Some Identities of the Twisted $q$-Genocchi Numbers and Polynomials with Weight $\alpha$ and $q$-Bernstein Polynomials with Weight $\alpha$

H. Y. Lee, N. S. Jung, and C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea

Correspondence should be addressed to H. Y. Lee, normaliz1@naver.com

Received 7 July 2011; Accepted 22 August 2011

Academic Editor: John Rassias

Copyright © 2011 H. Y. Lee et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Recently mathematicians have studied some interesting relations between $q$-Genocchi numbers, $q$-Euler numbers, polynomials, Bernstein polynomials, and $q$-Bernstein polynomials. In this paper, we give some interesting identities of the twisted $q$-Genocchi numbers, polynomials, and $q$-Bernstein polynomials with weighted $\alpha$.

1. Introduction

Throughout this paper, let $p$ be a fixed odd prime number. The symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and let $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$. As a well-known definition, the $p$-adic absolute value is given by $|x|_p = p^{-r}$, where $x = p^r t/s$ with $(t, p) = (s, p) = (t, s) = 1$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$.

We assume that $\text{UD}(\mathbb{Z}_p)$ is the space of the uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, Kim defined the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \quad (1.1)$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x + n)$ be translation. As a well known equation, by (1.1), we have

$$q^n \int_{\mathbb{Z}_p} f(x+n) d\mu_{-q}(x) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \quad (1.2)$$
compared [1–4]. Throughout this paper we use the notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q},
\]

(cf. [1–16]). \(\lim_{q \to 1} [x]_q = x\) for any \(x\) with \(|x|_p \leq 1\) in the present \(p\)-adic case. To investigate relation of the twisted \(q\)-Genocchi numbers and polynomials with weight \(\alpha\) and the Bernstein polynomials with weight \(\alpha\), we will use useful property for \([x]_{q^\alpha}\) as follows;

\[
[x]_{q^\alpha} = 1 - [1 - x]_{q^{\alpha - 1}},
\]

\[
[1 - x]_{q^{\alpha - 1}} = 1 - [x]_{q^\alpha}.
\]

The twisted \(q\)-Genocchi numbers and polynomials with weight \(\alpha\) are defined by the generating function as follows, respectively:

\[
G_{n,q,w}^{(\alpha)} = n \int_{\mathbb{Z}_p} \phi_w(x) [x]_q^{n-1} d\mu_{-q}(x),
\]

\[
G_{n,q,w}^{(\alpha)}(x) = n \int_{\mathbb{Z}_p} \phi_w(y) [y + x]_q^{n-1} d\mu_{-q}(y).
\]

In the special case, \(x = 0\), \(G_{n,q,w}^{(\alpha)}(0) = G_{n,q,w}^{(\alpha)}\) are called the \(n\)th twisted \(q\)-Genocchi numbers with weight \(\alpha\) (see [9]).

Let \(C_{p^n} = \{w \mid w^{p^n} = 1\}\) be the cyclic group of order \(p^n\) and let

\[
T_p = \lim_{n \to \infty} C_{p^n} = \bigcup_{n \geq 1} C_{p^n},
\]

see [9, 12–15].

Kim defined the \(q\)-Bernstein polynomials with weight \(\alpha\) of degree \(n\) as follows:

\[
B_{n,k}^{(\alpha)}(x) = \binom{n}{k} [x]_{q^\alpha}^{k} [1 - x]_{q^\alpha}^{n-k}, \quad \text{where} \ x \in [0,1], \ n, k \in \mathbb{Z}_+, \]

compare [4, 7].

