z_m) < \varepsilon_x, \quad t \in \left[s_q^{(m)}, t_q^{(m)}\right] \tag{3.39}$$

for $m \geq N_0$, $q \geq K^{(m)}$. For the positive ε_y satisfying (3.16) and (3.38), we have the following two cases:

(i) $y(t, z_m) \geq \varepsilon_y$ for all $t \in [s_q^{(m)}, s_q^{(m)} + P]$;

(ii) there exists $\tau_{q_1}^{(m)} \in [s_q^{(m)}, s_q^{(m)} + P]$, such that $y(\tau_{q_1}^{(m)}, z_m) < \varepsilon_y$.

If (i) holds, by (3.38) and (3.39) we have

$$\begin{aligned}
\varepsilon_y &\leq y\left(s_q^{(m)} + P, z_m\right) \\
&\leq y\left(s_q^{(m)}, z_m\right) \exp \int_{s_q^{(m)}}^{s_q^{(m)}+P} \left(\gamma(t) g(t, \varepsilon_x)\varepsilon_x - \mu(t) - h(t)\varepsilon_y\right) dt \\
&\leq M_y \exp \int_0^P \left(\gamma(t) g(t, \varepsilon_x)\varepsilon_x - \mu(t) - h(t)\varepsilon_y\right) dt \\
&< \varepsilon_y.
\end{aligned} \tag{3.40}$$

This is a contradiction.

If (ii) holds, we now claim that

$$y(t, z_m) \leq \varepsilon_y \exp(\alpha\omega), \quad t \in \left(\tau_{q_1}^{(m)}, t_q^{(m)}\right]. \tag{3.41}$$

Otherwise, there exists $\tau_{q_2}^{(m)} \in (\tau_{q_1}^{(m)}, t_q^{(m)}]$ such that

$$y\left(\tau_{q_2}^{(m)}, z_m\right) > \varepsilon_y \exp(\alpha\omega). \tag{3.42}$$

By the continuity of $y(t, z_m)$, there must exist $\tau_{q_3}^{(m)} \in (\tau_{q_1}^{(m)}, \tau_{q_2}^{(m)})$ such that

$$\begin{gathered} y\left(\tau_{q_3}^{(m)}, z_m\right) = \varepsilon_y, \\ y(t, z_m) > \varepsilon_y, \quad \text{for } t \in \left(\tau_{q_3}^{(m)}, \tau_{q_2}^{(m)}\right). \end{gathered} \tag{3.43}$$

Denote $P^{(m)}$ as the nonnegative integer, such that $\tau_{q_2}^{(m)} \in (\tau_{q_3}^{(m)} + P^{(m)}\omega, \tau_{q_3}^{(m)} + (P^{(m)} + 1)\omega]$. By (3.15), we have

$$\begin{aligned} \varepsilon_y \exp(\alpha\omega) &< y\left(\tau_{q_2}^{(m)}, z_m\right) \\ &< y\left(\tau_{q_3}^{(m)}, z_m\right) \exp \int_{\tau_{q_3}^{(m)}}^{\tau_{q_2}^{(m)}} \left(\gamma(t) g(t, \varepsilon_x)\varepsilon_x - \mu(t) - h(t)\varepsilon_y\right) dt \\ &= \varepsilon_y \exp\left\{ \int_{\tau_{q_3}^{(m)}}^{\tau_{q_3}^{(m)} + P^{(m)}\omega} + \int_{\tau_{q_3}^{(m)} + P^{(m)}\omega}^{\tau_{q_2}^{(m)}} \right\} \left(\gamma(t) g(t, \varepsilon_x)\varepsilon_x - \mu(t) - h(t)\varepsilon_y\right) dt \\ &< \varepsilon_y \exp(\alpha\omega). \end{aligned} \tag{3.44}$$

This contradiction establishes that (3.41) is true, particularly (3.41) holds for $t \in [s_q^{(m)} + P, t_q^{(m)}]$.

By (3.31) and (3.38), we have

$$\begin{aligned} \frac{\eta_x}{m^2} &= x\left(t_q^{(m)}, z_m\right) \\ &\geq x\left(s_q^{(m)} + P, z_m\right) \exp \int_{s_q^{(m)} + P}^{t_q^{(m)}} \left(f(t, \varepsilon_x) - M_g \varepsilon_y \exp(\alpha\omega)\right) dt \\ &= x\left(s_q^{(m)} + P, z_m\right) \exp \int_{s_q^{(m)} + P}^{\tau_q^{(m)}} \phi_\varepsilon(t) dt \\ &> \frac{\eta_x}{m^2}, \end{aligned} \tag{3.45}$$

which is also a contradiction. This completes the proof of Proposition 3.6. □

Proposition 3.7. *Suppose f satisfies (A1)–(A4), g satisfies (A5)–(A7), and* (2.2) *holds. Then there exists a positive constant η_y such that*

$$\lim_{t \to \infty} \sup\, y(t) > \eta_y \tag{3.46}$$

for all solutions $(x(t), y(t))$ of (1.3) *with positive initial values.*

Proof. By assumption (A5) and (2.2) we can choose a constant $\varepsilon_1 > 0$ such that

$$\overline{\varphi_{\varepsilon_1}(t)} > 0, \tag{3.47}$$

where

$$\varphi_{\varepsilon_1}(t) = \gamma(t)g(t, x^*(t) - \varepsilon_1)(x^*(t) - \varepsilon_1) - \mu(t) - h(t)\varepsilon_1. \tag{3.48}$$

