Research Article

Summability of Sequences and Selection Properties

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We prove that some classes of summable sequences of positive real numbers satisfy several selection principles related to a special kind of convergence.

1. Introduction

By \(\mathbb{N}, \mathbb{R},\) and \(\mathbb{R}\) we denote the set of natural numbers, real numbers, and the extended real line \(\mathbb{R} \cup \{-\infty, \infty\},\) respectively.

Let \(S\) denote the set of sequences \(a = (a_n)_{n \in \mathbb{N}}\) of positive real numbers.

We begin with the following definitions of selection principles.

Let \(\mathcal{A}\) and \(\mathcal{B}\) be nonempty subsets of \(S.\) Then the symbol \(S_1(\mathcal{A}, \mathcal{B})\) denotes the selection principle.

For each sequence \((a_n : n \in \mathbb{N})\) of elements of \(\mathcal{A}\) there is a sequence \(b = (b_n)_{n \in \mathbb{N}} \in \mathcal{B}\) such that \(b_n \in a_n\) for each \(n \in \mathbb{N}.\)

The following infinitely long game \(G_1(\mathcal{A}, \mathcal{B})\) is naturally associated to the previous selection principle.

Two players, ONE and TWO, play a round for each positive integer. In the \(n\)-th round ONE chooses a sequence \(a_n \in \mathcal{A},\) and TWO responds by choosing an element \(b_n \in a_n.\) TWO wins a play \((a_1, b_1; \ldots; a_n, b_n; \ldots)\) if \(b = (b_n)_{n \in \mathbb{N}} \in \mathcal{B};\) otherwise, ONE wins.

Another selection principle \(S_{\text{fin}}(\mathcal{A}, \mathcal{B})\) is defined as follows.

For each sequence \((a_n : n \in \mathbb{N})\) of elements of \(\mathcal{A}\) there is a sequence \(b \in \mathcal{B}\) such that \(b \cap a_n\) is finite for each \(n \in \mathbb{N}.\)

It is clear how the corresponding game \(G_{\text{fin}}(\mathcal{A}, \mathcal{B})\) is defined.
A strategy of a player is a function \( \sigma \) from the set of all finite sequences of moves of the other player into the set of admissible moves of the strategy owner.

A strategy \( \sigma \) for the player TWO is a coding strategy if TWO remembers only the most recent move by ONE and by TWO before his next move. More precisely the moves of TWO are \( b_1 = \sigma(a_1, \emptyset); b_n = \sigma(a_n, b_{n-1}), n \geq 2 \).

In this paper we introduce also the following game. Let \( i \in \mathbb{N} \) be a fixed (but arbitrary) natural number. We define the game \( G_{1}^{(\infty)}(\mathcal{A}, \mathcal{B}) \) for two players, ONE and TWO, who play a round for each \( n \in \mathbb{N} \). In the \( i \)-th round ONE plays a sequence \( a_i = (a_{i,m})_{m \in \mathbb{N}} \in \mathcal{A} \), and TWO responds by choosing a finite set \( F_i = \{ a_{i,m_1}, \ldots, a_{i,m_k} \} \). In the \( n \)-th round, \( n \neq i \), ONE plays a sequence \( a_n = (a_{n,m})_{m \in \mathbb{N}} \in \mathcal{A} \), and TWO responds by choosing an element \( a_{n,m_n} \in a_n \). TWO wins a play if the sequence \( b = (a_{1,m_1}, \ldots, a_{i-1,m_i}, a_{i,m_i}, \ldots, a_{i,m_k}, a_{i+1,m_{i+1}}, \ldots) \) belongs to \( \mathcal{B} \); otherwise, ONE wins.

For more information on selection principles and games see the survey papers in [1, 2] and references therein.

In a number of papers by the authors published in the last few years it was demonstrated that some subclasses \( \mathcal{A} \) and \( \mathcal{B} \) of \( \mathcal{S} \) satisfy certain selection principles and game theoretical statements (for \( \mathcal{A} \) and \( \mathcal{B} \) classes of divergent sequences related to celebrated Karamata’s theory of regular variation [3–6] see [7–12], and for \( \mathcal{A} \) and \( \mathcal{B} \) classes of sequences converging to 0 see [13]). For other results concerning sequences and sequence spaces see [14–16].