In this paper, we investigate some properties for the twisted \(q\)-Genocchi numbers and polynomials with weight \(\alpha\). By using these properties, we give some interesting identities on the twisted \(q\)-Genocchi polynomials with weight \(\alpha\) and \(q\)-Bernstein polynomials with weight \(\alpha\).
2. Some Identities on the Twisted $q$-Genocchi Polynomials with Weight $\alpha$ and $q$-Bernstein Polynomials with Weight $\alpha$

From (1.8), we can derive the following recurrence formula for the twisted $q$-Genocchi numbers with weight $\alpha$:

$$
G_{0,q,w}^{(a)} = 0, \quad qwG_{n,q,w}^{(a)}(1) + G_{n,q,w}^{(a)} = \begin{cases} 
[2]_q & \text{if } n = 1, \\
0 & \text{if } n > 1, 
\end{cases} 
$$

(2.1)

$$
G_{0,q,w}^{(a)} = 0, \quad qw \left( 1 + q^a G_{q,w}^{(a)} \right)^n + q^a G_{n,q,w}^{(a)} = \begin{cases} 
q^n [2]_q & \text{if } n = 1, \\
0 & \text{if } n > 1, 
\end{cases} 
$$

(2.2)

$$
q^{a x} G_{n+1,q,w}^{(a)}(x) = \left( [x]_q^a + q^{ax} G_{q,w}^{(a)} \right)^{n+1} 
$$

(2.3)

with usual convention about replacing $(G_{q,w}^{(a)})^n$ by $G_{n,q,w}^{(a)}$.

By (1.5), we easily get

$$
G_{n,q,w}^{(a)}(x) = n[2]_q \left( \frac{1}{1 - q^a} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{a x l} \frac{1}{1 + w q^{a l+1}}. 
$$

(2.4)

By (2.4), we obtain the theorem below.

**Theorem 2.1.** Let $n \in \mathbb{Z}_+$. For $w \in T_p$, one has

$$
G_{n,q,w}^{(a)}(x) = (-1)^{n-1} w^{-1} q^{a(1-n)} G_{n,q^{-1},w^{-1}}^{(a)}(1 - x). 
$$

(2.5)

By (2.1), (2.2), and (2.3) we note that

$$
G_{n,q,w}^{(a)} = -qwG_{n,q,w}^{(a)}(1) 
$$

$$
= -nwo G_{1,q,w}^{(a)} + w^2 q^{2-2a} \sum_{l=2}^{n} \binom{n}{l} q^{a l} G_{l,q,w}^{(a)}(1) 
$$

$$
= -nwo G_{1,q,w}^{(a)} + w^2 q^{2-2a} \sum_{l=2}^{n} \binom{n}{l} q^{a l} \left( 1 + q^a G_{q,w}^{(a)} \right)^l 
$$

(2.6)

$$
= -nwo G_{1,q,w}^{(a)} + w^2 q^{2-2a} [2]_q^{a l} + q^{2a} G_{q,w}^{(a)} - nwo q^2 G_{1,q,w}^{(a)} 
$$

$$
= -nwo G_{1,q,w}^{(a)} + w^2 q^2 G_{n,q,w}^{(a)}(2) - nwo q^2 G_{1,q,w}^{(a)}. 
$$

Therefore, by (2.6), we obtain the theorem below.
Theorem 2.2. For $n \in \mathbb{N}$ with $n > 1$, one has

$$G^{(a)}_{n,q,w}(2) = w^{-2}q^{-2}G^{(a)}_{n,q,w} - \frac{n[2]_q}{1 + qw} + \frac{n[2]_q}{1 + qw}.$$  \hfill (2.7)

By (1.6) and Theorem 2.2,

$$\frac{G^{(a)}_{n+1,q,w}(2)}{n + 1} = \int_{\mathbb{Z}_p} \phi_w(y) [y + 2]_q^n d\mu_{-q}(y)$$

$$= \frac{1}{n + 1} \left( \frac{(n + 1)[2]_q}{1 + qw} + \frac{(n + 1)w^{-1}q^{-1}[2]_q}{1 + qw} \right) + w^{-2}q^{-2} \frac{G^{(a)}_{n+1,q,w}}{n + 1} \hfill (2.8)$$

$$= \frac{[2]_q}{1 + qw} + w^{-1}q^{-1} \frac{[2]_q}{1 + qw} + w^{-2}q^{-2} \frac{G^{(a)}_{n+1,q,w}}{n + 1}.$$

Hence, we obtain the corollary below.

