Research Article

Non-Self-Adjoint Singular Sturm-Liouville Problems with Boundary Conditions Dependent on the Eigenparameter

Elgiz Bairamov¹ and M. Seyyit Seyyidoglu²

¹ Department of Mathematics, Science Faculty, Ankara University, 06100 Ankara, Turkey

² Department of Mathematics, Science and Art Faculty, Usak University, 64200 Campus-Uşak, Turkey

Correspondence should be addressed to M. Seyyit Seyyidoglu, seyyit.seyyidoglu@usak.edu.tr

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Let A denote the operator generated in $L_2(\mathcal{R}_+)$ by the Sturm-Liouville problem: $-y'' + q(x)y = \lambda^2 y$, $x \in \mathcal{R}_+ = [0, \infty)$, $(y'/y)(0) = (\beta_1\lambda + \beta_0)/(\alpha_1\lambda + \alpha_0)$, where q is a complex valued function and $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{C}$, with $\alpha_0\beta_1 - \alpha_1\beta_0 \neq 0$. In this paper, using the uniqueness theorems of analytic functions, we investigate the eigenvalues and the spectral singularities of A . In particular, we obtain the conditions on q under which the operator A has a finite number of the eigenvalues and the spectral singularities.

1. Introduction

Let L denote the non-self-adjoint Sturm-Liouville operator generated in $L_2(\mathcal{R}_+)$ by the differential expression

$$l(y) = -y'' + q(x)y, \quad x \in \mathcal{R}_+ \quad (1.1)$$

and the boundary condition $y(0) = 0$, where q is a complex valued function. The spectral analysis of L with continuous and discrete spectrum was studied by Naïmark [1]. In this article, the spectrum of L was investigated and shown that it is composed of the eigenvalues, the continuous spectrum and the spectral singularities. The spectral singularities of L are poles of the resolvent which are imbedded in the continuous spectrum and are not the eigenvalues.

If the function q satisfies the Naïmark condition, that is,

$$\int_0^\infty e^{\varepsilon x} |q(x)| dx < \infty \quad (1.2)$$

for some $\varepsilon > 0$, then L has a finite number of the eigenvalues and spectral singularities with finite multiplicities.

The results of Naïmark were extended to the Sturm-Liouville operators on the entire real axis by Kemp [2] and to the differential operators with a singularity at the zero point by Gasymov [3]. The spectral analysis of dissipative Sturm-Liouville operators with spectral singularities was considered by Pavlov [4]. A very important development in the spectral analysis of L was made by Lyance [5, 6]. He showed that the spectral singularities play an important role in the spectral theory of L . He also investigated the effect of the spectral singularities in the spectral expansion. The spectral singularities of the non-self-adjoint Sturm-Liouville operator generated in $L_2(\mathcal{R}_+)$ by (1.1) and the boundary condition

$$\int_0^\infty K(x)y(x)dx + \alpha y'(0) - \beta y(0) = 0, \quad (1.3)$$

in which $K \in L_2(\mathcal{R}_+)$ is a complex valued function and $\alpha, \beta \in \mathbb{C}$, was studied in detail by Krall [7–9].

Some problems of spectral theory of differential and difference operators with spectral singularities were also investigated in [10–16]. Note that, the boundary conditions used in [1–17] are independent of spectral parameter. In recent years, various problems of the spectral theory of regular Sturm-Liouville problem whose boundary conditions depend on spectral parameter have been examined in [18–22].

Let us consider the boundary value problem

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathcal{R}_+, \quad (1.4)$$

$$\frac{y'}{y}(0) = \frac{\beta_1 \lambda + \beta_0}{\alpha_1 \lambda + \alpha_0}, \quad (1.5)$$

where q is a complex valued function and $\alpha_0, \alpha_1, \beta_0, \beta_1$ are complex numbers such that $\alpha_0 \beta_1 - \alpha_1 \beta_0 \neq 0$. By A we will denote the operator generated in $L_2(\mathcal{R}_+)$ by (1.4) and (1.5). In this paper we discuss the discrete spectrum of A and prove that the operator A has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity if

$$\lim_{x \rightarrow \infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x^\delta} |q'(x)| dx < \infty \quad (1.6)$$

for some $\varepsilon > 0$ and $1/2 \leq \delta < 1$. We also show that the analogue of the Naïmark condition for A is the form

$$\lim_{x \rightarrow \infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x} |q'(x)| dx < \infty \quad (1.7)$$

for some $\varepsilon > 0$.