Consider the following equation with positive parameter α:

$$\dot{x} = x\left(f(t,x) - 2M_g\alpha\right). \tag{3.49}$$

By Lemma 3.2, (3.49) has a unique positive ω-periodic solution $x_\alpha^*(t)$ which is globally asymptotically stable since $\alpha < \varepsilon_0/(2M_g)$. Let $x_\alpha(t)$ be the solution of (3.49) with initial condition $x_\alpha(0) = x^*(0)$ in which $x^*(t)$ is the unique periodic solution of (1.4) given by Lemma 2.1. Hence, for the above ε_1, there exists sufficiently large $T_2 > T_1$, such that

$$|x_\alpha(t) - x_\alpha^*(t)| < \frac{\varepsilon_1}{4}, \quad \text{for } t \geq T_2. \tag{3.50}$$

By the continuity of the solution in the parameter, we have $x_\alpha(t) \to x^*(t)$ uniformly in $[T_2, T_2 + \omega]$ as $\alpha \to 0$. Hence, for $\varepsilon_1 > 0$ there exists $\alpha_0 > 0$, such that

$$|x_\alpha(t) - x^*(t)| < \frac{\varepsilon_1}{4}, \quad \text{for } t \in [T_2, T_2 + \omega], \quad 0 < \alpha < \alpha_0. \tag{3.51}$$

So, we have

$$|x_\alpha^*(t) - x^*(t)| < \frac{\varepsilon_1}{2}, \quad t \in [T_2, T_2 + \omega]. \tag{3.52}$$

Notice that $x_\alpha(t)$ and $x^*(t)$ are all ω-periodic, hence

$$|x_\alpha^*(t) - x^*(t)| < \frac{\varepsilon_1}{2}, \quad t \geq 0,\ 0 < \alpha < \alpha_0. \tag{3.53}$$

Choosing a constant α_1 $(0 < \alpha_1 < \alpha_0,\ 2\alpha_1 < \varepsilon_1)$, we have

$$x_{\alpha_1}^*(t) \geq x^*(t) - \frac{\varepsilon_1}{2}, \quad t \geq 0. \tag{3.54}$$

Suppose that (3.46) is not true. Then there exists $z \in R_+^2$, such that

$$\lim_{t\to\infty} \sup\ y(t,z) < \alpha_1, \tag{3.55}$$

where $(x(t,z), y(t,z))$ is the solution of (1.3) with $(x(0,z), y(0,z)) = z$. So, there exist $T_3 \geq T_2$, such that

$$y(t,z) < 2\alpha_1 < \varepsilon_1, \quad t \geq T_3 \tag{3.56}$$

and hence

$$\dot{x} \geq x(t,z)\left(f(t,x) - 2M_g\alpha_1\right). \tag{3.57}$$

Let $u(t)$ be the solution of (3.49) with $\alpha = \alpha_1$ and $u(T_3) = x(T_3, z)$, then

$$x(t,z) \geq u(t), \quad t \geq T_3. \tag{3.58}$$

By the global asymptotic stability of $x^*_{\alpha_1}(t)$, for the given $\varepsilon = \varepsilon_1/2$, there exists $T_4 \geq T_3$, such that

$$\left|u(t) - x^*_{\alpha_1}(t)\right| < \frac{\varepsilon_1}{2}, \quad t \geq T_4. \tag{3.59}$$

Hence

$$x(t,z) \geq u(t) > x^*_{\alpha_1}(t) - \frac{\varepsilon_1}{2}, \quad t \geq T_4 \tag{3.60}$$

and hence

$$x(t,z) > x^*(t) - \varepsilon_1, \quad t \geq T_4 \tag{3.61}$$

from (3.54). This implies

$$\dot{y}(t,z) \geq y(t,z)\left(\gamma(t)g(t, x^*(t) - \varepsilon_1)(x^* - \varepsilon_1) - \mu(t) - h(t)\varepsilon_1\right) = \varphi_{\varepsilon_1} y(t,z), \quad t \geq T_4. \tag{3.62}$$

Integrating the above inequality from T_4 to t yields

$$y(t,z) \geq y(T_4, z) \exp \int_{T_4}^{t} \varphi_{\varepsilon_1}(t)dt. \tag{3.63}$$

$y(t,z) \to \infty$ as $t \to \infty$ from (3.47) which is a contradiction. This completes the proof of Proposition 3.7. □