Corollary 2.3. For $n \in \mathbb{N}$, one has

$$\int_{\mathbb{Z}_p} \phi_w(y) [y + 2]_q^n d\mu_{-q}(y) = \frac{[2]_q}{1 + qw} + w^{-1}q^{-1} \frac{[2]_q}{1 + qw} + w^{-2}q^{-2} \frac{G^{(a)}_{n+1,q,w}}{n + 1}. \hfill (2.9)$$

By fermionic integral on $\mathbb{Z}_p$, Theorems 2.1 and 2.2, we note that

$$\int_{\mathbb{Z}_p} \phi_w(x) [1 - x]_q^n d\mu_{-q}(x) = (-1)^n q^{an} \int_{\mathbb{Z}_p} \phi_w(x) [x - 1]_q^n d\mu_{-q}(x)$$

$$= (-1)^n q^{an} \frac{G^{(a)}_{n+1,q,w}(-1)}{n + 1}$$

$$= w^{-1} \frac{G^{(a)}_{n+1,q,e^{-1},w^{-1}}(2)}{n + 1}$$

$$= w^{-1} \left( \frac{[2]_q^{-1}}{1 + q^{-1}w^{-1}} + wq^{-1} \frac{[2]_q^{-1}}{1 + q^{-1}w^{-1}} + w^2q^{-2} \frac{G^{(a)}_{n+1,q,e^{-1},w^{-1}}}{n + 1} \right)$$

$$= \frac{[2]_q}{1 + qw} + wq^{-1} \frac{[2]_q}{1 + qw} + w^2q^{-2} \frac{G^{(a)}_{n+1,q,e^{-1},w^{-1}}}{n + 1}. \hfill (2.10)$$

Therefore, we have the theorem below.
Theorem 2.4. For $n \in \mathbb{N}$ with $n > 1$, one has
\[
\int_{Z_p} \phi_w(x)[1-x]_q^n d\mu_q(x) = \frac{[2]_q}{1 + q\omega} + \omega q \frac{[2]_q}{1 + q\omega} + \omega q^2 \frac{G^{(a)}_{n+1,q^{-1},w^{-1}}}{n + 1}. \tag{2.11}
\]

By (1.4), Theorem 2.4, we take the fermionic $p$-adic invariant integral on $Z_p$ for one $q$-Bernstein polynomials as follows:
\[
\int_{Z_p} \phi_w(x) B_{n,k}(x, q) d\mu_q(x) = \int_{Z_p} \phi_w(x) \binom{n}{k} [x]_q^n [1-x]_q^n d\mu_q(x)
= \binom{n}{k} \int_{Z_p} \phi_w(x) [x]_q^n \left(1 - [x]_q^n \right)^{n-k} d\mu_q(x) \tag{2.12}
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \frac{G^{(a)}_{k+l+1,q,w}}{k+l+1}.
\]

By symmetry of $q$-Bernstein polynomials with weight $a$ of degree $n$, we get the following formula:
\[
\int_{Z_p} \phi_w(x) B_{n,k}(x, q) d\mu_q(x)
= \int_{Z_p} \phi_w(x) \binom{n}{k} [x]_q^n [1-x]_q^n d\mu_q(x)
= \int_{Z_p} \phi_w(x) \binom{n}{k} [1-x]_q^n \left(1 - [1-x]_q^n \right)^{n-k} d\mu_q(x) \tag{2.13}
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \int_{Z_p} \phi_w(x) [1-x]_q^n d\mu_q(x)
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \left( \frac{[2]_q}{1 + q\omega} + \omega q \frac{[2]_q}{1 + q\omega} + \omega q^2 \frac{G^{(a)}_{n-l+1,q^{-1},w^{-1}}}{n-l+1} \right).
\]

Therefore, by (2.12) and (2.13), we have the theorem below.