2. Jost Solution of (1.4)

We will denote the solution of (1.4) satisfying the condition

$$\lim_{x \rightarrow \infty} y(x, \lambda) e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathcal{C}}_+ := \{\lambda : \lambda \in \mathcal{C}, \operatorname{Im} \lambda \geq 0\}, \quad (2.1)$$

by $e(x, \lambda)$. The solution $e(x, \lambda)$ is called the Jost solution of (1.4). Under the condition

$$\int_0^\infty x |q(x)| dx < \infty, \quad (2.2)$$

the Jost solution has a representation

$$e(x, \lambda) = e^{i\lambda x} + \int_x^\infty K(x, t) e^{i\lambda t} dt \quad (2.3)$$

for $\lambda \in \overline{\mathcal{C}}_+$, where the kernel $K(x, t)$ satisfies

$$\begin{aligned} K(x, t) = & \frac{1}{2} \int_{(x+t)/2}^\infty q(\xi) d\xi + \frac{1}{2} \int_x^{(x+t)/2} \int_{t+x-\xi}^{t+\xi-x} K(\xi, \eta) q(\xi) d\eta d\xi \\ & + \frac{1}{2} \int_{(x+t)/2}^\infty \int_\xi^{t+\xi-x} K(\xi, \eta) q(\xi) d\eta d\xi. \end{aligned} \quad (2.4)$$

Moreover, $K(x, t)$ is continuously differentiable with respect to its arguments and

$$|K(x, t)| \leq c \int_{(x+t)/2}^\infty |q(\xi)| d\xi, \quad (2.5)$$

$$|K_x(x, t)|, |K_t(x, t)| \leq \frac{1}{4} \left| q\left(\frac{x+t}{2}\right) \right| + c \int_{(x+t)/2}^\infty |q(\xi)| d\xi, \quad (2.6)$$

where $c > 0$ is a constant [23, Chapter 3].

The solution $e(x, \lambda)$ is analytic with respect to λ in $\mathcal{C}_+ := \{\lambda : \lambda \in \mathcal{C}, \operatorname{Im} \lambda > 0\}$ and continuous on the real axis.

Let $\mathcal{AC}(\mathcal{R}_+)$ denote the class of complex valued absolutely continuous functions in $\mathcal{R}_+$. In the sequel we will need the following.

Lemma 2.1. *If*

$$q \in \mathcal{AC}(\mathcal{R}_+), \quad \lim_{x \rightarrow \infty} q(x) = 0, \quad \int_0^\infty x^2 |q'(x)| dx < \infty, \quad (2.7)$$

then $K_{xt}(x, t) := (\partial^2 / \partial t \partial x) K(x, t)$ exists and

$$\begin{aligned} K_{xt}(x, t) = & -\frac{1}{8} q' \left(\frac{x+t}{2} \right) - \frac{1}{4} K \left(\frac{x+t}{2}, \frac{x+t}{2} \right) q \left(\frac{x+t}{2} \right) \\ & - \frac{1}{2} \int_x^{(x+t)/2} [K_t(\xi, t+x-\xi) + K_t(\xi, t-x+\xi)] q(\xi) d\xi \\ & - \frac{1}{2} \int_{(x+t)/2}^\infty K_t(\xi, t-x+\xi) q(\xi) d\xi. \end{aligned} \quad (2.8)$$

The proof of the lemma is the direct consequence of (2.4).

From (2.5)–(2.8) we find that

$$|K_{xt}(0, t)| \leq c \left[\left| q \left(\frac{t}{2} \right) \right| + \left| q' \left(\frac{t}{2} \right) \right| + \int_{t/2}^\infty |q(\xi)| d\xi \right], \quad (2.9)$$

where $c > 0$ is a constant.

3. The Green Function and the Continuous Spectrum

Let $\varphi(x, \lambda)$ denote the solution of (1.4) subject to the initial conditions $\varphi(0, \lambda) = \alpha_0 + \alpha_1 \lambda$, $\varphi'(0, \lambda) = \beta_0 + \beta_1 \lambda$. Therefore $\varphi(x, \lambda)$ is an entire function of λ .