Proposition 3.8. *Suppose f satisfies (A1)–(A4), g satisfies (A5)–(A7), and* (2.2) *holds. Then there exists a positive constant γ_y, such that*

$$\lim_{t\to\infty} \inf y(t) \geq \gamma_y \tag{3.64}$$

for all solutions $(x(t), y(t))$ of (1.3) *with positive initial values.*

Proof. Otherwise, there exists a sequence $\{z_m\} \subset R_+^2$, such that

$$\lim_{t\to\infty} \inf y(t, z_m) < \frac{\eta_y}{(m+1)^2}, \quad m = 1, 2, \ldots. \tag{3.65}$$

However

$$\lim_{t\to\infty} \sup y(t, z_m) > \eta_y, \quad m = 1, 2, \ldots, \tag{3.66}$$

from Proposition 3.7. Hence there are two time sequence $\{s_q^{(m)}\}$ and $\{t_q^{(m)}\}$ satisfying the following conditions:

$$\begin{gathered} 0 < s_1^{(m)} < t_1^{(m)} < s_2^{(m)} < t_1^{(m)} < \cdots < s_q^{(m)} < t_q^{(m)} < \cdots, \\ s_q^{(m)} \longrightarrow \infty, \quad t_q^{(m)} \longrightarrow \infty, \quad \text{as } q \longrightarrow \infty, \\ y\left(t_q^{(m)}, z_m\right) = \frac{\eta_y}{(m+1)^2} < y(t, z_m) < \frac{\eta_y}{m+1} = y\left(s_q^{(m)}, z_m\right), \quad t \in \left(s_q^{(m)}, t_q^{(m)}\right). \end{gathered} \tag{3.67}$$

By Proposition 3.4, for a given integer $m > 0$ there is a $T_1^{(m)} > 0$, such that

$$y(t, z_m) \le M_y, \quad \text{for } t \ge T_1^{(m)}. \tag{3.68}$$

Because of $s_q^{(m)} \to \infty$ as $q \to \infty$, there is a positive integer $K^{(m)}$, such that $s_q^{(m)} > T_1^{(m)}$ as $q \ge K^{(m)}$, and hence

$$\dot{y}(t, z_m) \ge y(t, z_m)\left(-\mu(t) - h(t)M_y\right) \tag{3.69}$$

for $q \ge K^{(m)}$ and $t \in [s_q^{(m)}, t_q^{(m)}]$. Integrating the above inequality from $s^{(m)}$ to $t_q^{(m)}$, we have

$$y\left(t_q^{(m)}, z_m\right) \ge y\left(s_q^{(m)}, z_m\right) \exp \int_{s_q^{(m)}}^{t_q^{(m)}} \left(-\mu(t) - h(t)M_y\right) dt \tag{3.70}$$

or

$$\int_{s_q^{(m)}}^{t_q^{(m)}} \left(\mu(t) + h(t)M_y\right) dt \ge \ln(m+1), \quad q \ge K^{(m)}. \tag{3.71}$$

Because of the boundedness of the function $\mu(t) + h(t)M_y$, we know that

$$t_q^{(m)} - s_q^{(m)} \longrightarrow \infty, \quad \text{as } m \longrightarrow \infty,\ q \ge K^{(m)}. \tag{3.72}$$

By (3.47), there is a constant $P > 0$ and a positive integer N_0 such that

$$\frac{\eta_y}{m+1} < \alpha_1 < \varepsilon_1, \quad t_q^{(m)} - s_q^{(m)} > 2P, \tag{3.73}$$

$$\int_0^q \psi_{\varepsilon_1}(t)dt > 0 \tag{3.74}$$

for $m \geq N_0$, $q \geq K^{(m)}$ and $a \geq P$. Further,

$$y(t, z_m) < \alpha_1 < \varepsilon_1, \quad t \in \left[s_q^{(m)}, t_q^{(m)}\right], \ m \geq N_0, \ \ q \geq K^{(m)}. \tag{3.75}$$

In addition,

$$\dot{x}(t, z_m) \geq x(t, z_m)\left(f(t, x(t, z_m)) - M_g\alpha_1\right). \tag{3.76}$$

Let $u(t)$ be the solution of (3.49) with $\alpha = \alpha_1$ and $u(s_q^{(m)}) = x(s_q^{(m)}, z_m)$. Then

$$x(t, z_m) \geq u(t), \quad t \in \left[s_q^{(m)}, t_q^{(m)}\right] \tag{3.77}$$

by Lemma 3.3. Further, by Propositions 3.4 and 3.6, we can choose $K_1^{(m)} > K^{(m)}$ such that

$$\gamma_x \leq x\left(s_q^{(m)}, z_m\right) \leq M_x \tag{3.78}$$

for $q \geq K_1^{(m)}$. For $\alpha = \alpha_1$, (3.49) has a unique positive ω-periodic solution $x_{\alpha_1}^*(t)$ which is globally asymptotically stable. In addition, by the periodicity of (3.49), the periodic solution $x_{\alpha_1}(t)$ is uniformly asymptotically stable with respect to the compact set $\Omega = \{x : \gamma_x \leq x \leq M_x\}$. Hence, for the given ε_1 in Proposition 3.7, there exists $T_0 > P$ which is independent of m and q, such that

$$u(t) \geq x_{\alpha_1}^*(t) - \frac{\varepsilon_1}{2}, \quad t \geq T_0 + s_q^{(m)}. \tag{3.79}$$

