Research Article

Existence and Asymptotic Behavior of Boundary Blow-Up Solutions for Weighted $p(x)$-Laplacian Equations with Exponential Nonlinearities

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This paper investigates the following $p(x)$-Laplacian equations with exponential nonlinearities: $-\Delta_{p(x)}u + \rho(x)e^{\beta(x,u)} = 0$ in $\Omega$, $u(x) \to +\infty$ as $d(x, \partial \Omega) \to 0$, where $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2} \nabla u)$ is called $p(x)$-Laplacian, $\rho(x) \in C(\Omega)$. The asymptotic behavior of boundary blow-up solutions is discussed, and the existence of boundary blow-up solutions is given.

1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. On the background of this class of problems, we refer to [1–3]. Many results have been obtained on this kind of problems, for example, [4–18]. On the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions, we refer to [4, 5, 8]. On the existence of solutions for $p(x)$-Laplacian equation Dirichlet problems in bounded domain, we refer to [7, 9, 15, 18]. In this paper, we consider the following $p(x)$-Laplacian equations with exponential nonlinearities

$$-\Delta_{p(x)}u + \rho(x)e^{\beta(x,u)} = 0, \quad \text{in } \Omega,$$

$$u(x) \to +\infty, \quad \text{as } d(x, \partial \Omega) \to 0,$$

(P)
where $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2} \nabla u)$ and $\Omega = B(0,R) \subset \mathbb{R}^N$ is a bounded radial domain $(B(0,R) = \{ x \in \mathbb{R}^N \mid |x| < R \})$. Our aim is to give the asymptotic behavior and the existence of boundary blow-up solutions for problem (P).

Throughout the paper, we assume that $p(x), \rho(x)$, and $f(x,u)$ satisfy the following.

(H1) $p(x) \in C^1(\overline{\Omega})$ is radial and satisfies

$$1 < p^- \leq p^+ < +\infty, \quad \text{where } p^- = \inf_{\Omega} p(x), \quad p^+ = \sup_{\Omega} p(x). \quad (1.1)$$

(H2) $f(x,u)$ is radial with respect to $x$, $f(x,\cdot)$ is increasing, and $f(x,0) = 0$ for any $x \in \Omega$.

(H3) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|f(x,t)| \leq C_1 + C_2|t|^\gamma(x), \quad \forall (x,t) \in \Omega \times \mathbb{R}, \quad (1.2)$$

where $C_1, C_2$ are positive constants and $0 \leq \gamma \in C(\overline{\Omega})$.

(H4) $\rho(x) \in C(\Omega)$ is a radial nonnegative function, and there exists a constant $\sigma \in [R/2,R)$ such that

$$\rho_0(R-r)^{-\beta(r)} \leq \rho(r) \leq \rho_1(R-r)^{-\beta_1(r)} \quad \text{for } r \in [\sigma,R) \text{ uniformly}, \quad (1.3)$$

where $\rho_0$ and $\rho_1$ are positive constants and $\beta(r)$ and $\beta_1(r)$ are Lipschitz continuous on $[\sigma,R]$, which satisfy $\beta(r) \leq \beta_1(r) < p(r)$ for any $r \in [\sigma,R]$.

The operator $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2} \nabla u)$ is called $p(x)$-Laplacian. Specifically, if $p(x) \equiv p$ (a constant), (P) is the well-known $p$-Laplacian problem. If $f(x,u)$ can be represented as $h(x)f(u)$, on the boundary blow-up solutions for the following $p$-Laplacian equations ($p$ is a constant):

$$-\Delta_p u + h(x)f(u) = 0, \quad \text{in } \Omega, \quad (1.4)$$

we refer to [19–26], and the following generalized Keller-Osserman condition is crucial

$$\int_1^\infty \frac{1}{(F(t))^{1/p}} dt < +\infty, \quad \text{where } F(t) = \int_0^t f(s)ds, \quad (1.5)$$

but the typical form of $p(x)$-Laplacian equation is

$$-\Delta_{p(x)} u + |u|^{p(x)-2} u = 0, \quad \text{in } \Omega, \quad (1.6)$$

and there are some differences between the results of (1.4) and (1.6) (see [16]).

On the boundary blow-up solutions for the following $p$-Laplacian equations with exponential nonlinearities ($p$ is a constant):

$$-\Delta_p u + e^{h(x)f(u)} = 0, \quad \text{in } \Omega, \quad (1.7)$$
we refer to [20–22], but the results on the boundary blow-up solutions for \( p(x) \)-Laplacian equations are rare (see [16]).

In [16], the present author discussed the existence and asymptotic behavior of boundary blow-up solutions for the following \( p(x) \)-Laplacian equations:

\[
-\Delta_{p(x)} u + f(x, u) = 0, \quad \text{in } \Omega, \\
u(x) \to +\infty, \quad \text{as } d(x, \partial \Omega) \to 0,
\]

on the condition that \( f(x, \cdot) \) satisfies polynomial growth condition.

If \( p(x) \) is a function, the typical form of \( (P) \) is the following:

\[
-\Delta_{p(x)} u + p(x) e^{u^{p(x)-2}u} = 0,
\]

and the method to construct subsolution and supersolution in [16] cannot give the exact asymptotic behavior of solutions for \( (P) \). Our results partially generalized the results of [20–22].

Because of the nonhomogeneity of \( p(x) \)-Laplacian, \( p(x) \)-Laplacian problems are more complicated than those of \( p \)-Laplacian ones (see [10]); another difficulty of this paper is that \( f(x, u) \) cannot be represented as \( h(x) f(u) \).

2. Preliminary

In order to deal with \( p(x) \)-Laplacian problems, we need some theories on the spaces \( L^{p(x)}(\Omega) \), \( W^{1,p(x)}(\Omega) \) and properties of \( p(x) \)-Laplacian, which we will use later (see [6, 11]). Let

\[
L^{p(x)}(\Omega) = \left\{ u \mid u \text{ is a measurable real-valued function, } \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \right\}.
\]

We can introduce the norm on \( L^{p(x)}(\Omega) \) by

\[
|u|_{p(x)} = \inf \left\{ \lambda > 0 \mid \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}.
\]

The space \( (L^{p(x)}(\Omega), |\cdot|_{p(x)}) \) becomes a Banach space. We call it generalized Lebesgue space. The space \( (L^{p(x)}(\Omega), |\cdot|_{p(x)}) \) is a separable, reflexive, and uniform convex Banach space (see [6, Theorems 1.10, 1.14]).

The space \( W^{1,p(x)}(\Omega) \) is defined by

\[
W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega) \right\}.
\]
and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega). \quad (2.4)$$

$W^{1,p(x)}_0(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are separable, reflexive, and uniform convex Banach spaces (see [6, Theorem 2.1]).

If $u \in W^{1,p(x)}_0(\Omega) \cap C(\Omega)$, $u$ is called a blow-up solution of (P) when it satisfies

$$\int_Q |\nabla u|^{p(x)-2}\nabla u \nabla q \, dx + \int_Q \rho(x)f(x,u)q \, dx = 0, \quad \forall q \in W^{1,p(x)}_0(Q), \quad (2.5)$$

for any domain $Q \subseteq \Omega$, and $\max(k-u,0) \in W^{1,p(x)}_0(\Omega)$ for every positive integer $k$.