Let us define the following functions:

$$D_\pm(\lambda) = (\alpha_0 + \alpha_1 \lambda) e_x(0, \pm \lambda) - (\beta_0 + \beta_1 \lambda) e(0, \pm \lambda) \quad \lambda \in \overline{\mathcal{C}}_\pm, \quad (3.1)$$

where $\overline{\mathcal{C}}_\pm = \{\lambda : \lambda \in \mathcal{C}, \pm \operatorname{Im} \lambda \geq 0\}$. It is obvious that the functions D_+ and D_- are analytic in $\mathcal{C}_+$ and $\mathcal{C}_- := \{\lambda : \lambda \in \mathcal{C}, \operatorname{Im} \lambda < 0\}$, respectively and continuous on the real axis.

Let

$$G(x, t; \lambda) = \begin{cases} G_+(x, t; \lambda), & \lambda \in \mathcal{C}_+, \\ G_-(x, t; \lambda), & \lambda \in \mathcal{C}_- \end{cases} \quad (3.2)$$

be the Green function of A (obtained by the standard techniques), where

$$G_\pm(x, t; \lambda) = \begin{cases} -\frac{e(x, \pm \lambda) \varphi(t, \lambda)}{D_\pm(\lambda)}, & 0 \leq t \leq x, \\ -\frac{e(t, \pm \lambda) \varphi(x, \lambda)}{D_\pm(\lambda)}, & x \leq t < \infty. \end{cases} \quad (3.3)$$

We will denote the continuous spectrum of A by σ_c . Using (3.1)–(3.3) in a way similar to Theorem 2 [17, page 303], we get the following:

$$\sigma_c = \mathcal{R}. \quad (3.4)$$

4. The Discrete Spectrum of the Operator A

Let us denote the eigenvalues and the spectral singularities of the operator A by σ_d and σ_{ss} respectively. From (2.3) and (3.1)–(3.4) it follows that

$$\begin{aligned} \sigma_d &= \{\lambda : \lambda \in \mathcal{C}_+, D_+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathcal{C}_-, D_-(\lambda) = 0\}, \\ \sigma_{ss} &= \{\lambda : \lambda \in \mathcal{R}^*, D_+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathcal{R}^*, D_-(\lambda) = 0\}, \end{aligned} \quad (4.1)$$

where $\mathcal{R}^* = \mathcal{R} - \{0\}$.

Definition 4.1. The multiplicity of a zero of D_+ (or D_-) in $\bar{\mathcal{C}}_+$ (or $\bar{\mathcal{C}}_-$) is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of A .

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of A we need to discuss the quantitative properties of the zeros of D_+ and D_- in $\bar{\mathcal{C}}_+$ and $\bar{\mathcal{C}}_-$, respectively. For the sake of simplicity we will consider only the zeros of D_+ in $\bar{\mathcal{C}}_+$. A similar procedure may also be employed for zeros of D_- in $\bar{\mathcal{C}}_-$.

Let us define

$$M_1^\pm = \{\lambda : \lambda \in \mathcal{C}_\pm, D_\pm(\lambda) = 0\}, \quad M_2^\pm = \{\lambda : \lambda \in \mathcal{R}, D_\pm(\lambda) = 0\}. \quad (4.2)$$

So we have, by (4.1), that

$$\sigma_d = M_1^+ \cup M_1^-, \quad \sigma_{ss} = M_2^+ \cup M_2^- - \{0\}. \quad (4.3)$$

Theorem 4.2. *Under the conditions in (2.7):*

- (i) *the discrete spectrum σ_d is a bounded, at most countable set and its limit points lie on the bounded subinterval of the real axis;*
- (ii) *the set σ_{ss} is a bounded and its linear Lebesgue measure is zero.*

Proof. From (2.3) and (3.1) we obtain that D_+ is analytic in $\mathcal{C}_+$, continuous on the real axis and has the form

$$D_+(\lambda) = i\alpha_1\lambda^2 + a\lambda + b + \int_0^\infty f(t)e^{i\lambda t} dt, \quad (4.4)$$

where

$$\begin{aligned} a &= i\alpha_0 - \alpha_1 K(0, 0) - \beta_1, \\ b &= -(\alpha_0 + i\beta_1)K(0, 0) - \beta_0 + i\alpha_1 K_x(0, 0), \\ f(t) &= -\beta_0 K(0, t) - i\beta_1 K_t(0, t) + \alpha_0 K_x(0, t) + i\alpha_1 K_{xt}(0, t). \end{aligned} \quad (4.5)$$

