Research Article

A Strong Convergence Theorem for a Common Fixed Point of Two Sequences of Strictly Pseudocontractive Mappings in Hilbert Spaces and Applications

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We prove a strong convergence theorem for a common fixed point of two sequences of strictly pseudocontractive mappings in Hilbert spaces. We also provide some applications of the main theorem to find a common element of the set of fixed points of a strict pseudocontraction and the set of solutions of an equilibrium problem in Hilbert spaces. The results extend and improve the recent ones announced by Marino and Xu (2007) and others.

1. Introduction

Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \rightarrow C$ be a self-mapping of C. Recall that T is said to be a strict pseudocontraction if there exists a constant $0 \leqslant k < 1$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\| + k\|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

for all $x, y \in C$. (We also say that T is a k-strict pseudocontraction if T satisfies (1.1)). We use $F(T)$ to denote the set of fixed points of T (i.e., $F(T) = \{x \in C : Tx = x\}$). Note that the class

of strict pseudocontractions strictly includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leqslant \|x - y\| \tag{1.2}$$

for all $x, y \in C$. That is, T is nonexpansive if and only if T is a 0-strict pseudocontraction.

In 1953, Mann [1] introduced the following iterative scheme:

$$x_0 \in C \quad \text{chosen arbitrarily,}$$
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

where the sequence $\{\alpha_n\}$ is chosen in $[0, 1]$. Mann's iteration process (1.3) has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results was proved by Reich [2]. In an infinite-dimensional Hilbert space, the Mann's iteration (1.3) can conclude only weak convergence [3, 4]. In 1967, Browder and Petryshyn [5] established the first convergence result for a k-strict pseudocontraction in a real Hilbert space. They proved weak and strong convergence theorems by using (1.3) with a constant control sequence $\{\alpha_n\} \equiv \alpha$ for all n. However, this scheme has only weak convergence even in a Hilbert space. Therefore, many authors try to modify the normal Mann's iteration process to have strong convergence; see, for example, [6–10] and the references therein.

Attempts to modify (1.3) so that strong convergence is guaranteed have been made. In 2003, Nakajo and Takahashi [9] proposed the following modification of (1.3) for a single nonexpansive mapping T by using the hybrid projection method in a Hilbert space H

$$x_0 \in C \quad \text{chosen arbitrarily,}$$
$$y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$
$$C_n = \{z \in C : \|z - y_n\| \leqslant \|z - x_n\|\},$$
$$Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geqslant 0\},$$
$$x_{n+1} = P_{C_n \cap Q_n}(x), \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ defined by (1.4) converges strongly to $P_{F(T)}(x)$.

In 2007, Marino and Xu [11] proved the following strong convergence theorem by using the hybrid projection method for a strict pseudocontraction. They defined a sequence as follows:

$$x_0 \in C \quad \text{chosen arbitrarily,}$$

$$y_n = \alpha_n x_n + (1 - \alpha_n) T x_n,$$

$$C_n = \left\{ z \in C : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + (1 - \alpha_n)(k - \alpha_n)\|x_n - Tx_n\|^2 \right\}, \qquad (1.5)$$

$$Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geqslant 0 \},$$

$$x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n = 0, 1, 2, \ldots,$$

They proved that if $0 \leqslant \alpha_n < 1$, then $\{x_n\}$ defined by (1.5) converges strongly to $P_{F(T)}(x_0)$.

Motivated and inspired by the above-mentioned results, it is the purpose of this paper to improve and generalize the algorithm (1.5) to the new general process of two sequences of strictly pseudocontractive mappings in Hilbert spaces. Let C be a closed convex subset of a Hilbert space H and $T_n, S_n : C \to C$ two sequences of strictly pseudocontractive mappings such that $\bigcap_{n=0}^{\infty} F(T_n) \cap \bigcap_{n=0}^{\infty} F(S_n) \neq \emptyset$. Define $\{x_n\}$ in the following ways:

$$x_0 \in C \quad \text{chosen arbitrarily,}$$

$$y_n = \alpha_n x_n + (1 - \alpha_n) z_n,$$

$$z_n = \beta_n T_n x_n + (1 - \beta_n) S_n x_n,$$

$$\begin{aligned}
\widehat{C}_n &= \Big\{ z \in C : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + (1 - \alpha_n)\beta_n(k_T^n - \alpha_n)\|x_n - T_n x_n\|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n)(k_S^n - \alpha_n)\|x_n - S_n x_n\|^2 - (1 - \alpha_n)^2 \beta_n(1 - \beta_n)\|T_n x_n - S_n x_n\|^2 \Big\},
\end{aligned}$$

$$Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geqslant 0 \},$$

$$x_{n+1} = P_{\widehat{C}_n \cap Q_n}(x_0),$$

$$(1.6)$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$.