Thus

$$u(t) \geq x^*(t) - \varepsilon_1, \quad t \geq T_0 + s_q^{(m)} \tag{3.80}$$

from (3.54). By (3.72), there exists a positive integer $N_1 \geq N_0$ such that $t_q^{(m)} > s_q^{(m)} + 2T_0 > s_q^{(m)} + 2P$ for $m \geq K_1^{(m)}$ and $q \geq K_1^{(m)}$. So we have

$$x(t, z_m) \geq x^*(t) - \varepsilon_1, \quad t \in \left[s_q^{(m)} + T_0, t_q^{(m)}\right] \tag{3.81}$$

since $m \geq N_1$ and $q \geq K_1^{(m)}$. Hence,

$$\dot{y}(t, z_m) \geq \varphi_{\varepsilon_1}(t) y(t, z_m), \quad t \in \left[s_q^{(m)} + T_0, t_q^{(m)}\right] \tag{3.82}$$

from (3.75) and (3.81). Integrationg the above inequality from $s_q^{(m)} + T_0$ to $t_q^{(m)}$ yields

$$y\left(t_q^{(m)}, z_m\right) \geq y\left(s_q^{(m)} + T_0, z_m\right) \exp \int_{s_q^{(m)}+T_0}^{t_q^{(m)}} \varphi_{\varepsilon_1}(t) dt, \tag{3.83}$$

and hence

$$\frac{\eta_y}{(m+1)^2} \geq \frac{\eta_y}{(m+1)^2} \exp \int_{s_q^{(m)}+T_0}^{t_q^{(m)}} \varphi_{\varepsilon_1}(t) dt > \frac{\eta_y}{(m+1)^2} \tag{3.84}$$

from (3.74). This is a contradiction. This completes the proof of Proposition 3.8. The result of Theorem 2.2 follows from Propositions 3.4–3.8. □

Proof of Theorem 2.3. Suppose $x^*(t)$ is the periodic solution of (1.4). For any solution $(x(t), y(t))$ of (1.3) with a positive initial value, there are two possible cases:

(i) for all $t \geq 0$, $x(t) > x^*(t)$;

(ii) there is a $\widehat{T} > 0$, such that $x(\widehat{T}) \leq x^*(t)$.

Now, in the above two cases, we prove $\lim_{t \to 0} y(t) = 0$, respectively.

(i) Suppose $v(t)$ is a solution of (1.4) with $v(0) = x(0)$. Comparing (1.3) and (1.4) we have $v(t) \geq x(t) \geq x^*(t)$ and $\lim_{t \to 0}(v(t) - x^*(t)) = 0$. Hence

$$\lim_{t \to \infty} (x(t) - x^*(t)) = 0. \tag{3.85}$$

From system (1.3), for all $t_0 \in [0, \infty)$, we have

$$\begin{aligned} x(t_0 + \omega) - x(t_0) &= \int_{t_0}^{t_0+\omega} x(s) f(s, x(s)) ds - \int_{t_0}^{t_0+\omega} x(s) g(s, x) y \, ds \\ x^*(t_0 + \omega) - x^*(t_0) &= \int_{t_0}^{t_0+\omega} x^*(s) f(s, x^*) ds = 0. \end{aligned} \tag{3.86}$$

So

$$\begin{aligned} x(t_0+\omega)-x(t_0) &= \int_{t_0}^{t_0+\omega}\left[xf(s,x)-x^*f(s,x^*)\right]ds-\int_{t_0}^{t_0+\omega}xg(s,x)y\,ds \\ &= \int_{t_0}^{t_0+\omega}\left\{(x-x^*)f(s,x)+x^*\left[f(s,x)-f(s,x^*)\right]\right\}ds-\int_{t_0}^{t_0+\omega}xg(s,x)y\,ds \\ &= \int_{t_0}^{t_0+\omega}(x-x^*)\left[f(s,x)+x^*\frac{\partial f}{\partial x}(s,x^*+\theta(x-x^*))\right]ds-\int_{t_0}^{t_0+\omega}xg(s,x)y\,ds, \end{aligned} \tag{3.87}$$

where $0<\theta<1$. By the boundedness of x, x^*, and the continuity of f and $\partial f/\partial x$, there must exist a constant $A>0$, such that

$$\left|f(s,x)+x^*\frac{\partial f}{\partial x}(s,x^*+\theta(x-x^*))\right|\le A. \tag{3.88}$$