Let $W^{1,p(x)}_{0,\text{loc}}(\Omega) = \{ u \mid$ there is an open domain $Q \subseteq \Omega$ such that $u \in W^{1,p(x)}_0(Q) \}$, and define $A : W^{1,p(x)}_{0,\text{loc}}(\Omega) \cap C(\Omega) \to (W^{1,p(x)}_0(\Omega))^*$ as

$$\langle Au, \varphi \rangle = \int_{\Omega} (|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + \rho(x)f(x,u)\varphi) \, dx, \quad \forall u \in W^{1,p(x)}_{0,\text{loc}}(\Omega) \cap C(\Omega), \forall \varphi \in W^{1,p(x)}_{0,\text{loc}}(\Omega). \quad (2.6)$$

**Lemma 2.1** (see [9, Theorem 3.1]). Let $h \in W^{1,p(x)}(\Omega) \cap C(\Omega)$, and $X = h + W^{1,p(x)}_0(\Omega) \cap C(\Omega)$. Then, $A : X \to (W^{1,p(x)}_0(\Omega))^*$ is strictly monotone.

Letting $g \in (W^{1,p(x)}_{0,\text{loc}}(\Omega))^*$, if $\langle g, \varphi \rangle \geq 0$, for all $\varphi \in W^{1,p(x)}_{0,\text{loc}}(\Omega)$ with $\varphi \geq 0$ a.e. in $\Omega$, then denote $g \geq 0$ in $(W^{1,p(x)}_{0,\text{loc}}(\Omega))^*$; correspondingly, if $-g \geq 0$ in $(W^{1,p(x)}_{0,\text{loc}}(\Omega))^*$, then denote $g \leq 0$ in $(W^{1,p(x)}_{0,\text{loc}}(\Omega))^*$.

**Definition 2.2.** Let $u \in W^{1,p(x)}_{0,\text{loc}}(\Omega) \cap C(\Omega)$. If $Au \geq 0$ $(Au \leq 0)$ in $(W^{1,p(x)}_{0,\text{loc}}(\Omega))^*$, then $u$ is called a weak supersolution (weak subsolution) of (P).

Copying the proof of [14], we have the following.

**Lemma 2.3** (comparison principle). Let $u, v \in W^{1,p(x)}_{0,\text{loc}}(\Omega) \cap C(\Omega)$ satisfy

$$Au - Av \geq 0, \quad in (W^{1,p(x)}_0(\Omega))^*. \quad (2.7)$$

Let $\varphi(x) = \min\{ u(x) - v(x), 0 \}$. If $\varphi(x) \in W^{1,p(x)}_0(\Omega)$ (i.e., $u \geq v$ on $\partial \Omega$), then $u \geq v$ a.e. in $\Omega$.

**Lemma 2.4** (see [8, Theorem 1.1]). Under the conditions (H1) and (H3), if $u \in W^{1,p(x)}(\Omega)$ is a bounded weak solution of $-\Delta_{p(x)}u + \rho(x)e^{f(x,u)} = 0$ in $\Omega$, then $u \in C^{1,\delta}_{\text{loc}}(\Omega)$, where $\delta \in (0,1)$ is a constant.
3. Asymptotic Behavior of Boundary Blow-Up Solutions

If \( u \) is a radial solution for (P), then (P) can be transformed into

\[
\left( r^{N-1} |u'|^{p(r)-2} u' \right)' = r^{N-1} \rho(r) e^{f(r,u)}, \quad r \in (0, R),
\]

\[
u(0) = u_0, \quad u'(0) = 0, \quad u'(r) \geq 0, \quad \text{for } 0 < r < R.
\]  

It means that \( u(r) \) is increasing.

**Theorem 3.1.** If \( f(r,u) \) satisfies

\[
f(r,u) \geq au^s \quad \text{(as } u \to +\infty) \text{ for } r \in [\sigma, R) \text{ uniformly,}
\]

where \( \sigma \) is defined in \((H_4)\) and \( a \) and \( s \) are positive constants, then there exists a supersolution \( \Phi_1(x) \) which satisfies \( \Phi_1(x) \to +\infty \) (as \( d(x, \partial \Omega) \to 0 \)), such that for every solution \( u \) of problem (P), one has \( u(x) \leq \Phi_1(x) \).

**Proof.** Define the function \( g(r,a,\lambda) \) on \([0, R_1)\) as

\[
g(r,a,\lambda) = \begin{cases} 
\left( a \ln \frac{1}{(R-r)^{1-\theta} - \lambda} \right)^{1/s} + k, & R_0 \leq r < R_1, \\
& \\
k - \int_{R_0}^{r} \left[ a^\frac{1}{s} (1-\theta) (R-R_0)^{-\theta} \left( \ln \frac{1}{(R-R_0)^{1-\theta} - \lambda} \right)^{(1/s)-1} \right]^{(p(R_0)-1)/(p(t)-1)} \\
& \times \left[ \frac{(R_0)^{N-1}}{t^{N-1}} \sin \epsilon(t-\sigma) \right]^{1/(p(t)-1)} dt, & \sigma < r < R_0, \\
& \\
k - \int_{\sigma}^{R_0} \left[ a^\frac{1}{s} (1-\theta) (R-R_0)^{-\theta} \left( \ln \frac{1}{(R-R_0)^{1-\theta} - \lambda} \right)^{(1/s)-1} \right]^{(p(R_0)-1)/(p(t)-1)} \\
& \times \left[ \frac{(R_0)^{N-1}}{t^{N-1}} \sin \epsilon(t-\sigma) \right]^{1/(p(t)-1)} dt, & r \leq \sigma,
\end{cases}
\]

(3.3)
where $\theta < \beta(R)/p(R)$, $a > (1/\alpha)\sup_{|x| \leq R} p(x)$ are constants, $R_0 \in (\sigma, R)$, $R - R_0$ is small enough, parameter $\lambda \in [0, (R - R_0)^{1-\theta}/2]$, $R_1$ satisfies $(R - R_1)^{1-\theta} - \lambda = 0$, $\varepsilon = \pi/2(R_0 - \sigma)$

\[
k = \left[ \frac{2p^+((1 + s)/s + 1/(1 - \theta)) + (\beta^+)/(1 - \theta)}{a} \ln \frac{2}{(R - R_0)^{(1 - \theta)}} \right]^{1/s}
\]

\[
+ \int_{\sigma}^{R_0} \left[ \frac{2a^{1/s}(1 - \theta)}{s(R - R_0)} \left( \ln \frac{2}{(R - R_0)^{(1 - \theta)}} \right)^{(1/s) - 1} \right]^{(p(R_0) - 1)/(p(r) - 1)} \left( \frac{R - r}{R - R_0} \right)^{1-\theta} \frac{(R - r)^{1-\theta}}{\ln \left( \frac{1}{(R - R_0)^{(1 - \theta)}} \right)} \sin \left( \frac{\pi}{2(R_0 - \sigma)} \right) \left( \frac{R - r}{R - R_0} \right)^{1-\theta} \frac{(R - r)^{1-\theta}}{\ln \left( \frac{1}{(R - R_0)^{(1 - \theta)}} \right)} \sin \left( \frac{\pi}{2(R_0 - \sigma)} \right) \frac{(R - r)^{1-\theta}}{\ln \left( \frac{1}{(R - R_0)^{(1 - \theta)}} \right)} \sin \left( \frac{\pi}{2(R_0 - \sigma)} \right)
\]