We prove that the algorithm (1.6) converges strongly to a common fixed point of two sequences of strictly pseudocontractive mappings $\{T_n\}$ and $\{S_n\}$ provided that $\{T_n\}, \{S_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ satisfy some appropriate conditions, and then we apply the result for finding a common element of the set of fixed points of a strict pseudocontraction and the set of solutions of an equilibrium problem in Hilbert spaces. Our results extend and improve the corresponding ones announced by Marino and Xu [11] and others.

Throughout the paper, we will use the following notation:

(i) $\to$ for strong convergence and $\rightharpoonup$ for weak convergence,

(ii) $\omega_w(x_n) = \{ x : \exists x_{n_r} \rightharpoonup x \}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

This section collects some definitions and lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Lemma 2.1. *Let H be a real Hilbert space. There holds the following identity:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$ *for all $x, y \in H$.*

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$ *for all $t \in [0,1]$, for all $x, y \in H$.*

Lemma 2.2. *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$\left\{ v \in C : \|y - v\|^2 \leqslant \|x - v\|^2 + \langle z, v \rangle + a \right\} \tag{2.1}$$

is convex (and closed).

Recall that given a closed convex subset C of a real Hilbert space H, the nearest point projection P_C from H onto C assigns to each $x \in H$ its nearest point denoted by $P_C x$ which is a unique point in C with the property

$$\|x - P_C x\| \leqslant \|x - z\| \quad \forall z \in C. \tag{2.2}$$

Lemma 2.3. *Let C be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$. Then, $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, z - y \rangle \geqslant 0 \quad \forall y \in C. \tag{2.3}$$

Lemma 2.4 (Martinez-Yanes and Xu [8]). *Let C be a closed convex subset of real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leqslant \|u - q\| \quad \forall n. \tag{2.4}$$

Then, $x_n \to q$.

Given a closed convex subset C of a real Hilbert space H and a mapping $T : C \to C$. Recall that T is said to be a quasistrict pseudocontraction if $F(T)$ is nonempty and there exists a constant $0 \leqslant k < 1$ such that

$$\|Tx - p\|^2 \leqslant \|x - p\|^2 + k\|x - Tx\|^2 \tag{2.5}$$

for all $x \in C$ and $p \in F(T)$.

Proposition 2.5 (Marino and Xu [11, Proposition 2.1]). *Assume C is a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a self-mapping of C.*

(i) *If T is a k-strict pseudocontraction, then T satisfies Lipschitz condition*

$$\|Tx - Ty\| \leqslant \frac{1+k}{1-k}\|x - y\| \quad \forall x, y \in C. \tag{2.6}$$

(ii) *If T is a k-strict pseudocontraction, then the mapping $I - T$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \hat{x}$ and $(I - T)x_n \to 0$, then $(I - T)\hat{x} = 0$.*

(iii) *If T is a k-quasistrict pseudocontraction, then the fixed-point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.6 (Plubtieng and Ungchittrakool [12, Lemma 3.1]). *Let C be a nonempty subset of a Banach space E and $\{T_n\}$ a sequence of mappings from C into E. Suppose that for any bounded subset B of C there exists continuous increasing function h_B from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $h_B(0) = 0$ and*

$$\lim_{k,l \to \infty} \rho_l^k = 0, \tag{2.7}$$

where

$$\rho_l^k := \sup\{h_B(\|T_k z - T_l z\|) : z \in B\} < \infty, \tag{2.8}$$

for all $k, l \in \mathbb{N}$. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to some point of E. Moreover, let T be a mapping from C into E defined by

$$Tx = \lim_{n \to \infty} T_n x \quad \forall x \in C. \tag{2.9}$$

Then, $\lim_{n \to \infty} \sup\{h_B(\|Tz - T_n z\|) : z \in B\} = 0$.

Lemma 2.7. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let $\{T_n\}$ be a sequence such that for each n, T_n is k^n-strict pseudo contraction from C into H with $\limsup_{n \to \infty} k^n < 1$ and*

$$Tx = \lim_{n \to \infty} T_n x \quad \forall x \in C. \tag{2.10}$$

Then, $F(T)$ is closed and convex so that the projection $P_{F(T)}$ is well defined.

Proof. To see that $F(T)$ is closed, assume that $\{p_n\}$ is a sequence in $F(T)$ such that $p_n \to \hat{p}$. Since T_n is a k^n-quasistrict pseudocontraction, we get, for each n,

$$
\begin{aligned}
\|T\hat{p} - p_n\|^2 = \|T\hat{p} - Tp_n\|^2 &= \left\| \lim_{m\to\infty} T_m\hat{p} - \lim_{m\to\infty} T_mp_n \right\|^2 = \lim_{m\to\infty} \|T_m\hat{p} - T_mp_n\|^2 \\
&\leqslant \limsup_{m\to\infty} \left(\|\hat{p} - p_n\|^2 + k^m \|(\hat{p} - T_m\hat{p}) - (p_n - T_mp_n)\|^2 \right) \\
&\leqslant \|\hat{p} - p_n\|^2 + \left(\limsup_{m\to\infty} k^m \right) \left(\limsup_{m\to\infty} \|(\hat{p} - T_m\hat{p}) - (p_n - T_mp_n)\|^2 \right) \\
&= \|\hat{p} - p_n\|^2 + \left(\limsup_{m\to\infty} k^m \right) \left\| \left(\hat{p} - \lim_{m\to\infty} T_m\hat{p}\right) - \left(p_n - \lim_{m\to\infty} T_mp_n\right) \right\|^2 \\
&= \|\hat{p} - p_n\|^2 + \left(\limsup_{m\to\infty} k^m \right) \|\hat{p} - T\hat{p}\|^2.
\end{aligned}
\tag{2.11}
$$