Theorem 2.5. For $n \in \mathbb{N}$ with $n > 1$, one has
\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \frac{G^{(a)}_{k+l+1,q,w}}{k+l+1} = \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \left( \frac{[2]_q}{1 + q\omega} + \omega q \frac{[2]_q}{1 + q\omega} + \omega q^2 \frac{G^{(a)}_{n-l+1,q^{-1},w^{-1}}}{n-l+1} \right). \tag{2.14}
\]
Also, we note that

\[
\int_{\mathbb{Z}_p} \phi_w(x) B_{n,k}(x, q) d\mu_{-q}(x)
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{G_{k+l+1,q,w}^{(a)}}{k+l+1}
\]

\[
= \binom{n}{k} \int_{\mathbb{Z}_p} \phi_w(x) [1 - x]_{q^{-1}}^{n-k}[x]_{q^{-1}}^k d\mu_{-q}(x)
\]

\[
= \binom{n}{k} \int_{\mathbb{Z}_p} \phi_w(x) [1 - x]_{q^{-1}}^{n-k} \left( 1 - [1 - x]_{q^{-1}}^k \right) d\mu_{-q}(x)
\]

\[
= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} \phi_w(x) [1 - x]_{q^{-1}}^{n-l} d\mu_{-q}(x)
\]

\[
= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left( \frac{[2]_{q^{-1}}}{1 + q \omega} + wt \frac{[2]_{q^{-1}}}{1 + q \omega} + wt^2 \frac{G_{n-l+1,q^{-1},w^{-1}}^{(a)}}{n - l + 1} \right).
\] (2.15)

Therefore, we have the theorem below.

**Theorem 2.6.** For \( n, k \in \mathbb{Z}_+ \) with \( n > k + 1 \), one has

\[
\int_{\mathbb{Z}_p} \phi_w(x) B_{k,n}(x, q) d\mu_{-q}(x)
= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left( \frac{[2]_{q^{-1}}}{1 + q \omega} + wt \frac{[2]_{q^{-1}}}{1 + q \omega} + wt^2 \frac{G_{n-l+1,q^{-1},w^{-1}}^{(a)}}{n - l + 1} \right).
\] (2.16)

By (2.11) and Theorem 2.6, we have the theorem below.

**Theorem 2.7.** Let \( n, k \in \mathbb{Z}_+ \) with \( n > k + 1 \). Then one has

\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{G_{k+l+1,q,w}^{(a)}}{k+l+1}
= \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left( \frac{[2]_{q^{-1}}}{1 + q \omega} + wt \frac{[2]_{q^{-1}}}{1 + q \omega} + wt^2 \frac{G_{n-l+1,q^{-1},w^{-1}}^{(a)}}{n - l + 1} \right).
\] (2.17)
Let \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k + 1 \). Then we get

\[
\int_{\mathbb{Z}_p} \phi_w(x) B_{n_1,k}^{(a)}(x,q) B_{n_2,k}^{(a)}(x,q) \, d\mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} \frac{2k}{l} (-1)^{2k-l} \int_{\mathbb{Z}_p} \phi_w(x) [1-x]^{n_1+n_2-l} \, d\mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \left( \frac{[2]_q}{1+qw} + \frac{[2]_q}{1+qw} + \frac{[2]_q}{1+qw} \frac{G_{n_1+n_2-l+1}^{(a),q^{-1},w^{-1}}}{n_1 + n_2 - l + 1} \right).
\]

Therefore, we obtain the theorem below.

**Theorem 2.8.** For \( n_1, n_2, k \in \mathbb{Z}_+ \), one has

\[
\int_{\mathbb{Z}_p} \phi_w(x) B_{n_1,k}^{(a)}(x,q) B_{n_2,k}^{(a)}(x,q) \, d\mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \left( \frac{[2]_q}{1+qw} + \frac{[2]_q}{1+qw} + \frac{[2]_q}{1+qw} \frac{G_{n_1+n_2-l+1}^{(a),q^{-1},w^{-1}}}{n_1 + n_2 - l + 1} \right),
\]

\[
= \begin{cases} 
\frac{[2]_q}{1+qw} + \frac{[2]_q}{1+qw} + \frac{[2]_q}{1+qw} \frac{G_{n_1+n_2-l+1}^{(a),q^{-1},w^{-1}}}{n_1 + n_2 - l + 1}, & \text{if } k = 0, \\
\frac{[2]_q}{1+qw} + \frac{[2]_q}{1+qw} + \frac{[2]_q}{1+qw} \frac{G_{n_1+n_2-l+1}^{(a),q^{-1},w^{-1}}}{n_1 + n_2 - l + 1}, & \text{if } k > 0,
\end{cases}
\]