(3.4)

Obviously, for any positive constant $a$, we have $g(r, a, \lambda) \in C^1[0, R_1]$. When $R_0 < r < R_1 < R$, we have

\[
g' = \frac{a^{1/s} \left( \ln \frac{1}{(R - r)^{1-\theta} - \lambda} \right)^{(1/s) - 1} (1 - \theta)(R - r)^{-\theta}}{(R - r)^{-\theta} - \lambda},
\]

\[
|g'|^{p(r)-2} g' = \left[ \frac{(1 - \theta)a^{1/s} s}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R - r)^{1-\theta} - \lambda} \right)^{(1/s) - 1} \frac{(R - r)^{-\theta} - \lambda}{(R - r)^{-\theta} - \lambda}^{p(r)-1},
\]

\[
(p^{-1} |g'|^{p(r)-2} g')' = p^{-1} \left[ \frac{(1 - \theta)a^{1/s} s}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R - r)^{1-\theta} - \lambda} \right)^{(1/s) - 1} \frac{(R - r)^{-\theta} - \lambda}{(R - r)^{-\theta} - \lambda}^{p(r)-1}
\times \frac{(p(r) - 1)(R - r)^{-\theta} - \lambda}{(R - r)^{1-\theta} - \lambda}^{p(r)} \left[ (1 - \theta) + \Pi(r) \right],
\]

(3.5)

where

\[
\Pi(r) = \frac{r^{-1} \left[ (1 - \theta)a^{1/s} s \right]^{p(r)-1}}{(p(r) - 1)r^{-1} \left[ (1 - \theta)a^{1/s} s \right]^{p(r)-1} (R - r)^{1-\theta} - \lambda (R - r) + (1/s - 1)(1 - \theta)} \left( \ln \frac{1}{(R - r)^{1-\theta} - \lambda} \right)
\]

\[
+ \frac{\frac{R - r}{(R - r)^{1-\theta} - \lambda} (R - r) \left( \frac{1/s - 1}{p(r) - 1} \right) \ln \frac{1}{(R - r)^{1-\theta} - \lambda}}{\frac{R - r}{(R - r)^{1-\theta} - \lambda} (R - r) \ln \frac{1}{(R - r)}}
\]

\[
+ \frac{\frac{R - r}{(R - r)^{1-\theta} - \lambda} (R - r) \ln \frac{1}{(R - r)}}{\frac{R - r}{(R - r)^{1-\theta} - \lambda} (R - r) \ln \frac{1}{(R - r)}}
\]

\[
+ \frac{\frac{R - r}{(R - r)^{1-\theta} - \lambda} (R - r) \ln \frac{1}{(R - r)}}{\frac{R - r}{(R - r)^{1-\theta} - \lambda} (R - r) \ln \frac{1}{(R - r)}}
\]

(3.6)
If $(R - R_0)$ is small enough, it is easy to see that

$$\left| \Pi(r) \right| \leq \ln \frac{1}{(R-r)^{1-\theta} - \lambda}, \quad \text{for } \lambda \in \left[ 0, \frac{(R-R_0)^{1-\theta}}{2} \right] \text{ uniformly,} \tag{3.7}$$

and then

$$\left( r^{N-1} |g'|^{p(r)-2} g' \right)' \leq r^{N-1} \left[ \frac{(1-\theta)a^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} - \lambda} \right)^{(1/s)-1}(p(r)-1)+1 \times \frac{(p(r)-1)(R-r)^{-\theta p(r)}}{[(R-r)^{1-\theta} - \lambda]^{p(r)}} \quad \forall r \in (R_0, R_1). \tag{3.8}$$

Thus, when $0 < R - R_0$ is small enough, from (3.5) and (3.8), for $\lambda \in [0, (R-R_0)^{1-\theta}/2]$ uniformly, we have

$$\left( r^{N-1} |g'|^{p(r)-2} g' \right)' \leq 2r^{N-1} \left[ \frac{(1-\theta)a^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} - \lambda} \right)^{(1/s)-1}(p(r)-1)+1 \times \frac{(p(r)-1)(R-r)^{-\theta p(r)}}{[(R-r)^{1-\theta} - \lambda]^{p(r)}} \leq r^{N-1} \rho(r) \left( \frac{1}{(R-r)^{1-\theta} - \lambda} \right)^a = r^{N-1} \rho(r) e^{a \delta} \leq r^{N-1} \rho(r) e^{f(r, g)}, \quad \forall r \in (R_0, R_1). \tag{3.9}$$

Thus, when $0 < R - R_0$ is small enough, the following inequality is valid for $\lambda \in [0, (R-R_0)^{1-\theta}/2]$ uniformly:

$$\left( r^{N-1} |g'|^{p(r)-2} g' \right)' \leq r^{N-1} \rho(r) f(r, g), \quad \forall r \in (R_0, R_1). \tag{3.10}$$
Obviously, if \( R - R_0 \) is small enough, then \( g \geq \left[ (2p^+((s + 1)/s + 1/(1 - \theta)) + |\beta|^{-1}/(1 - \theta) \right) \ln(2/(R - R_0)^{1-\theta}) \left]^{1/s} \) is large enough. Since \( \lambda \in [0, (R - R_0)^{1-\theta}/2] \),

\[
\left( r^{N-1} |g' p(r) g' \right)'
= \varepsilon(R_0)^{N-1} \left[ \frac{a^{1/s}(1 - \theta)(R - R_0)^{-\theta}}{s(R - R_0)} \left( \ln \left( \frac{1}{(R - R_0)^{1-\theta} - \lambda} \right) \right)^{(1/s) - 1} \right]^{(p(R_0) - 1)} \cos(\varepsilon(r - \sigma))
\leq \varepsilon(R_0)^{N-1} \left[ \frac{a^{1/s}(1 - \theta)(R - R_0)^{-\theta}}{s(1/2)(R - R_0)^{1-\sigma}} \left( \ln \left( \frac{2}{(R - R_0)^{1-\theta}} \right) \right)^{(1/s) + 1} \right]^{(p(R_0) - 1)}
\leq \varepsilon(R_0)^{N-1} \left[ \frac{2a^{1/s}(1 - \theta)}{s} \left( \frac{2}{R - R_0} \right)^{(1/s) + 1} \right]^{(p(R_0) - 1)}
\leq \varepsilon(R_0)^{N-1} \left[ \frac{2a^{1/s}(1 - \theta)}{s} \left( \frac{2}{R - R_0} \right) \right]^{(s+1)/(s)(1-\theta)+1} \leq r^{N-1} \rho(r) e^{f(r,s)} , \quad \sigma < r < R_0.
\]

Thus,

\[
\left( r^{N-1} |g' p(r) g' \right)'
\leq r^{N-1} \rho(r) e^{f(r,s)} , \quad \sigma < r < R_0.
\]

(3.11)

Obviously,

\[
\left( r^{N-1} |g' p(r) g' \right)'
= 0 \leq r^{N-1} \rho(r) e^{f(r,s)} , \quad 0 \leq r < \sigma.
\]

(3.13)

Since \( g(x, a, \lambda) = g(|x|, a, \lambda) \) is a \( C^1 \) function on \( B(0, R_1) \), if \( 0 < R - R_0 \) is small enough \( (R_0 \text{ depends on } R, p, s, a) \), from (3.10), (3.12), and (3.13), for any \( \lambda \in [0, (R - R_0)^{1-\theta}/2] \), we can see that \( g(|x|, a, \lambda) \) is a supersolution for (P) on \( B(0, R_1) \), and then \( g(|x|, a, 0) \) is a supersolution for (P).