Taking the limit as $n \to \infty$ yields $\|T\hat{p} - \hat{p}\|^2 \leqslant \kappa \|\hat{p} - T\hat{p}\|^2$, where $\kappa := \limsup_{m\to\infty} k^m$. Since $0 \leqslant \kappa < 1$, we have $T\hat{p} = \hat{p}$. $\qquad\square$

3. Main Result

In this section, we prove a strong convergence theorem by using the hybrid projection method (some authors call this the *CQ* method) for finding a common element of the set of fixed points of two sequences of strictly pseudocontractive mappings in Hilbert spaces.

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H. For each n, let $T_n, S_n : C \to C$ be k_T^n, k_S^n-strict pseudocontractions for some $0 \leqslant k_T^n, k_S^n < 1$ with $\limsup_{n\to\infty} k_T^n, \limsup_{n\to\infty} k_S^n < 1$, respectively, and assume that $\bigcap_{n=0}^{\infty} F(T_n) \cap \bigcap_{n=0}^{\infty} F(S_n) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$x_0 \in C \quad \text{chosen arbitrarily,}$$

$$y_n = \alpha_n x_n + (1 - \alpha_n) z_n,$$

$$z_n = \beta_n T_n x_n + (1 - \beta_n) S_n x_n,$$

$$
\begin{aligned}
\widehat{C}_n = \Big\{ z \in C : &\|y_n - z\|^2 \leqslant \|x_n - z\|^2 + (1 - \alpha_n)\beta_n (k_T^n - \alpha_n) \|x_n - T_n x_n\|^2 \\
&+ (1 - \alpha_n)(1 - \beta_n)(k_S^n - \alpha_n)\|x_n - S_n x_n\|^2 - (1 - \alpha_n)^2 \beta_n (1 - \beta_n) \|T_n x_n - S_n x_n\|^2 \Big\},
\end{aligned}
$$

$$Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geqslant 0 \},$$

$$x_{n+1} = P_{\widehat{C}_n \cap Q_n}(x_0).$$

$$\tag{3.1}$$

Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $0 \leqslant \alpha_n < 1$ and $0 < a \leqslant \beta_n \leqslant b < 1$ for all n. Suppose that for any bounded subset B of C there exists an increasing, continuous, and convex function h_B from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $h_B(0) = 0$, and

$$\lim_{k,l \to \infty} \sup\{h_B(\|T_k z - T_l z\|) : z \in B\} = 0 = \lim_{k,l \to \infty} \sup\{h_B(\|S_k z - S_l z\|) : z \in B\}. \tag{3.2}$$

Let $T, S : C \to C$ such that $Tx = \lim_{n \to \infty} T_n x$ and $Sx = \lim_{n \to \infty} S_n x$ for all $x \in C$, respectively, and suppose that $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$ and $F(S) = \bigcap_{n=0}^{\infty} F(S_n)$. Then, $\{x_n\}$ converges strongly to a common fixed point $q = P_{F(T) \cap F(S)}(x_0)$.

Proof. It is not hard to check that $\widehat{C}_n$ and Q_n are closed and convex for all n (via **Lemma 2.2** and the properties of the inner product). Then, if $\widehat{C}_n \cap Q_n$ is nonempty for all n, the sequence $\{x_n\}$ is well defined. Now, we will show that $\bigcap_{n=0}^{\infty} F(T_n) \cap \bigcap_{n=0}^{\infty} F(S_n) \subset \widehat{C}_n$ for all n. Let $p \in \bigcap_{n=0}^{\infty} F(T_n) \cap \bigcap_{n=0}^{\infty} F(S_n)$, we observe that