In this paper our selections are related to special kinds of convergence of series. More precisely, we start by a sequence of summable sequences and during the selection process we control not only the convergence of series, but also the nature of that convergence.

### 2. Results

We use the following notations for the classes of sequences we deal with:

\[
\mathcal{E}^1 = \left\{ a \in \mathcal{S} : \sum_{n=1}^{\infty} a_n < \infty \right\},
\]

\[
\mathcal{E}^{1,S} = \left\{ a \in \mathcal{S} : \sum_{n=1}^{\infty} a_n = S \right\}, \quad \text{for } S \in (0, \infty],
\]

\[
\mathcal{E}^{1,(\alpha, \beta)} = \left\{ a \in \mathcal{E}^{1,S} : S \in (\alpha, \beta) \right\}, \quad \text{for } \alpha, \beta \in (0, \infty),
\]

\[
\mathcal{E}^{1,\{\alpha, \beta\}} = \mathcal{E}^{1,(\alpha, \beta)} \cup \mathcal{E}^{1,\beta}, \quad \text{for } \alpha, \beta \in (0, \infty).
\]

Notice that the sequence \( x = (x_n)_{n \in \mathbb{N}}, x_n = S/2^n \), belongs to the class \( \mathcal{E}^{1,S} \), so that all the classes above are nonempty.

**Theorem 2.1.** For each \( S \in (0, \infty) \) and each \( \varepsilon = \varepsilon(S) \in (0, S) \) TWO has a winning coding strategy in the game \( G_{1}^{(\infty)}(\mathcal{E}^{1,S}, \mathcal{E}^{1,(S-\varepsilon,S)}) \).

**Proof.** Let \( \sigma \) denote a strategy of TWO, and let \( S > 0 \) and \( \varepsilon = \varepsilon(S) \in (0, S) \) be fixed. Suppose that in the first round ONE chooses a sequence \( x_1 = (x_{1,m})_{m \in \mathbb{N}} \) from \( \mathcal{E}^{1,S} \). There is \( k \in \mathbb{N} \)
such that \( \sum_{m=k+1}^{\infty} x_{1,m} < \varepsilon / 2 \), and thus \( M = S - \sum_{m=1}^{k} x_{1,m} \in (0, \varepsilon / 2) \). Player TWO plays \( \sigma(x_1) = \{ x_{1,1}, \ldots, x_{1,k} \} \) — a finite subset of \( x_1 \).

In the second round ONE chooses a sequence \( x_2 = (x_{2,m})_{m \in \mathbb{N}} \in \mathcal{E}^{1,S} \), and then TWO responds by choosing \( \sigma(x_2, \sigma(x_1)) = x_{2,m_2} \) such that \( x_{2,m_2} < M / 2 \) (which is possible because \( \lim_{m \to \infty} x_{2,m} = 0 \)).

In the \( n \)-th round, \( n \geq 3 \), ONE chooses \( x_n = (x_{n,m})_{m \in \mathbb{N}} \in \mathcal{E}^{1,S} \), and TWO’s response is \( \sigma(x_n, x_{n-1,m_{n-1}}) = x_{n,m_n} \) such that \( x_{n,m_n} < x_{n-1,m_{n-1}} / 2^{n-1} < M / 2^{n-1} \), and so on.

Set \( y_n = x_{1,n} \) for \( n \leq k \) and \( y_n = x_{n-k+1,m_{n-k+1}} \) for \( n > k \). Let us prove \( y = (y_n)_{n \in \mathbb{N}} \in \mathcal{E}^{1,(S-\varepsilon,S)} \). We have

\[
\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{k} y_n + \sum_{n=k+1}^{\infty} y_n = \sum_{m=1}^{k} x_{1,m} + \sum_{n=k+1}^{\infty} y_n = S - M + \sum_{n=k+1}^{\infty} y_n < S - M + M \left( \sum_{i=1}^{\infty} \frac{1}{2^i} \right) = S.
\]

On the other hand,

\[
\sum_{n=1}^{\infty} y_n > \sum_{m=1}^{k} x_{1,m} = S - M > S - \frac{\varepsilon}{2}.
\]

That is, \( y \in \mathcal{E}^{1,(S-\varepsilon,S)} \). \( \square \)

**Corollary 2.2.** For each \( S \in (0, \infty) \) and each \( \varepsilon = \varepsilon(S) \in (0, S) \) the selection principle \( S_{\text{fin}}(\mathcal{E}^{1,S}, \mathcal{E}^{1,(S-\varepsilon,S)}) \) is true.