For all $\varepsilon>0$, by (3.85), there exists a $T^0>0$, such that $0<x(t)-x^*(t)<\varepsilon$, for all $t\ge T^0$. Hence

$$\begin{aligned} \int_{t_0}^{t_0+\omega}xg(s,x)y\,ds &\le -x(t_0+\omega)+x(t_0)+A\omega\varepsilon \\ &= -[x(t_0+\omega)-x^*(t_0+\omega)]+[x(t_0)-x^*(t_0)]+A\omega\varepsilon \\ &< 0+\varepsilon+A\omega\varepsilon \\ &= (A\omega+1)\varepsilon \end{aligned} \tag{3.89}$$

for $t_0\ge T^0$. Denote $G^L=\min_{0\le t\le\omega}x^*(t)g(t,x^*(t))$, $y_{t_0}^L=\min_{t_0\le t\le t_0+\omega}y(t)$. Then

$$0<G^L\int_{t_0}^{t_0+\omega}y\,ds\le\int_{t_0}^{t_0+\omega}xg(s,x)y\,ds\le(A\omega+1)\varepsilon. \tag{3.90}$$

Hence

$$0<y_{t_0}^L\le\frac{1}{\omega}\int_{t_0}^{t_0+\omega}y(s)ds\le\left(\frac{A}{G^L}+\frac{1}{(\omega G^L)}\right)\varepsilon. \tag{3.91}$$

This implies

$$\lim_{t_0\to\infty}y_{t_0}^L=\lim_{t_0\to\infty}\int_{t_0}^{t_0+\omega}y(s)ds=0. \tag{3.92}$$

On the other hand,

$$y(t) = y(t_0)\exp\int_{t_0}^{t_0+\omega}\left[\gamma(s)x(s)g(s,x(s)) - \mu(s) - h(s)y(s)\right]ds. \tag{3.93}$$

By the boundedness of x, y, g, and the continuity of γ, μ, and h, there exists a $C > 0$ such that

$$\left|\gamma(t)g(t,x)x - \mu(t) - h(t)y(t)\right| \le C, \quad t \ge 0. \tag{3.94}$$

Hence

$$\left|\ln\frac{y(t)}{y(t_0)}\right| \le C\int_{t_0}^{t_0+\omega} ds = C\omega, \quad \text{for } t \in [0,\omega]. \tag{3.95}$$

For all $t, s \in [0,\omega]$, we have

$$\left|\ln\frac{y(t)}{y(s)}\right| = \left|\ln\frac{y(t)}{y(t_0)} - \ln\frac{y(s)}{y(t_0)}\right| \le 2C\omega. \tag{3.96}$$

It follows that

$$\exp(-2C\omega) \le \frac{y(t)}{y(s)} \le \exp(2C\omega), \quad |t-s| \le \omega. \tag{3.97}$$

Considering (3.92), we have

$$\lim_{t\to\infty} y(t) = 0. \tag{3.98}$$

(ii) Consider the second case. We claim that once there is a $\widehat{T} > 0$ such that $x(\widehat{T}) \le x^*(\widehat{T})$, then

$$x(t) < x^*(t), \quad \forall t > \widehat{T}. \tag{3.99}$$

Otherwise, suppose $\widetilde{T} = \min_{t>\widehat{T}}\{t : x(t) = x^*(t)\}$. Then

$$\dot{x}\left(\widetilde{T}\right) = x'\left(\widetilde{T}-0\right) = \lim_{t\to\widetilde{T}-0}\frac{x(t)-x\left(\widetilde{T}\right)}{t-\widetilde{T}} = \lim_{t\to\widetilde{T}-0}\frac{x(t)-x^*\left(\widetilde{T}\right)}{t-\widetilde{T}} \ge \lim_{t\to\widetilde{T}-0}\frac{x^*(t)-x^*\left(\widetilde{T}\right)}{t-\widetilde{T}} = \dot{x}^*\left(\widetilde{T}\right). \tag{3.100}$$

This is a contradiction since from (1.3) and (1.4) we know $\dot{x}(t) < \dot{x}^*(t)$ at the same point $(t,x) \in R_+^2$. So (3.99) holds.

Now we show for all $\varepsilon > 0$, there is a $T^{(1)} > \widehat{T} \geq 0$ such that

$$y\left(T^{(1)}\right) < \varepsilon. \tag{3.101}$$

Otherwise, suppose for all $t \geq \widehat{T}$, $y(t) \geq \varepsilon$. Then

$$\dot{x} \leq x\left(f(t,x) - g(t,x)\varepsilon\right), \quad t \geq \widehat{T}. \tag{3.102}$$