Using (2.5), (2.6), and (2.9) we get that $f \in L_1(\mathcal{R}_+)$. So

$$D_+(\lambda) = i\alpha_1 \lambda^2 + a\lambda + b + o(1), \quad \lambda \in \overline{\mathcal{C}}_+, \quad |\lambda| \rightarrow \infty. \quad (4.6)$$

From (4.3), (4.6) and uniqueness theorem for analytic functions [24], we get (i) and (ii). $\square$

Theorem 4.3. *If*

$$q \in \mathcal{AC}(\mathcal{R}_+), \quad \lim_{x \rightarrow \infty} q(x) = 0, \quad \int_0^\infty x^3 |q'(x)| dx < \infty, \quad (4.7)$$

then

$$\sum_v |l_v| \ln \frac{1}{|l_v|} < \infty, \quad (4.8)$$

where $|l_v|$ is the lengths of the boundary complementary intervals of σ_{ss} .

Proof. From (2.5), (2.6), (2.9), (4.4) and (4.7) we see that D_+ is continuously differentiable on $\mathcal{R}$. Since the function D_+ is not identically equal to zero, by Beurling's theorem we obtain (4.8) [25]. $\square$

Theorem 4.4. *Under the conditions*

$$q \in \mathcal{AC}(\mathcal{R}_+), \quad \lim_{x \rightarrow \infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x} |q'(x)| dx < \infty, \quad \varepsilon > 0, \quad (4.9)$$

the operator A has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

Proof. (2.5), (2.7), (2.9), (4.4) and (4.9) imply that the function D_+ has an analytic continuation to the half-plane $\text{Im} \lambda > -\varepsilon/2$. Hence the limit points of its zeros on $\overline{\mathcal{C}}_+$ cannot lie in $\mathcal{R}$. Therefore using Theorem 4.2, we have the finiteness of zeros of D_+ in $\overline{\mathcal{C}}_+$. Similarly we find that the function D_- has a finite number of zeros with finite multiplicity in $\overline{\mathcal{C}}_-$. Then the proof of the theorem is the direct consequence of (4.3).

Note that the conditions in (4.9) are analogous to the Naïmark condition (1.2) for the operator A .

It is clear that the condition (4.9) guarantees the analytic continuation of D_+ and D_- from the real axis to the lower and the upper half-planes respectively. So the finiteness of the eigenvalues and the spectral singularities of A are obtained as a result of these analytic continuations.

Now let suppose that

$$q \in \mathcal{AC}(\mathcal{R}_+), \quad \lim_{x \rightarrow \infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x^\delta} |q'(x)| dx < \infty, \quad (4.10)$$

for some $\varepsilon > 0$ and $1/2 \leq \delta < 1$, which is weaker than (4.9). It is obvious that under the condition (4.10) the function D_+ is analytic in $\mathcal{C}_+$ and infinitely differentiable on the real axis. But D_+ does not have analytic continuation from the real axis to the lower half-plane. Similarly, D_- does not have analytic continuation from the real axis to the upper half-plane either. Consequently, under the conditions in (4.10) the finiteness of the eigenvalues and the spectral singularities of A cannot be shown in a way similar to Theorem 4.4.

Let us denote the sets of limit points of M_1^+ and M_2^+ by M_3^+ and M_4^+ respectively and the set of all zeros of D_+ with infinite multiplicity in $\overline{\mathcal{C}}_+$ by M_∞^+ . Analogously define the sets M_3^-, M_4^- and M_∞^- .