Notice that one can prove a refinement of Theorem 2.1 (and Corollary 2.2) in the sense that it is possible to have additional control of selections giving the sequence \( y \). For this we need the following definitions and notation.

**Definition 2.3** (see [13]). A sequence \( (x_n)_{n \in \mathbb{N}} \in \mathcal{S} \) is said to belong to the class \( \text{Tr}(\mathbb{R}_{-\infty,a}) \) if for each \( \lambda \geq 1 \) it satisfies

\[
\lim_{n \to \infty} \frac{x_{[n+1]}}{x_n} = 0,
\]

where \( [r] \) denotes the integer part of \( r \in \mathbb{R} \).

**Definition 2.4** (see [9]). For a sequence \( x = (x_n)_{n \in \mathbb{N}} \in \mathcal{S} \), the Landau-Hurwicz sequence \( w(x) = (w_n(x))_{n \in \mathbb{N}} \) of \( x \) is defined by

\[
w_n(x) := \sup \{|x_m - x_k| : m \geq n, k \geq n\}, \quad n \in \mathbb{N}.
\]
Given a sequence \( x = (x_n)_{n \in \mathbb{N}} \in S \) we denote by \( S_x = (S_n(x))_{n \in \mathbb{N}} \) the sequence defined by

\[
S_n(x) = \sum_{i=1}^{n} x_i, \quad n \in \mathbb{N}. \tag{2.6}
\]

Let \( \mathcal{C}_1^{(\alpha, \beta)} \) be the set of all sequences \( a = (a_n)_{n \in \mathbb{N}} \in \mathcal{C}_1^{(\alpha, \beta)} \) such that \( \omega(S_n) \in \text{Tr}(\mathbb{R}_{-\infty, \alpha}) \).

**Theorem 2.5.** For each \( S \in (0, \infty) \) and each \( \varepsilon = \varepsilon(S) \in (0, S) \) TWO has a winning coding strategy in the game \( S^{(n=1)}(\mathcal{C}_1^{(\alpha, \beta)}) \).

**Proof.** The strategy \( \sigma \) of player TWO and the sequence \( y = (y_n)_{n \in \mathbb{N}} \) are actually from the proof of Theorem 2.1. Therefore, \( y \in \mathcal{C}_1^{(S-\varepsilon, S)} \). Besides, since, by construction, the series

\[
\sum_{n=1}^{\infty} \frac{y_{n+1}}{y_n}
\]

is convergent, we have

\[
\lim_{n \to \infty} \left( \sum_{k=n}^{\infty} \frac{y_{k+1}}{y_k} \right) = 0. \tag{2.8}
\]

Consider now the sequence \( S_y = (S_n(y))_{n \in \mathbb{N}} \). This sequence is convergent (by the \( d'\)Alembert criterion), and let \( S(y) \) be its limit. It remains to prove \( \omega(S_y) = (\omega_n(S_y))_{n \in \mathbb{N}} \in \text{Tr}(\mathbb{R}_{-\infty, \alpha}) \). It is enough to prove

\[
\lim_{n \to \infty} \frac{\omega_{n+1}(S_y)}{\omega_n(S_y)} = 0. \tag{2.9}
\]

First, notice that

\[
\omega_n(S_y) = S(y) - S_n(y), \quad n \in \mathbb{N}. \tag{2.10}
\]

Thus we get

\[
\lim_{n \to \infty} \frac{\omega_{n+1}(S_y)}{\omega_n(S_y)} = \lim_{n \to \infty} \frac{S(y) - S_{n+1}(y)}{S(y) - S_n(y)} = 1 - \lim_{n \to \infty} \frac{y_{n+1}}{y_{n+1} + y_{n+2} + \cdots} = 0. \tag{2.11}
\]