$$\begin{aligned}
\|z_n - p\|^2 &= \|\beta_n(T_n x_n - p) + (1 - \beta_n)(S_n x_n - p)\|^2 \\
&= \beta_n \|T_n x_n - p\|^2 + (1 - \beta_n)\|S_n x_n - p\|^2 - \beta_n(1 - \beta_n)\|T_n x_n - S_n x_n\|^2 \\
&\leqslant \beta_n\left(\|x_n - p\|^2 + k_T^n \|x_n - T_n x_n\|^2\right) + (1 - \beta_n)\left(\|x_n - p\|^2 + k_S^n \|x_n - S_n x_n\|^2\right) \\
&\quad - \beta_n(1 - \beta_n)\|T_n x_n - S_n x_n\|^2 \\
&= \|x_n - p\|^2 + \beta_n k_T^n \|x_n - T_n x_n\|^2 + (1 - \beta_n) k_S^n \|x_n - S_n x_n\|^2 \\
&\quad - \beta_n(1 - \beta_n)\|T_n x_n - S_n x_n\|^2,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\|x_n - z_n\|^2 &= \|\beta_n(x_n - T_n x_n) + (1 - \beta_n)(x_n - S_n x_n)\|^2 \\
&= \beta_n \|x_n - T_n x_n\|^2 + (1 - \beta_n)\|x_n - S_n x_n\|^2 - \beta_n(1 - \beta_n)\|T_n x_n - S_n x_n\|^2.
\end{aligned} \tag{3.4}$$

By (3.3) and (3.4) we obtain

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(z_n - p)\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - z_n\|^2 \\
&\leqslant \|x_n - p\|^2 + (1 - \alpha_n)\beta_n(k_T^n - \alpha_n)\|x_n - T_n x_n\|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n)(k_S^n - \alpha_n)\|x_n - S_n x_n\|^2 \\
&\quad - (1 - \alpha_n)^2 \beta_n(1 - \beta_n)\|T_n x_n - S_n x_n\|^2.
\end{aligned} \tag{3.5}$$

Thus, we have $F(T) \cap F(S) \subset \widehat{C}_n$ for all n. Next, we will show that $F(T) \cap F(S) \subset Q_n$ for all n. If $n = 0$, then $F(T) \cap F(S) \subset C = Q_0$. Assume that $F(T) \cap F(S) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\widehat{C}_n \cap Q_n$, by Lemma 2.3 we have

$$\langle x_{n+1} - z, x_0 - x_{n+1} \rangle \geqslant 0 \quad \forall z \in \widehat{C}_n \cap Q_n. \tag{3.6}$$

Noting that $F(T) \cap F(S) \subset \widehat{C}_n \cap Q_n$ by the induction assumption, it implies that $F(T) \cap F(S) \subset Q_{n+1}$, thus by induction $F(T) \cap F(S) \subset Q_n$ for all n. Hence, $F(T) \cap F(S) \subset \widehat{C}_n \cap Q_n$ for all n. So, $\{x_n\}$ is well defined.

Notice that the definition of Q_n actually implies $x_n = P_{Q_n}(x_0)$. This together with the fact $F(T) \cap F(S) \subset Q_n$ further implies

$$\|x_n - x_0\| \leqslant \|p - x_0\| \quad \forall p \in F(T) \cap F(S). \tag{3.7}$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leqslant \|q - x_0\| \quad \forall n, \tag{3.8}$$

where $q = P_{F(T) \cap F(S)}(x_0)$.

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geqslant 0$. This together with Lemma 2.1(i) and Lemma 2.3 implies

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\
&\leqslant \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned} \tag{3.9}$$

It turns out that

$$\|x_{n+1} - x_n\| \longrightarrow 0. \tag{3.10}$$

By the fact $x_{n+1} \in \widehat{C}_n$, we get

$$\begin{aligned}
\|x_{n+1} - y_n\|^2 &\leqslant \|x_{n+1} - x_n\|^2 + (1 - \alpha_n)\beta_n(k_T^n - \alpha_n)\|x_n - T_n x_n\|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n)(k_S^n - \alpha_n)\|x_n - S_n x_n\|^2 \\
&\quad - (1 - \alpha_n)^2 \beta_n (1 - \beta_n)\|T_n x_n - S_n x_n\|^2.
\end{aligned} \tag{3.11}$$

Observe that

$$\begin{aligned}
\left\| x_{n+1} - y_n \right\|^2 &= \left\| \alpha_n(x_{n+1} - x_n) + (1 - \alpha_n)(x_n - z_n) \right\|^2 \\
&= \alpha_n \| x_{n+1} - x_n \|^2 + (1 - \alpha_n)\| x_{n+1} - z_n \|^2 - \alpha_n(1 - \alpha_n)\| x_n - z_n \|^2, \\
\left\| x_{n+1} - z_n \right\|^2 &= \left\| \beta_n(x_{n+1} - T_n x_n) + (1 - \beta_n)(x_{n+1} - S_n x_n) \right\|^2 \\
&= \beta_n \| x_{n+1} - T_n x_n \|^2 + (1 - \beta_n)\| x_{n+1} - S_n x_n \|^2 - \beta_n(1 - \beta_n)\| T_n x_n - S_n x_n \|^2.
\end{aligned} \tag{3.12}$$