By simple calculation, we easily see that

\[
\int_{\mathbb{Z}_p} \phi_w(x) B_{n_1,k}^{(a)}(x,q) B_{n_2,k}^{(a)}(x,q) \, d\mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \int_{\mathbb{Z}_p} \phi_w(x) [x]^{2k+l} \, d\mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \frac{G_{2k+l+1}^{(a),q,w}}{2k+l+1}, \quad \text{where } n_1, n_2, k \in \mathbb{Z}_+.
\]

Therefore, by (2.20) and Theorem 2.8, we obtain the theorem below.
Theorem 2.9. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then one has

$$
\sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} \left( \frac{[2]_q}{1 + qw} + \frac{[2]_q}{1 + qw} + wq^2 \frac{G_{n_1+n_2-l+1,q^{-1},w^{-1}}^{(a)}}{n_1 + n_2 - l + 1} \right) = \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1 + n_2 - 2k}{l} \frac{G_{2k+l+1,q,w}^{(a)}}{2k + l + 1}.
$$

(2.21)

For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$, $n_1 + n_2 + \cdots + n_s > sk + 1$, and let $\sum_{i=1}^s n_i = m$, then by the symmetry of $q$-Bernstein polynomials with weight $\alpha$, we see that

$$
\int_{z_p} \phi_w(x) \prod_{i=1}^s B_{k,n_i}^{(a)}(x,q) d\mu_{-q}(x)
= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \int_{z_p} \phi_w(x) [1-x]^{m-l} d\mu_{-q}(x)
= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \left( \frac{[2]_q}{1 + qw} + \frac{[2]_q}{1 + qw} + wq^2 \frac{G_{m-l+1,q^{-1},w^{-1}}^{(a)}}{m - l + 1} \right).
$$

(2.22)

Therefore, we have the theorem below.

Theorem 2.10. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$, one has

$$
\int_{z_p} \phi_w(x) \prod_{i=1}^s B_{k,n_i}^{(a)}(x,q) d\mu_{-q}(x)
= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \left( \frac{[2]_q}{1 + qw} + \frac{[2]_q}{1 + qw} + wq^2 \frac{G_{m-l+1,q^{-1},w^{-1}}^{(a)}}{m - l + 1} \right),
$$

where $n_1 + \cdots + n_s = m$.

In the same manner as in (2.15), we can get the following relation:

$$
\int_{z_p} \phi_w(x) \prod_{i=1}^s B_{k,n_i}^{(a)}(x,q) d\mu_{-q}(x)
= \prod_{i=1}^s \binom{n_i}{k} \int_{z_p} \phi_w(x) [x]_q^m \sum_{l=0}^{m-sk} (-1)^l \binom{m-sk}{l} (-1)^l [x]_q^l d\mu_{-q}(x)
= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{m-sk} (-1)^l \binom{m-sk}{l} \frac{G_{sk+l+1,q,w}^{(a)}}{sk + l + 1},
$$

where $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $m = n_1 + n_2 + \cdots + n_s > sk + 1$. 
By Theorem 2.10 and (2.13), we have the following corollary.

**Corollary 2.11.** Let \( m \in \mathbb{N} \). For \( n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+ \) with \( n_1 + \cdots + n_s > mk + 1 \), one has

\[
\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} \left( \frac{[2]_q}{1 + qw} + \frac{[2]_q}{1 + qw} + wq \frac{G_{m-l+1,q}^{(a)}}{m-l+1} \right) = \sum_{l=0}^{sk} (-1)^{l} \binom{m-sk}{l} \frac{C_{sk+l}^{(a)}}{sk + l + 1},
\]

(2.25)

where \( n_1 + \cdots + n_s = m \).

**References**