That is (2.6), since by (2.8) and the fact that for \( n \) sufficiently large it holds

\[
\frac{y_{n+2}}{y_{n+1}} + \frac{y_{n+3}}{y_{n+1}} + \cdots \leq \frac{y_{n+2}}{y_{n+1}} + \frac{y_{n+3}}{y_{n+1}} + \cdots, \tag{2.12}
\]
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we have

$$\lim_{n \to \infty} \frac{y_{n+1}}{y_{n+1} + y_{n+2} + \cdots} = \lim_{n \to \infty} \frac{1}{1 + (y_{n+2}/y_{n+1}) + (y_{n+3}/y_{n+1}) + \cdots} = 1.$$  \hspace{1cm} (2.13)

The theorem is proved. \hfill \Box

**Corollary 2.6.** The selection principle $S_{\text{fin}}(\ell^{1,S}, \ell^{1,1-\epsilon,S})$ is true.

The following two theorems give other selection results for defined classes of sequences: one of the $S_{\text{fin}}$-type and the other of the $S_1$-type.

**Theorem 2.7.** For each $S \in (0, \infty]$ the selection principle $S_{\text{fin}}(\ell^{1,S}, \ell^{1,\infty})$ is satisfied.

*Proof.* Consider first the case $S \in (0, \infty)$. Let $(x_n : n \in \mathbb{N})$, $x_n = (x_{n,m})_{m \in \mathbb{N}}$, be a sequence of elements of $\ell^{1,S}$. For each $n \in \mathbb{N}$ let $z_n = x_{n,i}$, $i \leq k = k(n)$, be a finite subset of $x_n$ such that $S/2 < \sum_{i=1}^{k(n)} z_i < S$. Arrange now $z_{n,r} \in \mathbb{N}, p \in \{1,2,\ldots,k(n)\}$, in the sequence $y = (y_j)_{j \in \mathbb{N}}$ in which the position of an element is determined first by $n$ and then by $p$, that is,

$$y = (z_{1,1}, \ldots, z_{k(n),1}; \ldots; z_{1,2}, ..., z_{k(n),2}; \ldots).$$ \hspace{1cm} (2.14)

We have

$$n \cdot \frac{S}{2} < \sum_{m=1}^{n} \sum_{i=1}^{k(m)} z_{m,i} = \sum_{j=1}^{k(n)} y_j,$$ \hspace{1cm} (2.15)

where $y_{k(n)}$ is the last element of $x_n$ belonging to the sequence $y$. Therefore,

$$\sum_{j=1}^{\infty} y_j = \lim_{n \to \infty} \sum_{j=1}^{k(n)} y_j > \lim_{n \to \infty} \left( n \cdot \frac{S}{2} \right) = \infty.$$ \hspace{1cm} (2.16)

That is, $y \in \ell^{1,\infty}$.

Suppose now that $S = \infty$. This case is treated similarly to the previous case, but here we require $\sum_{i=1}^{k(n)} z_i > 1$ for each $n \in \mathbb{N}$; the sequence $y = (y_j)_{j \in \mathbb{N}}$ is formed in a similar way as in the first case. So we have $n \cdot 1 < \sum_{j=1}^{k(n)} y_j$ for each $n \in \mathbb{N}$, hence

$$\sum_{j=1}^{\infty} y_j = \lim_{n \to \infty} \sum_{j=1}^{k(n)} y_j > \lim_{n \to \infty} (n \cdot 1) = \infty.$$ \hspace{1cm} (2.17)

That is, $y \in \ell^{1,\infty}$. The theorem is proved. \hfill \Box

**Theorem 2.8.** For each $S \in (0, \infty)$ and each $\alpha > 0$ the selection principle $S_1(\ell^{1,S}, \ell^{1,0,\alpha})$ is true.

