Research Article

Hyers-Ulam Stability of Power Series Equations

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We prove the Hyers-Ulam stability of power series equation $\sum_{n=0}^{\infty} a_n x^n = 0$, where a_n for $n = 0, 1, 2, 3, \dots$ can be real or complex.

1. Introduction and Preliminaries

A classical question in the theory of functional equations is that “when is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$.” Such a problem was formulated by Ulam [1] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [2]. It gave rise to the *Hyers-Ulam stability* for functional equations.

In 1978, Th. M. Rassias [3] provided a generalization of Hyers’ theorem by proving the existence of unique linear mappings near approximate additive mappings. On the other hand, J. M. Rassias [4–6] considered the Cauchy difference controlled by a product of different powers of norm. This new concept is known as generalized Hyers-Ulam stability of functional equations (see also [7–10] and references therein).

Recently, Li and Hua [11] discussed and proved the Hyers-Ulam stability of a polynomial equation

$$x^n + \alpha x + \beta = 0, \quad (1.1)$$

where $x \in [-1, 1]$ and proved the following.

Theorem 1.1. *If $|\alpha| > n$, $|\beta| < |\alpha| - 1$ and $y \in [-1, 1]$ satisfies the inequality*

$$|y^n + \alpha y + \beta| \leq \varepsilon, \quad (1.2)$$

then there exists a solution $v \in [-1, 1]$ of (1.1) such that

$$|y - v| \leq K\varepsilon, \quad (1.3)$$

where $K > 0$ is constant.

They also asked an open problem whether the real polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad (1.4)$$

has Hyers-Ulam stability for the case that this real polynomial equation has some solution in $[a, b]$.

In this paper we establish the Hyers-Ulam-Rassias stability of power series with real or complex coefficients. So we prove the generalized Hyers-Ulam stability of equation

$$f(z) = 0, \quad (1.5)$$

where f is any analytic function. First we give the definition of the generalized Hyers-Ulam stability.

Definition 1.2. Let p be a real number. We say that (1.7) has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\varepsilon > 0$, $y \in [-1, 1]$ if

$$\left| \sum_{n=0}^{\infty} a_n y^n \right| \leq \varepsilon \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n} \right), \quad (1.6)$$

then there exists some $x \in [-1, 1]$ satisfying

$$\sum_{n=0}^{\infty} a_n x^n = 0 \quad (1.7)$$

such that $|y - x| \leq K\varepsilon$. For the complex coefficients, $[-1, 1]$ can be replaced by closed unit disc

$$D = \{z \in \mathbb{C}; |z| \leq 1\}. \quad (1.8)$$

2. Main Results

The aim of this work is to investigate the generalized Hyers-Ulam stability for (1.7).

Theorem 2.1. *If*

$$\sum_{n=0, n \neq 1}^{\infty} |a_n| < |a_1|, \quad (2.1)$$

$$\sum_{n=2}^{\infty} n|a_n| < |a_1|, \quad (2.2)$$

then there exists an exact solution $v \in [-1, 1]$ of (1.7).

Proof. If we set

$$g(x) = \frac{-1}{a_1} \left(\sum_{n=0, n \neq 1}^{\infty} a_n x^n \right), \quad (2.3)$$

for $x \in [-1, 1]$, then we have

$$\begin{aligned} |g(x)| &= \frac{1}{|a_1|} \left| \sum_{n=0, n \neq 1}^{\infty} a_n x^n \right| \\ &\leq \frac{1}{|a_1|} \left(\sum_{n=0, n \neq 1}^{\infty} |a_n| \right) \\ &\leq 1 \end{aligned} \quad (2.4)$$

by (2.1).

Let $X = [-1, 1]$, $d(x, y) = |x - y|$. Then (X, d) is a complete metric space and g map X to X . Now, we will show that g is a contraction mapping from X to X . For any $x, y \in X$, we have

$$\begin{aligned} d(g(x), g(y)) &= \left| \frac{1}{a_1} (-a_0 - a_2 x^2 - \dots) - \frac{1}{a_1} (-a_0 - a_1 y^2 - \dots) \right| \\ &\leq \frac{1}{|a_1|} |x - y| \left\{ \sum_{n=2}^{\infty} n|a_n| \right\}. \end{aligned} \quad (2.5)$$