Defining the function \( g_m(|x|, a - e) = g(r, a - e, 1/m) \) on \( [0, R_{1/m}] \), where \( a - e > (1/\alpha) \sup_{|x| \leq R} p(x) \), then \( g_m(|x|, a - e) \) is a supersolution for (P) on \( B(0, R - (1/m)) \). If \( u \) is a solution for (P), according to the comparison principle, we get that \( g_m(|x|, a - e) \geq u(x) \) for any \( x \in B(0, R_{1/m}) \). For any \( x \in B(0, R) \setminus B(0, R_0) \), we have \( g_m(|x|, a - e) \geq g_{m+1}(|x|, a - e) \), when \( m \) is large enough. Thus

\[
u(x) \leq \lim_{m \to +\infty} g_m(|x|, a - e), \quad \forall x \in B(0, R) \setminus B(0, R_0).
\]

(3.14)
When \( d(x, \partial \Omega) > 0 \) is small enough, we have
\[
\lim_{m \to +\infty} g_m(|x|, a - e) < \left( a \ln \frac{1}{(R - r)^{1 - \sigma}} \right)^{1/s} + k \leq g(|x|, a, 0). \tag{3.15}
\]

According to the comparison principle, we get that \( g(|x|, a, 0) \geq u(x) \), for all \( x \in B(0, R) \); then \( \Phi_1(x) = \Phi_1(|x|) = g(|x|, a, 0) \) is a radial upper control function of all of the solutions for (P), and \( \Phi_1(x) = \Phi_1(|x|) \) is a radial supersolution for (P). The proof is completed. \( \square \)

**Theorem 3.2.** If \( f(r, u) \) satisfies
\[
f(r, u) \to -\infty \quad \text{(as } u \to -\infty \text{) for } r \in [\sigma, R] \text{ uniformly,}
\]
\[
f(r, u) \leq \delta u^s \quad \text{(as } u \to +\infty \text{) for } r \in [\sigma, R] \text{ uniformly,}
\]
where \( \sigma \) is defined in (H_4) and \( \delta \) and \( s \) are positive constants, then there exists a subsolution \( \Phi_2(x) \) which satisfies \( \Phi_2(x) \to +\infty \) (as \( d(x, \partial \Omega) \to 0 \)), such that for every solution \( u(x) \) for problem (P), one has \( u(x) \geq \Phi_2(x) \).

**Proof.** We will prove this theorem in the following two cases.

(i) \( \beta_1(R) > 0 \).

(ii) \( \beta_1(R) \leq 0 \).

**Case 1** (\( \beta_1(R) > 0 \)). Let \( z_1 \) be a radial solution of
\[
-\Delta_{p(x)} z_1(x) = -\mu, \quad \text{in } \Omega_1 = B(0, \sigma), \quad z_1 = 0, \quad \text{on } \partial \Omega_1, \tag{3.17}
\]
where \( \mu > 2(\max_{r \in [0, R_0]} p(r) + 1)^{2(2^{-1})/(2^{-1})} \) is a positive constant. We denote \( z_1 = z_1(r) = z_1(|x|) \). Then, \( z_1 \) satisfies
\[
-\left( r^{N-1} |z_1|^{p(r)-2} z_1' \right)' = -r^{N-1} \mu, \quad z_1(\sigma) = 0, \quad z_1'(0) = 0,
\]
\[
z_1' = \left| \frac{r \mu}{N} \right|^{1/(p(r)-1)}, \quad z_1 = -\int_{r}^{\sigma} \left| \frac{r \mu}{N} \right|^{1/(p(r)-1)} dr. \tag{3.18}
\]

Denote \( h_b(r, \lambda) \) on \([\sigma, R_0]\) as
\[
h_b(r, \lambda) = \int_{r}^{R_0} \left\{ \frac{(R_0)^{N-1}}{t^{N-1}} \frac{t - \sigma}{R_0 - \sigma} \left( \frac{b(1 - \theta)(R - R_0)^{-\theta}}{s((R - R_0)^{1-\theta} + \lambda)} \right) \left( \frac{b \ln \frac{1}{(R - R_0)^{1-\theta} + \lambda}}{(R - R_0)^{1-\theta} + \lambda} \right)^{(1/s) - 1} \right\}^{p(R_b)-1} \left( \frac{(R_0)^{N-1}}{t^{N-1}} \frac{t - \sigma}{R_0 - \sigma} \left( \frac{\sigma \mu}{N} \right) \right)^{1/(p(t)-1)} dt. \tag{3.19}
\]
It is easy to see that

\[-h'_b(\sigma, \lambda) = z'_1(\sigma) = \left[ \frac{\psi_{\lambda R}}{N} \right]^{1/(p(\sigma)-1)},\]
\[-h'_b(0, \lambda) = \frac{b(1 - \theta)(R - R_0)^{-\theta}}{s \left( (R - R_0)^{1-\theta} + \lambda \right)} \left( b \ln \frac{1}{(R - R_0)^{1-\theta} + \lambda} \right)^{(1/s)-1}. \tag{3.20}\]

Define the function \( v(r, b, \lambda) \) on \([0, R]\) as

\[v(r, b, \lambda) = \begin{cases} 
\left( b \ln \frac{1}{(R - r)^{1-\theta} + \lambda} \right)^{1/s} - k^*, & R_0 \leq r < R, \\
\left( b \ln \frac{1}{(R - R_0)^{1-\theta} + \lambda} \right)^{1/s} - k^* - h_b(r, \lambda), & \sigma < r < R_0, \\
- \int_{r}^{\sigma} \left( \frac{r \psi_{\lambda R}}{N} \right)^{1/(p(r)-1)} dr + \left( b \ln \frac{1}{(R - R_0)^{1-\theta} + \lambda} \right)^{1/s} - k^* - h_b(\sigma, \lambda), & r \leq \sigma,
\end{cases} \tag{3.21}\]

where \( \theta \in (\beta_1(R)/p(R), 1) \), \( b \in (0, 1/\delta) \inf_{x \geq R, p(x)} \) are constants, \( R_0 \in (\sigma, R), R - R_0 \) is small enough, parameter \( \lambda \in [0, (R - R_0)^{1-\theta}/2] \), and

\[k^* = M + \left( b \ln \frac{1}{(R - R_0)^{1-\theta}} \right)^{1/s}, \tag{3.22}\]

where \( M \) satisfies

\[(\sigma)^{N-1} \frac{1}{R_0 - \sigma} \geq r^{N-1} p(r) e^{f(r, \psi)}, \quad \forall y \leq -M, \forall r \in [0, R_0]. \tag{3.23}\]

Obviously, for any positive constant \( b \), \( v(r, b, \lambda) \in C^1[0, R] \).