Suppose $v(t)$ is a solution of (1.4) with $v(\widehat{T}) = x(\widehat{T})$. Then by Lemma 2.1 we have $v(t) \to x^*(t)$ $(t \to \infty)$, where $x^*(t)$ denotes the periodic solution of (1.4). Since $\dot{x}(\widehat{T}) < \dot{v}(\widehat{T})$, there exists a $\sigma > 0$ such that $x(t) < v(t)$ for $t \in (\widehat{T}, \widehat{T} + \sigma]$. Denote $\delta(t) = v(t) - x(t)$. From (3.99) we know $\delta(t) > 0$, for $t \geq \widehat{T}$. We show that there is an $\varepsilon_1 > 0$ such that $\lim_{t\to\infty} \inf \delta(t) \geq \varepsilon_1$. In fact,

$$\begin{aligned}
\dot{v} - \dot{x} &= vf(t,v) - \left(xf(t,x) - g(t,x)\varepsilon\right) \\
&= v\left(f(t,v) - f(t,x)\right) + f(t,x)(v - x) + xg(t,x)\varepsilon \\
&= \left(v\frac{\partial f}{\partial x}(t, x + \theta(v - x)) + f(t,x)\right)(v - x) + xg(t,x)\varepsilon
\end{aligned} \tag{3.103}$$

where $0 < \theta < 1$. By the boundedness of x, v, and the continuity of f and $\partial f/\partial x$, there exists an $L > 0$, such that $|v(\partial f/\partial x)(t, x + \theta(v - x)) + f(t,x)| < L$. Hence

$$\dot{\delta} > -L\delta + xg(t,x)\varepsilon. \tag{3.104}$$

Choosing $T^{(0)} > \widehat{T}$, such that $v(t) \geq x^*(t)/2$ for $t \geq T^{(0)}$, and letting

$$\varepsilon_1 = \min\left\{\max_{0\leq t\leq\sigma} \delta(t), \min_{0\leq t\leq\omega}\left(\frac{x^*(t)}{4}\right), \frac{\min_{0\leq t\leq\omega} g(t, x^*(t)/4\ )x^*(t)\varepsilon}{4L}\right\}. \tag{3.105}$$

Then whenever $\delta(t) \leq \varepsilon_1$ for $t \geq T^{(0)}$, we have $x(t) = v(t) - \delta(t) \geq x^*(t)/2 - x^*(t)/4 = x^*(t)/4$. By assumption (A6), $\dot{\delta}(t) > -L\varepsilon_1 + g(t, x^*(t)/4)x^*(t)/4 \geq 0$, and this implies $\lim_{t\to\infty} \inf \delta(t) > \varepsilon_1$. Choosing $T' > \widehat{T}$, such that $v(t) < x^*(t) + \varepsilon_1/2$ and $\delta(t) \geq \varepsilon_1$ for $t \geq T'$, we have

$$x(t) \leq v(t) - \varepsilon_1 \leq \left(x^*(t) + \frac{\varepsilon_1}{2}\right) - \varepsilon_1 = x^*(t) - \frac{\varepsilon_1}{2}, \quad \text{for } t \geq T'. \tag{3.106}$$

Hence, there exists an $\varepsilon_0 > 0$ such that

$$\overline{\gamma(t)g(t,x(t))x(t) - \mu(t) - h(t)\varepsilon} \leq \overline{\gamma(t)g\left(t, x^*(t) - \frac{\varepsilon_1}{2}\right)\left(x^*(t) - \frac{\varepsilon_1}{2}\right) - \mu(t)} \leq -\varepsilon_0 < 0 \tag{3.107}$$

for $t \geq T'$, by (2.3) and assumption (A5). So

$$\varepsilon \leq y(t) \leq y(T') \exp \int_{T'}^{t} [\gamma(s)g(s,x(s))x(s) - \mu(s) - h(s)\varepsilon]\, ds \longrightarrow 0 \quad (t \longrightarrow \infty). \tag{3.108}$$

This is a contradiction and it implies that (3.101) holds.

Second, we show that

$$y(t) \leq \varepsilon \exp(M(\varepsilon)\omega), \qquad \text{for } t \geq T^{(1)}, \tag{3.109}$$

where

$$M(\varepsilon) = \max_{0 \leq t \leq \omega} \{\gamma(t)g(t,x^*)(t)x^*(t) + \mu(t) + h(t)\varepsilon\}. \tag{3.110}$$

Otherwise, there exists $T^{(2)} > T^{(1)}$, such that

$$y\left(T^{(2)}\right) > \varepsilon \exp(M(\varepsilon)\omega). \tag{3.111}$$

By the continuity of $y(t)$, there must exist $T^{(3)} \in (T^{(1)}, T^{(2)})$ such that $y(T^{(3)}) = \varepsilon$ and $y(t) > \varepsilon$ for $t \in (T^{(3)},\ T^{(2)}]$. Let P_1 be the nonnegative integer, such that $T^{(2)} \in (T^{(3)} + P_1\omega,\ T^{(3)} + (P_1 + 1)\omega]$, and by (2.3), (A6) and (3.99), we have