With simple calculation by using (3.12) and (3.4), we have

$$\begin{aligned}
\left\| x_{n+1} - y_n \right\|^2 &= \alpha_n \| x_{n+1} - x_n \|^2 + (1 - \alpha_n)\beta_n \| x_{n+1} - T_n x_n \|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n)\| x_{n+1} - S_n x_n \|^2 - \alpha_n(1 - \alpha_n)\beta_n \| x_n - T_n x_n \|^2 \\
&\quad - \alpha_n(1 - \alpha_n)(1 - \beta_n)\| x_n - S_n x_n \|^2 - (1 - \alpha_n)^2 \beta_n(1 - \beta_n)\| T_n x_n - S_n x_n \|^2.
\end{aligned} \tag{3.13}$$

So, when we combine (3.11) and (3.13) and compute, we obtain

$$\begin{aligned}
&(1 - \alpha_n)\beta_n \| x_{n+1} - T_n x_n \|^2 + (1 - \alpha_n)(1 - \beta_n)\| x_{n+1} - S_n x_n \|^2 \\
&\quad \leqslant (1 - \alpha_n)\| x_{n+1} - x_n \|^2 + (1 - \alpha_n)\beta_n k_T^n \| x_n - T_n x_n \|^2 + (1 - \alpha_n)(1 - \beta_n)k_S^n \| x_n - S_n x_n \|^2.
\end{aligned} \tag{3.14}$$

Since $\alpha_n < 1$ for all n, we have

$$\begin{aligned}
&\beta_n \| x_{n+1} - T_n x_n \|^2 + (1 - \beta_n)\| x_{n+1} - S_n x_n \|^2 \\
&\quad \leqslant \| x_{n+1} - x_n \|^2 + \beta_n k_T^n \| x_n - T_n x_n \|^2 + (1 - \beta_n)k_S^n \| x_n - S_n x_n \|^2.
\end{aligned} \tag{3.15}$$

Notice that

$$\begin{aligned}
\left\| x_{n+1} - T_n x_n \right\|^2 &= \left\| (x_{n+1} - x_n) + (x_n - T_n x_n) \right\|^2 \\
&= \| x_{n+1} - x_n \|^2 + 2\langle x_{n+1} - x_n, x_n - T_n x_n \rangle + \| x_n - T_n x_n \|^2,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\left\| x_{n+1} - S_n x_n \right\|^2 &= \left\| (x_{n+1} - x_n) + (x_n - S_n x_n) \right\|^2 \\
&= \| x_{n+1} - x_n \|^2 + 2\langle x_{n+1} - x_n, x_n - S_n x_n \rangle + \| x_n - S_n x_n \|^2.
\end{aligned} \tag{3.17}$$

By (3.15), (3.16), and (3.17), we have

$$\begin{aligned}
&\beta_n(1 - k_T^n)\| x_n - T_n x_n \|^2 + (1 - \beta_n)(1 - k_S^n)\| x_n - S_n x_n \|^2 \\
&\quad \leqslant -2\beta_n\langle x_{n+1} - x_n, x_n - T_n x_n \rangle - 2(1 - \beta_n)\langle x_{n+1} - x_n, x_n - S_n x_n \rangle.
\end{aligned} \tag{3.18}$$

Since $\{x_n\}$, $\{T_n x_n\}$, and $\{S_n x_n\}$ are bounded, $0 < a \leqslant \beta_n \leqslant b < 1$ for all n and $\limsup_{n \to \infty} k_T^n$, $\limsup_{n \to \infty} k_S^n < 1$, it follows from (3.10) and (3.18) that

$$\lim_{n \to \infty} \|x_n - T_n x_n\| = 0 = \lim_{n \to \infty} \|x_n - S_n x_n\|. \tag{3.19}$$

Since $\{x_n\}$ is bounded, there exists a bounded subset B of C such that $\{x_n\} \subset B$. From Lemma 2.6, we are able to set $Tx = \lim_{m \to \infty} T_m x$ for all $x \in C$, and then observe that

$$\frac{1}{2}\|x_n - Tx_n\| \leqslant \frac{1}{2}\|x_n - T_n x_n\| + \frac{1}{2}\|T_n x_n - Tx_n\|. \tag{3.20}$$

Since h_B is an increasing, continuous, and convex function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $h_B(0) = 0$, we discover that

$$h_B\left(\frac{1}{2}\|x_n - Tx_n\|\right) \leqslant \frac{1}{2}h_B(\|x_n - T_n x_n\|) + \frac{1}{2}h_B(\|T_n x_n - Tx_n\|)$$
$$\leqslant \frac{1}{2}h_B(\|x_n - T_n x_n\|) + \frac{1}{2}\sup\{h_B(\|T_n z - Tz\|) : z \in B\}. \tag{3.21}$$

By Lemma 2.6 and the continuity of h_B, we have $\lim_{n \to \infty} h_B((1/2)\|x_n - Tx_n\|) = 0$. And then the properties of h_B yield

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0. \tag{3.22}$$

By the same argument, we have

$$\lim_{n \to \infty} \|x_n - Sx_n\| = 0. \tag{3.23}$$

Now Proposition 2.5 guarantees that $\omega_w(x_n) \subset F(T) \cap F(S)$. This fact, the inequality (3.8), and Lemma 2.4 ensure the strong convergence of $\{x_n\}$ to $q = P_{F(T) \cap F(S)}(x_0)$. $\qquad \square$

If $T_n = T$ and $S_n = S$ for all n, then $k_T^n = k_T$ and $k_S^n = k_S$ for all n. So, Theorem 3.1 reduces to the following corollary.