By computation, when \( r \in (R_0, R) \), we have

\[v' = v'(r, b, \lambda) = \frac{b^{1/s}}{s} \left( \ln \frac{1}{(R - r)^{1-\theta} + \lambda} \right)^{1/s-1} \frac{(1 - \theta)(R - r)^{-\theta}}{(R - r)^{1-\theta} + \lambda},\]
\[|v'|^{p(r)-2} v' = \left( \frac{1 - \theta}{s} b^{1/s} \right)^{(p(r)-1)} \left( \ln \frac{1}{(R - r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \frac{(R - r)^{-\theta(1-p(r)-1)}}{(R - r)^{1-\theta} + \lambda}^{p(r)-1}.\]
\[
\left( r^{N-1} |v'(p(r)-2)v' \right)' = r^{N-1} \left[ \frac{(1-\theta)b^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \times \frac{(p(r)-1)(R-r)^{-\theta(p(r)-1)}}{[R-r]^{1-\theta} + \lambda}^{p(r)-1} (\theta + \Lambda(r)),
\]

where

\[
\Lambda(r) = \frac{r^{N-1} \left[ (1-\theta)b^{1/s}/s \right]^{p(r)-1} (R-r) + (1/s-1)(1-\theta) \ln \left( 1/(R-r)^{1-\theta} \right) + \theta p'(r)(R-r) \ln \left( (R-r)^{1-\theta} + \lambda \right) + \frac{(1-\theta)}{(R-r)^{1-\theta} + \lambda} + \frac{p'(r)}{p(r)-1} (R-r) \ln \left( (R-r)^{1-\theta} + \lambda \right)}{(p(r)-1)r^{N-1} \left[ (1-\theta)b^{1/s}/s \right]^{p(r)-1} (R-r) + (1/s-1)(1-\theta) \ln \left( 1/(R-r)^{1-\theta} \right) + \theta p'(r)(R-r) \ln \left( (R-r)^{1-\theta} + \lambda \right) + \frac{(1-\theta)}{(R-r)^{1-\theta} + \lambda} + \frac{p'(r)}{p(r)-1} (R-r) \ln \left( (R-r)^{1-\theta} + \lambda \right)}.
\]

By computation, when \( R-R_0 \) is small enough, for \( \lambda \in [0, (R-R_0)^{1-\theta}/2] \) uniformly, we have

\[
\left( r^{N-1} |v'|^{p(r)-2}v' \right)' \geq r^{N-1} \left[ \frac{(1-\theta)b^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \times \frac{(p(r)-1)(R-r)^{-\theta(p(r)-1)}}{[R-r]^{1-\theta} + \lambda}^{p(r)-1} \theta \left( 1 - \frac{1}{2} \right) \\
\geq \frac{\theta}{2} r^{N-1} \left[ \frac{(1-\theta)b^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \times \frac{(p(r)-1)(R-r)^{-\theta(p(r)-1)}}{[R-r]^{1-\theta} + \lambda}^{p(r)} (R-r)^{1-\theta} \\
\geq \frac{\theta}{2} r^{N-1} \left[ \frac{(1-\theta)b^{1/s}}{s} \right]^{p(r)-1} \left( \ln \frac{1}{(R-r)^{1-\theta} + \lambda} \right)^{(1/s-1)(p(r)-1)} \times \frac{(p(r)-1)(R-r)^{-\theta(p(r)-1)}}{[R-r]^{1-\theta} + \lambda}^{p(r)} (R-r)^{1-\theta} \\
\geq \frac{\theta}{2} r^{N-1} \rho_1 (R-r)^{-\theta(p(r)-1)} (R-r)^{1-\theta} \\
\geq \frac{\theta}{2} r^{N-1} \rho_1 (R-r)^{-\theta(p(r)-1)} (R-r)^{1-\theta} e^{\delta v_0} \\
\geq \frac{\theta}{2} r^{N-1} \rho(r) e^{f(v,v)}, \quad \forall r \in (R_0, R).
\]

(3.26)
Then, for \( \lambda \in [0, (R - R_0)^{-\theta}/2] \) uniformly, we have

\[
\left( r^{N-1} |v'|^{p(r) - 2} v' \right)' \geq r^{N-1} \rho(r) e^{f(r, \omega)}, \quad \forall r \in (R_0, R). \tag{3.27}
\]

When \( R - R_0 \) is small enough, for all \( r \in (\sigma, R_0) \), since \( v \leq -M \), it is easy to see that

\[
\left( r^{N-1} |v'|^{p(r) - 2} v' \right)' \geq \left( r^{N-1} |h'|^{p(r) - 2} h' \right)'
\]

\[
= (R_0)^{N-1} \frac{1}{R_0 - \sigma} \left[ \frac{b(1 - \theta)(R - R_0)^{-\theta}}{s((R - R_0)^{1-\theta} + \lambda)} \left( b \ln \frac{1}{(R - R_0)^{1-\theta} + \lambda} \right)^{1/s-1} \right]^{(p_R)-1}
\]

\[
- (\sigma)^{N-1} \frac{1}{R_0 - \sigma} \left| \frac{\sigma \mu}{\lambda} \right|
\]

\[
\geq (\sigma)^{N-1} \frac{1}{R_0 - \sigma}
\]

\[
\geq r^{N-1} \rho(r) e^{f(r, \omega)}, \quad \forall r \in (\sigma, R_0). \tag{3.28}
\]

Then,

\[
\left( r^{N-1} |v'|^{p(r) - 2} v' \right)' \geq r^{N-1} \rho(r) e^{f(r, \omega)}, \quad \forall r \in (\sigma, R_0). \tag{3.29}
\]

Obviously,

\[
\left( r^{N-1} |v'|^{p(r) - 2} v' \right)' = r^{N-1} \mu \geq r^{N-1} \rho(r) e^{f(r, \omega)}, \quad \forall r \in (0, \sigma). \tag{3.30}
\]

Combining (3.27), (3.29), and (3.30), when \( R - R_0 \) is large enough, for any \( \lambda \in [0, (R - R_0)^{-\theta}/2] \), one can see that \( v(r, \alpha, \lambda) \) is a subsolution for (P).

Define the function \( v_m(r, b + e) \) on \( B(0, R) \) as

\[
v_m(r, b + e) = v_m \left( r, b + e, \frac{1}{m} \right), \tag{3.31}
\]

where \( e \) is a small enough positive constant such that \( (b + e) < (1/\delta)\inf_{|x| \geq R_0} p(x) \).

For any \( m = 1, 2, \ldots \), we can see that \( v_m(r, b + e) \in C^1((0, R)) \) is a subsolution for (P) on \( B(R_0, R) \). According to the comparison principle, we get that \( v_m(r, b + e) \leq u(x) \) for any \( x \in B(0, R) \). For any \( x \in B(0, R) \setminus B(0, R_0) \), we have \( v_m(|x|, b + e) \leq v_{m+1}(|x|, b + e) \). Thus

\[
u(x) \geq \lim_{m \to +\infty} v_m(|x|, b + e), \quad \forall x \in B(0, R) \setminus B(0, R_0). \tag{3.32}
\]

When \( d(x, \partial \Omega) \) is small enough, we have \( \lim_{m \to +\infty} v_m(|x|, b + e) > v(|x|, b, 0) \).
According to the comparison principle, we get that $v(|x|, b, 0) \leq u(x), \forall x \in B(0, R)$; then $\Phi_2(x) = \Phi_2(|x|) = v(|x|, b, 0)$ is a radial lower control function of all of the solutions for (P), and $\Phi_2(x)$ is a radial subsolution for (P).