$$\begin{aligned}
\varepsilon \exp(M(\varepsilon)\omega) &< y\left(T^{(2)}\right) \\
&< y\left(T^{(3)}\right) \exp \int_{T^{(3)}}^{T^{(2)}} [\gamma(s)g(s,x(s))x(s) - \mu(s) - h(s)\varepsilon]\, ds \\
&= \varepsilon \exp\left\{\int_{T^{(3)}}^{T^{(3)}+P_1\omega} + \int_{T^{(3)}+P_1\omega}^{T(2)}\right\} [\gamma(s)g(s,x)x - \mu(s) - h(s)\varepsilon]\, ds \\
&< \varepsilon \exp\left\{\int_{T^{(3)}}^{T^{(3)}+P_1\omega} + \int_{T^{(3)}+P_1\omega}^{T(2)}\right\} [\gamma(s)g(s,x^*)x^* - \mu(s) - h(s)\varepsilon]\, ds \\
&\leq \varepsilon \exp\left\{\int_{T(3)+P_1\omega}^{T^{(2)}} [\gamma(s)g(s,x^*)x^* - \mu(s) - h(s)\varepsilon]\, ds\right\} \\
&\leq \varepsilon \exp\left\{\int_{T(3)+P_1\omega}^{T^{(2)}} [\gamma(s)g(s,x^*)x^* + |\mu(s)| + h(s)\varepsilon]\, ds\right\} \\
&< \varepsilon \exp(M(\varepsilon)\omega),
\end{aligned} \tag{3.112}$$

which is a contradiction. This completes the proof of conclusion (i) of Theorem 2.3.

Now we prove the second conclusion. Since (i) holds and $g(t,x)$ is bounded, there is a $t_1 > 0$ and an $\varepsilon > 0$, such that $g(t,x)y < \varepsilon \le \varepsilon_0$ for $t \ge t_1$. Then we have

$$\dot{x} \ge x\big(f(t,x) - \varepsilon\big), \quad \text{for } t \ge t_1. \tag{3.113}$$

The following auxiliary equation,

$$\dot{v} = v\big(f(t,v) - \varepsilon\big) \tag{3.114}$$

has a globally asymptotically stable positive ω-periodic solution $x_\varepsilon^*(t)$ by Lemma 3.2. So by Lemma 3.3, we have

$$\lim_{t\to\infty} \inf x(t) \ge x_\varepsilon^*(t). \tag{3.115}$$

This completes the proof of conclusion (ii) of Theorem 2.3. □

Proof of Corollary 2.6. We claim that once $g(t,x)$ satisfies a local Lipschitz condition with respect to x, then the function

$$F(t,x,y) = \begin{pmatrix} x\big(f(t,x) - g(t,x)y\big) \\ y\big(\gamma(t)g(t,x)x - \mu(t) - h(t)y\big) \end{pmatrix} \tag{3.116}$$

satisfies a local Lipschitz condition with respect to x and y. Considering assumptions (A1)–(A7), this is clearly the case. So the uniqueness of solutions of system (1.3) is guaranteed, and by Lemma 2.5 we know that there exsits an ω-periodic solution $Z^*(t) = (x^*(t), y^*(t))$. From the proof of Theorem 15.5 in [37], the initial value of the periodic solution is the fix point of the mapping $T : z_0 \to z(\omega; 0, z_0)$ which is the limit point of a subsequence of the sequence $\{T^n z_0\}_1^\infty$, where $z_0 = (x_0, y_0)$ is the initial value of a bounded solution of system (1.3). Taking any positive x_0 and y_0, by Theorem 2.2 we know the solution started from this point is bounded and the limit of any subsequence of the sequence $\{T^n z_0\}_1^\infty$ is positive. So the periodic solution $(x^*(t), y^*(t))$ is positive. □

4. Examples

Example 4.1. Suppose $f(t,x) = 1 - (2 + \cos t)x$, $g(t,x) = 2xy/(2 + \sin t + x^2)$, $\gamma(t) = 0.5$, $\mu(t) = (1/10) - (1/20)\sin t$, $h(t) = 1$. The corresponding system is

$$\begin{aligned} \dot{x} &= x\left(1 - (2+\cos t)x - \frac{2xy}{2+\sin t + x^2}\right), \\ \dot{y} &= y\left(\frac{x^2}{2+\sin t + x^2} - \frac{1}{10} - \frac{1}{20}\sin t\right). \end{aligned} \tag{4.1}$$