Corollary 3.2. *Let C be a closed convex subset of a Hilbert space H. Let $T, S : C \to C$ be k_T, k_S-strict pseudocontractions for some $0 \leqslant k_T, k_S < 1$, respectively, and assume that $F(T) \cap F(S) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$x_0 \in C \quad \text{chosen arbitrarily},$$

$$y_n = \alpha_n x_n + (1 - \alpha_n) z_n,$$

$$z_n = \beta_n T x_n + (1 - \beta_n) S x_n,$$

$$
\begin{aligned}
\widehat{C}_n = \Big\{ z \in C : \|y_n - z\|^2 &\leqslant \|x_n - z\|^2 + (1 - \alpha_n)\beta_n(k_T - \alpha_n)\|x_n - Tx_n\|^2 \\
&+ (1 - \alpha_n)(1 - \beta_n)(k_S - \alpha_n)\|x_n - Sx_n\|^2 - (1 - \alpha_n)^2\beta_n(1 - \beta_n)\|Tx_n - Sx_n\|^2 \Big\},
\end{aligned}
\tag{3.24}
$$

$$Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geqslant 0\},$$

$$x_{n+1} = P_{\widehat{C}_n \cap Q_n}(x_0),$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 3.1. Then, $\{x_n\}$ converges strongly to a common fixed point $q = P_{F(T) \cap F(S)}(x_0)$.

In particular, if $T = S$, then $z_n = Tx_n$ and

$$
\begin{aligned}
\widehat{C}_n = \Big\{ z \in C : \|y_n - p\|^2 &\leqslant \|x_n - p\|^2 + (1 - \alpha_n)\beta_n(k_T - \alpha_n)\|x_n - Tx_n\|^2 \\
&+ (1 - \alpha_n)(1 - \beta_n)(k_T - \alpha_n)\|x_n - Tx_n\|^2 - (1 - \alpha_n)^2\beta_n(1 - \beta_n)\|Tx_n - Tx_n\|^2 \Big\}
\end{aligned}
\tag{3.25}
$$

$$= C_n.$$

So, Corollary 3.2 reduces to the following corollary.

Corollary 3.3 (Marino and Xu [11, Theorem 4.1]). *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a k-strict pseudocontraction for some $0 \leqslant k < 1$, and assume that the fixed-point set $F(T) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$x_0 \in C \quad \text{chosen arbitrarily},$$

$$y_n = \alpha_n x_n + (1 - \alpha_n) T x_n,$$

$$C_n = \Big\{ z \in C : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + (1 - \alpha_n)(k - \alpha_n)\|x_n - Tx_n\|^2 \Big\}
\tag{3.26}$$

$$Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geqslant 0\},$$

$$x_{n+1} = P_{C_n \cap Q_n}(x_0).$$

Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $0 \leqslant \alpha_n < 1$ for all n. Then, $\{x_n\}$ converges strongly to a fixed-point $q = P_{F(T)}(x_0)$.

If $T_n = T$ for all n and $\{S_n\}$ is a sequences of nonexpansive mappings, then $k_T^n = k$ and $k_S^n = 0$ for all n. So, Theorem 3.1 reduces to the following corollary.

Corollary 3.4. *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a k-strict pseudocontraction for some $0 \leqslant k < 1$, for each n and $S_n : C \to C$ a nonexpansive mapping, and assume that $F(T) \cap \bigcap_{n=0}^{\infty} F(S_n) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$
\begin{aligned}
x_0 &\in C \quad \text{chosen arbitrarily,} \\
y_n &= \alpha_n x_n + (1 - \alpha_n) z_n, \\
z_n &= \beta_n T x_n + (1 - \beta_n) S_n x_n, \\
\widehat{C}_n &= \Big\{ z \in C : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + (1 - \alpha_n)\beta_n(k - \alpha_n)\|x_n - Tx_n\|^2 \\
&\quad - \alpha_n(1 - \alpha_n)(1 - \beta_n)\|x_n - S_n x_n\|^2 - (1 - \alpha_n)^2 \beta_n (1 - \beta_n)\|Tx_n - S_n x_n\|^2 \Big\}, \\
Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geqslant 0 \}, \\
x_{n+1} &= P_{\widehat{C}_n \cap Q_n}(x_0),
\end{aligned}
\tag{3.27}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 3.1. Suppose that for any bounded subset B of C there exists an increasing, continuous, and convex function h_B from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $h_B(0) = 0$, and

$$
\lim_{k,l \to \infty} \sup\{ h_B(\|S_k z - S_l z\|) : z \in B \} = 0.
\tag{3.28}
$$

Let $S : C \to C$ be such that $Sx = \lim_{n \to \infty} S_n x$ for all $x \in C$, and suppose $F(S) = \bigcap_{n=0}^{\infty} F(S_n)$. Then, $\{x_n\}$ converges strongly to a common fixed point $q = P_{F(T) \cap F(S)}(x_0)$.