**Case 2 ($\beta_1(R) \leq 0$).** Let $\mu > 2(\max_{r \in [0,R_0]} \rho(r) + 1)^{2(p^*-1)/(p^-1)}$ be a positive constant. Denote $\sigma_b(r, \lambda)$ on $[\sigma, R_0]$ as

$$
\sigma_b(r, \lambda) = \int_r^{R_0} \left\{ \frac{(R_0)^{N-1}}{t^{N-1}} \frac{t - \sigma}{R_0 - \sigma} \left[ \frac{b}{s(R + \lambda - R_0)} \left( b \ln (R + \lambda - R_0)^{-1} \right)^{1/s} \right] \right\}^{p(R_0)-1} \\
+ \frac{(\sigma)^{N-1}}{t^{N-1}} \frac{R_0 - t}{R_0 - \sigma} \frac{\sigma \mu}{N} \right\}^{1/(p(\tau)-1)} dt
$$

(3.33)

It is easy to see that

$$
-\sigma'_b(\sigma, \lambda) = \zeta_1(\sigma) = \left[ \frac{\sigma \mu}{N} \right]^{1/(p(\tau)-1)}, \quad -\sigma'_b(\sigma, \lambda) = \frac{b}{s(R + \lambda - R_0)} \left( b \ln (R + \lambda - R_0)^{-1} \right)^{1/s-1}.
$$

(3.34)

Define the function $\eta(r, b, \lambda)$ on $B(0, R)$ as

$$
\eta(r, b, \lambda) = \begin{cases}
\left( b \ln (R + \lambda - r)^{-1} \right)^{1/s} - k^*, & R_0 \leq r < R, \\
\left( b \ln (R + \lambda - R_0)^{-1} \right)^{1/s} - k^* - \sigma_b(r, \lambda), & \sigma < r < R_0, \\
\int_{\sigma}^{r} \frac{\sigma \mu}{N} \right\}^{1/(p(\tau)-1)} dr + \left( b \ln (R + \lambda - R_0)^{-1} \right)^{1/s} - k^* - \sigma_b(\sigma, \lambda), & r \leq \sigma,
\end{cases}
$$

(3.35)

where $b \in (0, (1/\delta) \inf_{|x| \leq R_0} [p(x) - \beta_1(x)])$ is a constant, $R_0 \in (\sigma, R), R - R_0$ is small enough, parameter $\lambda \in [0, (R - R_0)/2]$, and

$$
k^* = M + \left( b \ln \frac{1}{R - R_0} \right)^{1/s},
$$

(3.36)

where $M$ is defined in (3.23).

Obviously, for any positive constant $b$, $\eta(r, b, \lambda) \in C^1[0, R]$. 

Similar to the proof of Case 1, when $R - R_0$ is small enough, we have

\[
\left( r^{N-1} |\eta'|^{\mu(r)-2} \eta' \right)'
\geq r^{N-1} \left( \frac{b^{1/s}}{s} \right)^{\mu(r)-1} (\mu(r) - 1) (R + \lambda - r) \left( \ln (R + \lambda - r)^{-1} \right)^{(1/s-1)(\mu(r)-1)} \left( 1 - \frac{1}{2} \right)^{1 - \frac{1}{2}}
\geq r^{N-1} \rho(r) e^{f(r,\eta)}, \quad \forall r \in (R_0, R). \tag{3.37}
\]

When $R - R_0$ is small enough, for all $r \in (\sigma, R_0)$, from the definition of $k^*$, it is easy to see that

\[
\left( r^{N-1} |\eta'|^{\mu(r)-2} \eta' \right)'
\geq (\sigma)^{N-1} \frac{1}{R_0 - \sigma} \geq r^{N-1} \rho(r) e^{f(r,\eta)}. \tag{3.38}
\]

Obviously

\[
\left( r^{N-1} |\eta'|^{\mu(r)-2} \eta' \right)'
= r^{N-1} \mu \geq r^{N-1} \rho(r) e^{f(r,\eta)}, \quad \forall r \in (0, \sigma). \tag{3.39}
\]

Combining (3.37), (3.38), and (3.39), when $R - R_0$ is large enough, for any $\lambda \in [0, (R - R_0) / 2]$, one can see that $\eta(r, a, \lambda)$ is a subsolution for (P).

Define the function $\eta_m(r, b + \varepsilon)$ on $B(0, R)$ as

\[
\eta_m(r, b + \varepsilon) = \eta \left( r, b + \varepsilon, \frac{1}{m} \right), \tag{3.40}
\]

where $\varepsilon$ is a small enough positive constant such that $(b + \varepsilon) < (1/\delta) \inf_{|x| \geq R_0} p(x)$.

We can see that $\eta_m(r, b + \varepsilon) \in C^1([0, R])$ is a subsolution for (P) for any $m = 1, 2, \ldots$. According to the comparison principle, we get that $\eta_m(r, b + \varepsilon) \leq u(x)$ for any $x \in B(0, R)$. For any $x \in B(0, R) \setminus B(0, R_0)$, we have $\eta_m(|x|, b + \varepsilon) \leq \eta_{m+1}(|x|, b + \varepsilon)$. Then,

\[
u(x) \geq \lim_{m \to +\infty} \eta_m(|x|, b + \varepsilon), \quad \forall x \in B(0, R) \setminus B(0, R_0). \tag{3.41}
\]

When $d(x, \partial \Omega)$ is small enough, we have

\[
\lim_{m \to +\infty} \eta_m(|x|, b + \varepsilon) > \eta(|x|, b, 0). \tag{3.42}
\]

According to the comparison principle, we get that $\eta(|x|, b, 0) \leq u(x)$, $\forall x \in B(0, R)$; then $\Phi_2(x) = \Phi_2(|x|) = \eta(|x|, b, 0)$ is a radial lower control function of all of the solutions for (P), and $\Phi_2(x) = \Phi_2(|x|)$ is a radial subsolution for (P).
Theorem 3.3. If \( f(r, u) \) satisfies

\[
\lim_{u \to +\infty} \frac{f(r, u)}{u^s} = \delta \quad \text{(as } u \to +\infty \text{) for } r \in [\sigma, R] \text{ uniformly,}
\]

(3.43)

where \( \sigma \) is defined in (H₄), \( \delta \) and \( s \) are positive constants, \( \rho(r) = \rho_0(R - r)^{-\beta(r)} \), where \( \beta(R) < p(R) \), then each solution \( u(x) \) for (P) satisfies

\[
\lim_{|x| \to R} \left( \frac{u(x)}{(p(R)/\delta) \left( \ln 1/(R - |x|)^{1-\theta} \right)^{1/s}} \right) = 1, \quad \text{where } \theta = \frac{\beta(R)}{p(R)}.
\]

(3.44)

Proof. It is easy to be seen from Theorems 3.1 and 3.2

4. The Existence of Boundary Blow-Up Solutions

Theorem 4.1. If \( \inf_{x \in \Omega} p(x) > N \) and \( f(r, u) \) satisfies

\[
f(r, u) \geq au^s \quad \text{(as } u \to +\infty \text{) for } r \in [\sigma, R] \text{ uniformly,}
\]

(4.1)

where \( \sigma \) is defined in (H₄), \( a \) and \( s \) are positive constants, then (P) possesses a boundary blow-up solution.