For $x, y \in [-1, 1]$, $x \neq y$, from (2.2), we obtain

$$d(g(x), g(y)) \leq \lambda d(x, y), \quad (2.6)$$

where

$$\lambda = \frac{\sum_{n=2}^{\infty} n|a_n|}{|a_1|} < 1. \quad (2.7)$$

Thus g is a contraction mapping from X to X . By the Banach contraction mapping theorem, there exists a unique $v \in X$, such that

$$g(v) = v. \quad (2.8)$$

Hence, (1.7) has a solution on $[-1, 1]$. $\square$

Theorem 2.2. *Under the conditions of Theorem 2.1, (1.7) has the generalized Hyers-Ulam stability.*

Proof. Let $\varepsilon > 0$ and $y \in [-1, 1]$ be such that

$$\left| \sum_{n=0}^{\infty} a_n y^n \right| \leq \varepsilon \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n} \right). \quad (2.9)$$

We will show that there exists a constant K independent of ε , v , and y such that

$$|y - v| \leq K\varepsilon \quad (2.10)$$

for some $v \in [-1, 1]$ satisfying (1.7).

Let us introduce the abbreviation $K = 2/(|a_1|^{1-p}(1-\lambda))$. Then

$$\begin{aligned} |y - v| &= |y - g(y) + g(y) - g(v)| \leq |y - g(y)| + |g(y) - g(v)| \\ &\leq \left| y - \left(\frac{-1}{a_1} \sum_{n=0, n \neq 1}^{\infty} a_n y^n \right) \right| + \lambda |y - v| \\ &= \frac{1}{|a_1|} \left| \sum_{n=0}^{\infty} a_n y^n \right| + \lambda |y - v|. \end{aligned} \quad (2.11)$$

Thus, we have

$$\begin{aligned} |y - v| &\leq \frac{1}{|a_1|(1-\lambda)} \left| \sum_{n=0}^{\infty} a_n y^n \right| \leq \frac{1}{|a_1|(1-\lambda)} \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n} \right) \varepsilon \\ &\leq \frac{1}{|a_1|(1-\lambda)} \left(\sum_{n=0}^{\infty} \frac{|a_1|^p}{2^n} \right) \varepsilon \\ &\leq K\varepsilon \end{aligned} \quad (2.12)$$

by (2.9) and so the result follows. $\square$

Next, for equation of complex power series

$$\sum_{n=0}^{\infty} a_n z^n = 0, \quad (2.13)$$

as an application of Rouché's theorem, we prove the following theorem which is much better than above result. In fact, we prove the following.

Theorem 2.3. *If*

$$\sum_{n=0, n \neq 1}^{\infty} |a_n| < |a_1|. \quad (2.14)$$

Then there exists an exact solution in open unit disc for (2.13).

Proof. If we set

$$g(z) = \frac{-1}{a_1} \left(\sum_{n=0, n \neq 1}^{\infty} a_n z^n \right), \quad (2.15)$$

for $|z| \leq 1$. Such as above we have

$$\begin{aligned} |g(z)| &= \frac{1}{|a_1|} \left| \sum_{n=0, n \neq 1}^{\infty} a_n z^n \right| \\ &\leq \frac{1}{|a_1|} \left(\sum_{n=0, n \neq 1}^{\infty} |a_n| \right), \quad \text{for } |z| \leq 1 \\ &< 1 \end{aligned} \quad (2.16)$$

by (2.14).

Since $|g(z)| < 1$ for $|z| = 1$, hence for $|g(z)| < |-z| = 1$ and by Rouché's theorem, we observe that $g(z) - z$ has exactly one zero in $|z| < 1$ which implies that g has a unique fixed point in $|z| < 1$. $\square$

Corollary 2.4. *Under the conditions of Theorem 2.1, (2.13) has the generalized Hyers-Ulam stability.*

For $R \geq 1$, we have the following corollary.

Corollary 2.5. *If*

$$\sum_{n=0, n \neq 1}^{\infty} |a_n| R^n < |a_1| R, \quad (2.17)$$

then there exists an exact solution in $\{z \in \mathbb{C}; |z| < R\}$ for (2.13).

The proof is similar to previous and details are omitted.

Remark 2.6. By the similar way, one can easily prove the generalized Hyers-Ulam stability of (1.7) on any finite interval $[a, b]$.

Remark 2.7. By replacing $a_n = f^{(n)}(0)$ in (2.14), we can prove the generalized Hyers-Ulam stability for (1.5).

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