4. Equilibrium Problem

In this section, we have an application of the main result for finding a common element of the set of fixed points of a strict pseudocontraction and the set of solutions of an equilibrium problem.

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let φ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\varphi : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$
\varphi(x, y) \geqslant 0 \quad \forall y \in C.
\tag{4.1}
$$

The set of solution of (4.1) is denoted by $\mathrm{EP}(\varphi)(= \{x \in C : \varphi(x, y) \geqslant 0 \text{ for all } y \in C\})$. Many problems in physics, optimization, and economics reduce to find some elements of $\mathrm{EP}(\varphi)$.

For solving the equilibrium problem for a bifunction $\varphi : C \times C \to \mathbb{R}$, let us assume that φ satisfies the following conditions:

(A1) $\varphi(x, x) = 0$ for all $x \in C$;

(A2) φ is monotone, that is, $\varphi(x, y) + \varphi(y, x) \leqslant 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \varphi(tz + (1 - t)x, y) \leqslant \varphi(x, y); \tag{4.2}$$

(A4) for each $x \in C$, $y \mapsto \varphi(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [13].

Lemma 4.1 (Blum and Oettli [13]). *Let C be a nonempty closed convex subset of H, and let φ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\varphi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geqslant 0 \quad \forall y \in C. \tag{4.3}$$

The following lemma was also given in [14].

Lemma 4.2 (Combettes and Hirstoaga [14]). *Assume that $\varphi : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r(x) = \left\{ z \in C : \varphi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geqslant 0, \ \forall y \in C \right\} \tag{4.4}$$

for all $z \in H$. Then, the following hold:

(1) S_r *is single-valued;*

(2) S_r *is firmly nonexpansive, that is, for any $x, y \in H$, $\|S_r x - S_r y\|^2 \leqslant \langle S_r x - S_r y, x - y \rangle$;*

(3) $F(S_r) = EP(\varphi)$;

(4) $EP(\varphi)$ *is closed and convex.*

The following corollary is an application of Corollary 3.4 in the case of finding a common element of the set of fixed points of a strict pseudocontraction and the set of solutions of an equilibrium problem.

Corollary 4.3. *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a k-strict pseudocontraction for some $0 \leqslant k < 1$ and φ a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4).*

Suppose that $F(T) \cap EP(\varphi) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by

$$x_0 \in C \quad \text{chosen arbitrarily,}$$

$$y_n = \alpha_n x_n + (1 - \alpha_n) z_n,$$

$$z_n = \beta_n T x_n + (1 - \beta_n) u_n,$$

$$u_n \in C \quad \text{such that} \quad \varphi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geqslant 0, \ \forall y \in C,$$

$$\widehat{C}_n = \left\{ z \in C : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + (1 - \alpha_n) \beta_n (k - \alpha_n) \|x_n - T x_n\|^2 \right.$$
$$\left. - \alpha_n (1 - \alpha_n)(1 - \beta_n) \|x_n - u_n\|^2 - (1 - \alpha_n)^2 \beta_n (1 - \beta_n) \|T x_n - u_n\|^2 \right\},$$

$$Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geqslant 0 \},$$

$$x_{n+1} = P_{\widehat{C}_n \cap Q_n}(x_0),$$

(4.5)

where $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 3.1 and $\{r_n\}_{n=0}^{\infty}$ is chosen so that $\{r_n\} \subset (0, \infty)$ with $\inf_n r_n > 0$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ converges strongly to a common fixed point $q = P_{F(T) \cap EP(\varphi)}(x_0)$.

Proof. Obviously, $u_n = S_{r_n} x_n$, where S_{r_n} are mappings as in Lemma 4.2. Next, we want to show that for any bounded subset B of C there exists an increasing, continuous, and convex function $h_B(= (\cdot)^2)$ from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $h_B(0) = 0$, and

$$\lim_{k,l \to \infty} \sup \{ h_B(\|S_{r_k} v - S_{r_l} v\|) : v \in B \} = 0.$$

(4.6)

Let B be a bounded subset of C. For each $v \in B$, let $v_n = S_{r_n} v$. Then, by Lemma 4.2, we have

$$\varphi(v_l, y) + \frac{1}{r_l} \langle y - v_l, v_l - v \rangle \geqslant 0 \quad \forall y \in C,$$

(4.7)

$$\varphi(v_k, y) + \frac{1}{r_k} \langle y - v_k, v_k - v \rangle \geqslant 0 \quad \forall y \in C.$$