Proof. In order to deal with the existence of boundary blow-up solutions, let us consider the problem

\[
-\Delta_p(x)u + \rho(r)e^{f(x,u)} = 0, \quad \text{in } \Omega_0,
\]

\[
u(x) = c, \quad \text{for } x \in \partial \Omega_0,
\]

(4.2)

where \( c \) is a positive constant and \( \Omega_0 \subset \Omega \) is a radial subdomain of \( \Omega \). Since \( \inf_{x \in \Omega} p(x) > N \), then \( W^{1,p(x)}(\Omega_0) \to C^2(\bar{\Omega}_0) \), where \( \alpha \in (0, 1) \). The relative functional of (4.2) is

\[
\varphi = \int_{\Omega_0} \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx + \int_{\Omega_0} F(x, u) dx,
\]

(4.3)

where \( F(x, u) = \int_0^u e^{f(x,t)} dt. \) Since \( \varphi \) is coercive in \( X := c + W^{1,p(x)}(\Omega_0) \), then \( \varphi \) possesses a nontrivial minimum point \( u \). So, problem (4.2) possesses a weak solution \( u \).

Since \( au^s \leq f(r, u) \leq C_1 + C_2 |u|^{r(x)} \), from Theorems 3.1 and 3.2, we get that (P) possesses a supersolution \( g^*(x) \) and a subsolution \( g_*(x) \), which satisfy \( g^*(x) \geq g_*(x) \), when \( d(x, \partial \Omega) \) (the distance from \( x \) to \( \partial \Omega \)) is small enough. According to the comparison principle, we get that \( g^*(x) \geq g_*(x) \) for any \( x \in \Omega. \)
Denote $D_j = \{ x \mid |x| < 1 - 1/(j+1)R \}$ ($j = 1, 2, \ldots$). Let us consider the problem

$$-\Delta_{p(x)} u_j + \rho(x) e^{f(u_j)} = 0, \quad \text{in } D_j,$$

$$u_j(x) = g_\ast(x), \quad \text{for } x \in \partial D_j,$$  

and the relative functional is

$$\varphi = \int_{D_j} \frac{1}{p(x)} |\nabla u_j(x)|^{p(x)} \, dx + \int_{D_j} \rho(x) F(x,u_j) \, dx.$$  

Let $g_\ast(x) = g_\ast(x)|_{D_j}$. Since the functional $\varphi$ is coercive in $X_j = g_\ast(x) + W_0^{1,p(x)}(D_j)$, then $\varphi$ has a nontrivial minimum point $u_j$. Therefore, problem (4.4) has a weak solution $u_j$.

According to the comparison principle, we get that $g_\ast(x) \leq u_j(x)$ for any $x \in D_j$ ($j = 1, 2, \ldots$). Since $u_j(x) = g_\ast(x)$ for any $x \in \partial D_j$, then $u_j(x) \leq u_{j+1}(x)$ for any $x \in \partial D_j$ ($j = 1, 2, \ldots$). According to the comparison principle, we get that $u_j(x) \leq u_{j+1}(x)$ for any $x \in D_j$ ($j = 1, 2, \ldots$).

Since $g_\ast(x)$ is a supersolution and $g_\ast(x) \geq g_\ast(x)$ for any $x \in \Omega$, so we have $u_j(x) = g_\ast(x)$ for any $x \in \partial D_j$ ($j = 1, 2, \ldots$). According to the comparison principle, we get that $u_j(x) \leq g_\ast(x)$ for any $x \in D_j$ ($j = 1, 2, \ldots$).

Since $g_\ast(x)$ and $g_\ast(x)$ are locally bounded, from Lemma 2.4, each weak solution of (4.4) is a $C^{1,\alpha}_{\text{loc}}$ function. The $C^{1,\alpha}$ interior regularity result implies that the sequences $\{u_j\}$ and $\{\nabla u_j\}$ are equicontinuous in $D_2$, and hence we can choose a subsequence, which we denoted by $\{u_j\}$, such that $u_j \rightharpoonup w_1$ and $\nabla u_j \rightharpoonup \nabla w_1$ uniformly on $D_1$ for some $w_1 \in C(D_1)$ and $\nabla w_1 \in (C^{a}(D_1))^N$. In fact, $\nabla w_1 = \nabla w_1 |_{D_1}$, and from the interior $C^{1,\alpha}$ estimate, we conclude that $\nabla w_1 \in (C^{\alpha}(D_1))^N$ for some $0 < \alpha < 1$. Thus, $w_1 \in W^{1,p(x)}(D_1) \cap C^{1,\alpha}(D_1)$. From the $C^{1,\alpha}$ interior regularity result, we see that $|\nabla u_j|^{p-1} |\nabla \varphi| \leq C |\nabla \varphi|$ on $D_1$, and since the function $\xi \rightarrow |\xi|^{p-2} \xi$ is continuous on $\mathbb{R}^N$, it follows that $|\nabla u_j(x)|^{p-2} \nabla u_j(x) \cdot \nabla \varphi(x) \rightarrow |\nabla w_1(x)|^{p-2} \nabla w_1(x) \cdot \nabla \varphi(x)$ for $x \in D_1$. Thus, by the dominated convergence theorem, we have

$$\int_{D_1} |\nabla u_j(x)|^{p-2} \nabla u_j(x) \cdot \nabla \varphi(x) \, dx \longrightarrow \int_{D_1} |\nabla w_1(x)|^{p-2} \nabla w_1(x) \cdot \nabla \varphi(x) \, dx, \quad \forall \varphi \in W_0^{1,p(x)}(D_1).$$  

Furthermore, since $0 \leq f(u_j) \leq f(u_{j+1})$ and $f(u_j(x)) \rightarrow f(w_1(x))$ for each $x \in D_1$, by the monotone convergence theorem, we obtain

$$\int_{D_1} \rho e^{f(u_j)} q \, dx \longrightarrow \int_{D_1} \rho e^{f(w_1)} q \, dx, \quad \forall q \in W_0^{1,p(x)}(D_1).$$  

Therefore, it follows that

$$\int_{D_1} |\nabla w_1(x)|^{p-2} \nabla w_1(x) \cdot \nabla q(x) \, dx + \int_{D_1} \rho e^{f(w_1)} q \, dx = 0, \quad \forall q \in W_0^{1,p(x)}(D_1),$$

and hence $w_1$ is a weak solution for $-\Delta_{p(x)} w_1 + \rho e^{f(w_1)} = 0$ on $D_1$.  

Thus, there exists a subsequence of \( \{ u_j \} \) which we denote it by \( \{ u^i_j \} \), such that \( u^i_j \to w_1 \) in \( D_1 \) (as \( j \to \infty \)), where \( w_1 \in W^{1,p(x)}(D_1) \cap C^{1,\alpha}(D_1) \) and satisfies

\[
\int_{D_1} |\nabla w_1|^{p(x)-2} \nabla w_1 \nabla q \, dx + \int_{D_1} \rho(x)e^{f(x,w_1)} q \, dx = 0, \quad \forall q \in W^{1,p(x)}_0(D_1). \tag{4.9}
\]

Similarly, we can prove that there exists a subsequence of \( \{ u_j^2 \} \) which we denote by \( \{ u^i_j \} \), such that \( u^i_j \to w_2 \) in \( D_2 \) (as \( j \to \infty \)), where \( w_2 \in W^{1,p(x)}(D_2) \cap C^{1,\alpha}(D_2) \) satisfies \( w_1 = w_2|_{D_1} \) and

\[
\int_{D_2} |\nabla w_2|^{p(x)-2} \nabla w_2 \nabla q \, dx + \int_{D_2} \rho(x)e^{f(x,w_2)} q \, dx = 0, \quad \forall q \in W^{1,p(x)}_0(D_2). \tag{4.10}
\]

Repeating the above steps, we can get a subsequence of \( \{ u_j^i \mid j = 1, 2, \ldots \} \) which we denote by \( \{ u^{i+1}_j \mid j = 1, 2, \ldots \} \) (\( i = 1, 2, \ldots \)) and satisfies the following,

(1) For any fixed \( i \), \( \{ u^{i+1}_j \} \) is a subsequence of \( \{ u_j^i \} \).