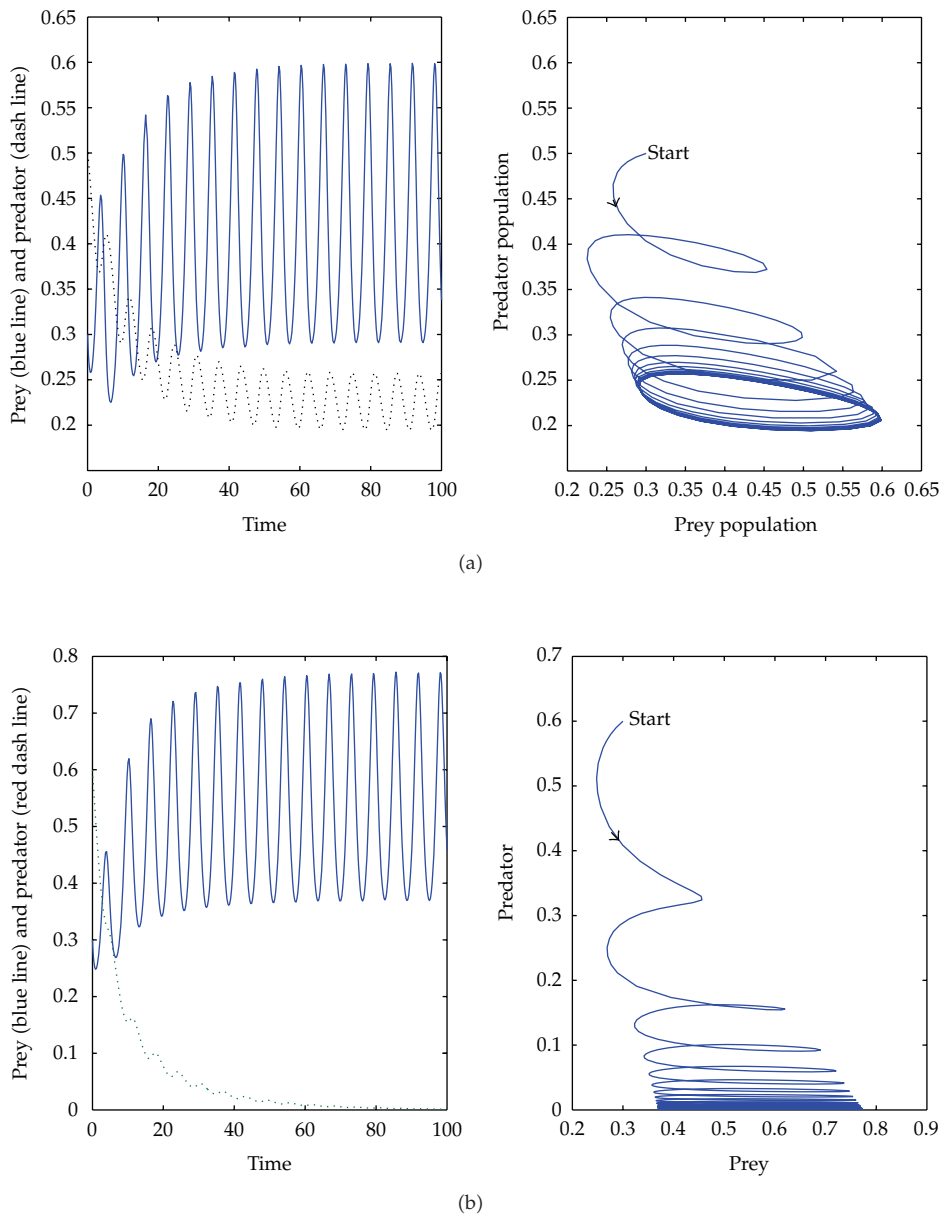

Figure 1: (a) Simulation results of Example 4.1 with initial values $x(0) = 0.3$, $y(0) = 0.5$. (b) Simulation results of Example 4.2 with initial values $x(0) = 0.3$, $y(0) = 0.6$.

We know that the periodic solution of system (3.1) is

$$x^*(t) = \frac{1 - \exp\left(-\int_0^\omega a(s)ds\right)}{\int_0^\omega b(t-s)\exp\left(-\int_0^\omega a(t-\tau)d\tau\right)ds}. \tag{4.2}$$

Using this formula we can compute the periodic solution of (4.1) in absence of predators

$$\begin{aligned} x^*(t) &= 2\,\frac{1-e^{-2\pi}}{(4e^{2\pi}+\cos(t)e^{2\pi}+\sin(t)e^{2\pi}-4-\cos(t)-\sin(t))e^{-2\pi}} \\ &\approx \frac{2}{(4+\cos(t)+\sin(t))}, \end{aligned}$$

$$\overline{\gamma(t)g(t,x^*(t))x^*(t)-\mu(t)} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{(x^*)^2}{2+\sin t+(x^*)^2}-\frac{1}{10}-\frac{1}{20}\sin t\right)dt \approx 0.05313 > 0. \tag{4.3}$$

System (4.1) is permanent.

Example 4.2. Taking $f(t,x)$, $g(t,x)$ and $\gamma(t)$ the same as in Example 4.1, and $\mu(t) = (1/5) - (1/10)\sin t$, we can compute

$$\overline{\gamma(t)g(t,x^*(t))x^*(t)-\mu(t)} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{(x^*)^2}{2+\sin t+(x^*)^2}-\frac{1}{5}-\frac{1}{10}\sin t\right)dt \approx -0.04687 < 0. \tag{4.4}$$

In this circumstance the predator population tends to extinction and the prey population keeps oscillating.

Simulation results of the two examples are shown in Figure 1.

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