It is clear from the boundary uniqueness theorem of analytic functions that [24]

$$\begin{aligned} M_1^\pm \cap M_\infty^\pm &= \emptyset, & M_3^\pm &\subset M_2^\pm, & M_4^\pm &\subset M_2^\pm, \\ M_\infty^\pm &\subset M_2^\pm, & M_3^\pm &\subset M_\infty^\pm, & M_4^\pm &\subset M_\infty^\pm, \end{aligned} \quad (4.11)$$

and $\mu(M_3^\pm) = \mu(M_4^\pm) = \mu(M_\infty^\pm) = 0$, where μ denote the Lebesgue measure on the real axis. $\square$

Theorem 4.5. *If (4.10) holds, then $M_\infty^+ = M_\infty^- = \emptyset$.*

Proof. We will prove that $M_\infty^+ = \emptyset$. The case $M_\infty^- = \emptyset$ is similar. Under the condition (4.10) D_+ is analytic in $\mathcal{C}_+$ all of its derivatives are continuous on the real axis and there exists $N > 0$ such that

$$\begin{aligned} \left| \frac{d^n}{d\lambda^n} D_+(\lambda) \right| &\leq B_n, \quad n = 0, 1, 2, \dots, \quad \lambda \in \overline{\mathcal{C}}_+, \quad |\lambda| < 2N, \\ B_0 &= 4|\alpha_1|N^2 + 2|a|N + |b| + \int_0^\infty |f(t)| dt, \\ B_1 &= 4|\alpha_1|N + |a| + \int_0^\infty t|f(t)| dt, \\ B_2 &= 2|\alpha_1| + \int_0^\infty t^2|f(t)| dt, \\ B_n &= \int_0^\infty t^n |f(t)| dt, \quad n \geq 3. \end{aligned} \quad (4.12)$$

From Theorem 4.2, we get that

$$\left| \int_{-\infty}^{-N} \frac{\ln|D_+(\lambda)|}{1+\lambda^2} d\lambda \right| < \infty, \quad \left| \int_N^{\infty} \frac{\ln|D_+(\lambda)|}{1+\lambda^2} d\lambda \right| < \infty. \quad (4.13)$$

Let us define the function

$$T(s) = \inf_n \frac{B_n s^n}{n!}. \quad (4.14)$$

Since the function D_+ is not equal to zero identically, by Pavlov's theorem [4],

$$\int_0^h \ln T(s) d\mu(M_{\infty,s}^+) > -\infty \quad (4.15)$$

holds, where $h > 0$ is a constant and $\mu(M_{\infty,s}^+)$ is the Lebesgue measure of s -neighborhood of M_{∞}^+ . Using (2.5), (2.6), (2.9) and (4.4) we obtain that

$$B_n \leq B d^n n! n^{n(1/\delta-1)}, \quad (4.16)$$

where B and d are constants depending on ε and δ . Substituting (4.16) in the definition of $T(s)$ we get

$$T(s) \leq B \exp \left\{ - \left(\frac{1}{\delta} - 1 \right) e^{-1} d^{-\delta/(1-\delta)} s^{-\delta/(1-\delta)} \right\}. \quad (4.17)$$

Now (4.15) and (4.17) imply that

$$\int_0^h s^{-\delta/(1-\delta)} d\mu(M_{\infty,s}^+) < \infty. \quad (4.18)$$

Since $\delta/(1-\delta) \geq 1$, consequently (4.18) holds for arbitrary s if and only if $\mu(M_{\infty,s}^+) = 0$ or $M_{\infty}^+ = \emptyset$. $\square$

Theorem 4.6. *Under the condition (4.10) the operator A has a finite number of the eigenvalues and the spectral singularities and each of them is of a finite multiplicity.*

Proof. To be able to prove the theorem we have to show that the functions D_+ and D_- have finite number of zeros with finite multiplicities in $\overline{C}_+$ and $\overline{C}_-$, respectively. We will prove it only for D_+ . The case of D_- is similar.

It follows from (4.11) that $M_3^+ = M_4^+ = \emptyset$. So the bounded sets M_1^+ and M_2^+ have no limit points, that is, the D_+ has only a finite number of zeros in $\overline{C}_+$. Since $M_{\infty}^+ = \emptyset$ these zeros are of a finite multiplicity. $\square$

Theorem 4.7. *If the condition (2.7) is satisfied then the set σ_{ss} is of the first category.*

Proof. From the continuity of D_+ it is clear that the set M_2^+ is closed and is a set of Lebesgue measure zero which is of type F_σ . According to Martin's theorem [26] there is measurable set whose metric density exists and is different from 0 and 1 at every point of M_2^+ . So, M_2^+ is of the first category from the theorem due to Goffman [27]. We also have obviously same things for M_2^- . Consequently σ_{ss} is of the first category by (4.3). $\square$

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