(2) For any fixed \( i \), \( u^{i+1}_j \to w_{i+1} \) in \( D_{i+1} \) (as \( j \to \infty \)), where \( w_{i+1} \in W^{1,p(x)}(D_{i+1}) \cap C^{1,\alpha}(D_{i+1}) \) satisfies \( w_i = w_{i+1}|_{D_i} \).

(3) For any fixed \( i \), \( w_i \) satisfies

\[
\int_{D_i} |\nabla w_i|^{p(x)-2} \nabla w_i \nabla q \, dx + \int_{D_i} \rho(x)e^{f(x,w_i)} q \, dx = 0, \quad \forall q \in W^{1,p(x)}_0(D_i). \tag{4.11}
\]

Thus, we can conclude that

(i) \( \{ u_j^i \} \) is a subsequence of \( \{ u_j \} \),

(ii) there exists a function \( w \in W^{1,p(x)}_{\text{loc}}(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\Omega) \) such that \( w_i = w|_{D_i} \), and for any \( x \in \Omega \), there exists a constant \( j_x \) such that when \( j \geq j_x \), \( u^i_j(x) \) is defined at \( x \), and \( \lim_{j \to \infty} u^i_j(x) = w(x) \),

(iii)

\[
\int_{\Omega} |\nabla w|^{p(x)-2} \nabla w \nabla q \, dx + \int_{\Omega} \rho(x)e^{f(x,w)} q \, dx = 0, \quad \forall q \in W^{1,p(x)}_{0,\text{loc}}(\Omega). \tag{4.12}
\]

Obviously, \( w \) is a boundary blow-up solution for (P).

This completes the proof.
In Theorem 4.1, when \( \inf_{x \in \Omega} p(x) > N \), the existence of solutions for (P) is given. In the following, we will consider the existence of solutions for (P) in the general case \( 1 < \inf_{x \in \Omega} p(x) \leq \sup_{x \in \Omega} p(x) < \infty \). We need to do some preparation. Let us consider

\[
\left( r^{N-1} |u'|^{p(r)-2} u' \right)' = r^{N-1} \rho(r)e^{f(r,u)}, \quad r \in (0, R_1),
\]

\[
u'(0) = 0, \quad u(R_1) = d,
\]

where \( R_1 \in (0, R) \) and \( d \) is a constant.

**Lemma 4.2.** If \( \Phi_2(R_1) \leq d \leq \Phi_1(R_1) \), where \( \Phi_1 \) and \( \Phi_2 \) are defined in Theorems 3.13.2, respectively, then (4.13) has a solution \( u \) satisfying

\[
\Phi_2(r) \leq u(r) \leq \Phi_1(r), \quad \forall r \in [0, R_1]. \tag{4.13}
\]

**Proof.** Denote

\[
h(r, u) = \begin{cases} 
  e^{f(r, \Phi_1(r))} + \arctan(u(r) - \Phi_1(r)), & u(r) > \Phi_1(r), \\
  e^{f(r,u)}, & \Phi_2(r) \leq u(r) \leq \Phi_1(r), \\
  e^{f(r, \Phi_2(r))} + \arctan(u(r) - \Phi_2(r)), & u(r) < \Phi_2(r).
\end{cases} \tag{4.14}
\]

Let \( \rho_E(t) = \rho(|t|) \), and \( h_E(t, u) = h(|t|, u) \), for all \( t \in [-R_1, R_1] \). Let us consider the even solutions of the following

\[
\left( |t|^{N-1} |u'|^{p(|t|)-2} u' \right)' = |t|^{N-1} \rho_E(t)h_E(t, u), \quad t \in (-R_1, R_1),
\]

\[
u(-R_1) = d, \quad u(R_1) = d.
\]

It is easy to see that \( u \) is an even solution for (4.15) if and only if \( u \) is even and

\[
u = d - \int_{-R_1}^{R_1} \left[ |t|^{1-N} \int_{0}^{1} |s|^{N-1} \rho(s)h(s, u(s))ds \right]^{1/(p(t)-1)} dt, \quad \forall r \in [0, R_1]. \tag{4.15}
\]

Denote \( \Psi(u, \mu) = \mu d - \mu \int_{-R_1}^{R_1} \left[ |t|^{1-N} \int_{0}^{1} |s|^{N-1} \rho(s)h(s, u(s))ds \right]^{1/(p(t)-1)} dt \). Similar to the proof of Lemma 2.3 of [18], for any \( \mu \in [0, 1] \), it is easy to see that \( \Psi(u, \mu) \) is compact continuous and bounded from \( C^1_{\bar{I}}[0, R_1] \) to \( C_E^1[0, R_1] \), where \( C_E^1[0, R_1] = \{ u \in C^1[0, R_1] \mid u \) is even \}. Thus, \( u = \Psi(u, 1) \) has a solution \( u \) in \( C^1_E[0, R_1] \) and satisfies \( u'(0) = \lim_{r \to 0} u'(r) = 0 \). Then, \( u(|t|) \) is an even solution for (4.15).

Denote \( \Phi_{1,E}(t) = \Phi_1(|t|), \Phi_{2,E}(t) = \Phi_2(|t|) \). From the definitions of \( \Phi_1 \) and \( \Phi_2 \), we can see that \( \Phi_1'(0) = 0 = \Phi_2'(0) \); therefore, \( \Phi_{1,E}(t) \) and \( \Phi_{2,E}(t) \) are supersolution and subsolution for (4.15), respectively.
Since $\Phi_2(R_1) \leq u(R_1) \leq \Phi_1(R_1)$ and $h_2(t, \cdot)$ is increasing, from the comparison principle, we have
\[
\Phi_2(t) \leq u(t) \leq \Phi_1(t), \quad \forall t \in [-R_1, R_1].
\]
(4.16)

It means that $u$ is a solution for (4.13) and $u$ satisfies
\[
\Phi_2(r) \leq u(r) \leq \Phi_1(r), \quad \forall r \in [0, R_1].
\]
(4.17)

Thus $u$ is a radial solution for (P). This completes the proof. 

**Theorem 4.3.** If $f(r, u)$ satisfies
\[
f(r, u) \geq au^s \quad \text{(as } u \to +\infty) \text{ for } r \in [\sigma, R] \text{ uniformly,}
\]
(4.18)
where $\sigma$ is defined in (H4) and $a$ and $s$ are positive constants, then (P) possesses a boundary blow-up solution.

**Proof.** From Lemma 4.2, we have that (4.4) has a weak solution $u_i(x) = u_i(|x|) = u_i(r)$. Similar to the proof of Theorem 4.1, we can obtain the existence of solutions for (P). 

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**References**

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