(4.8)

Put $y = v_k$ in (4.7) and $y = v_l$ in (4.8), we have

$$\varphi(v_l, v_k) + \frac{1}{r_l} \langle v_k - v_l, v_l - v \rangle \geqslant 0, \qquad \varphi(v_k, v_l) + \frac{1}{r_k} \langle v_l - v_k, v_k - v \rangle \geqslant 0.$$

(4.9)

So, from (A2), we have $\langle v_k - v_l, (v_l - v)/r_l - (v_k - v)/r_k \rangle \geqslant 0$ and hence $\langle v_k - v_l, v_l - v - (r_l/r_k)(v_k - v) \rangle \geqslant 0$. Thus,

$$\langle v_k - v_l, v_l - v_k \rangle + \left\langle v_k - v_l, \left(1 - \frac{r_l}{r_k}\right)(v_k - v) \right\rangle \geqslant 0.$$

(4.10)

Let $b := \inf_n r_n$. Thus, we have

$$
\begin{aligned}
(\|S_{r_k} v - S_{r_l} v\|)^2 &= \langle v_k - v_l, v_k - v_l \rangle \\
&\leqslant \left\langle v_k - v_l, \left(1 - \frac{r_l}{r_k}\right)(v_k - v) \right\rangle \\
&\leqslant \|v_k - v_l\| \frac{1}{b} |r_k - r_l| \|v_k - v\| \\
&= \frac{1}{b} \|S_{r_k} v - S_{r_l} v\| \|S_{r_k} v - v\| |r_k - r_l| \\
&\leqslant \frac{4}{b} \|v - p\| |r_k - r_l| \quad \text{where } p \in \text{EP}(\varphi) \\
&\leqslant \frac{4}{b} M |r_k - r_l| \quad \text{where } M := \sup\{\|v - p\| : v \in B\}.
\end{aligned}
\tag{4.11}
$$

Put $k > l$. Observe that

$$
(\|S_{r_k} v - S_{r_l} v\|)^2 \leqslant \frac{4}{b} M |r_k - r_l| \leqslant \frac{4}{b} M \sum_{n=l}^{k-1} |r_{n+1} - r_n| \leqslant \frac{4}{b} M \sum_{n=l}^{\infty} |r_{n+1} - r_n| < \infty,
\tag{4.12}
$$

for all $v \in B$, and then

$$
\rho_l^k = \sup\left\{ (\|S_{r_k} v - S_{r_l} v\|)^2 : v \in B \right\} \leqslant \frac{4}{b} M \sum_{n=l}^{\infty} |r_{n+1} - r_n| < \infty.
\tag{4.13}
$$

Let $l \to \infty$, we have $\lim_{k,l \to \infty} \rho_l^k = 0$. Then, by Lemma 2.6, we can define a mapping S by

$$
Sx = \lim_{n \to \infty} S_{r_n} x \quad \forall x \in C.
\tag{4.14}
$$

Next, we will show that

$$
F(S) = \bigcap_{n=0}^{\infty} F(S_{r_n}) = \text{EP}(\varphi).
\tag{4.15}
$$

Since $\{r_n\} \subset (0, \infty)$, $\inf_n r_n > 0$ and (4.14), it is easy to see that

$$
\text{EP}(\varphi) = \bigcap_{n=0}^{\infty} F(S_{r_n}) \subset F(S).
\tag{4.16}
$$

Let $p \in F(S)$. By the definition of S_r, we have

$$
\varphi(S_{r_n} p, y) + \frac{1}{r_n} \langle y - S_{r_n} p, S_{r_n} p - p \rangle \geqslant 0 \quad \forall y \in C.
\tag{4.17}
$$

By (A2), we have

$$\frac{1}{r_n}\langle y - S_{r_n}p, S_{r_n}p - p\rangle \geqslant \varphi(y, S_{r_n}p) \quad \forall y \in C. \tag{4.18}$$

From (4.14) and the lower semicontinuity of $\varphi(y, \cdot)$, we have $0 \geqslant \varphi(y, p)$ for all $y \in C$. Let $y \in C$ and set $x_t = ty + (1 - t)p$, for $t \in (0, 1]$. Then, we have

$$0 = \varphi(x_t, x_t) \leqslant t\varphi(x_t, y) + (1 - t)\varphi(x_t, p) \leqslant t\varphi(x_t, y). \tag{4.19}$$

So $\varphi(x_t, y) \geqslant 0$. Letting $t \downarrow 0$ and using (A3), we get $\varphi(p, y) \geqslant 0$ for all $y \in C$, and hence $p \in \mathrm{EP}(\varphi) = \bigcap_{n=1}^{\infty} F(S_{r_n})$. Hence, we have (4.15). Then, applying Corollary 3.4, $x_n \to q = P_{F(T) \cap \mathrm{EP}(\varphi)}(x_0)$. $\qquad\square$

Acknowledgments

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