*Proof.* Let $(x_n : n \in \mathbb{N})$, $x_n = (x_{n,m})_{m \in \mathbb{N}}$, be a sequence of elements in $\ell^{1,S}$. For each $n \in \mathbb{N}$ take $y_n = x_{n,m_n} \in x_n$ so that $y_1 \in (0,\alpha)$ (which is possible since $x_{1,m} \to 0$ as $m \to \infty$) and
\[ y_n < \alpha - \frac{y_1}{2^{n-1}} \text{ for } n \geq 2. \] Then the sequence \( y = (y_n)_{n \in \mathbb{N}} \) witnesses that the statement is true, because

\[
\sum_{n=1}^{\infty} y_n = y_1 + \sum_{n=2}^{\infty} y_n < y_1 + (\alpha - y_1) = \alpha.
\] (2.18)

That is, \( y \in \ell^{1, (0, \alpha)}. \)

For the next result we have to define the following selection principles [2, 17]. Notice that in [18] we developed an interesting technique for proving results concerning these selection principles and certain classes of sequences from \( S \). In [19] we proposed the use of this technique (and these selection principles) in other fields of mathematics and its applications.

Let, as before, \( \mathcal{A} \) and \( \mathcal{B} \) be certain nonempty subfamilies of \( S \). Then the symbol \( \alpha_i(\mathcal{A}, \mathcal{B}), i = 2, 3, 4 \), denotes the selection hypothesis that for each sequence \( (a_n : n \in \mathbb{N}) \) of elements of \( \mathcal{A} \) there is an element \( b \in \mathcal{B} \) such that:

\[ \alpha_2(\mathcal{A}, \mathcal{B}): \text{for each } n \in \mathbb{N} \text{ the set } a_n \cap b \text{ is infinite;} \]

\[ \alpha_3(\mathcal{A}, \mathcal{B}): \text{for infinitely many } n \in \mathbb{N} \text{ the set } a_n \cap b \text{ is infinite;} \]

\[ \alpha_4(\mathcal{A}, \mathcal{B}): \text{for infinitely many } n \in \mathbb{N} \text{ the set } a_n \cap b \text{ is nonempty.} \]

**Theorem 2.9.** For each \( S \in (0, \infty) \) and each \( \alpha > 0 \) the selection principles \( \alpha_i(\ell^{1,S}, \ell^{1, (0, \alpha)}), i = 2, 3, 4 \), are satisfied.

**Proof.** We prove that the principle \( \alpha_2 \) is true (hence also \( \alpha_3 \) and \( \alpha_4 \)). Let \( (x_n : n \in \mathbb{N}), x_n = (x_{n,m})_{m \in \mathbb{N}}, \) be a sequence of sequences from \( \ell^{1, S} \). Let \( m_1 \in \mathbb{N} \) be such that \( \sum_{m=m_1+1}^{\infty} x_{1,m} < \alpha/2 \). For \( k \leq 2 \) let \( m_k \) be a natural number such that \( \sum_{m=m_k+1}^{\infty} x_{k,m} < \alpha/2^k \). Consider the sequence \( y = (y_j)_{j \in \mathbb{N}} \) defined in this way:

\[
y = (x_1,m_1+1, x_1,m_2+1, \ldots; x_2,m_1+1, x_2,m_2+1, \ldots; x_k,m_1+1, x_k,m_2+1, \ldots). \tag{2.19}
\]

Then \( y \cap x_n \) is infinite for each \( n \in \mathbb{N} \). Further, \( y \in \ell^{1, (0, \alpha)} \) because

\[
0 < \sum_{j=1}^{\infty} y_j = \sum_{k=1}^{\infty} \sum_{m=m_k+1}^{\infty} x_{k,m} < \sum_{k=1}^{\infty} \frac{\alpha}{2^k} = \alpha. \tag{2.20}
\]

We construct now a new sequence \( z = (z_i) \) in the way described in Table 1. Evidently, \( z \cap x_n \) is infinite for each \( n \in \mathbb{N} \). Also, \( 0 < \sum_{i=1}^{\infty} z_i \leq \sum_{j=1}^{\infty} y_j < \alpha \), that is, \( z \in \ell^{1, (0, \alpha)} \). By a minor modification of the proof of Theorem 2.5 we obtain \( w(S_z) \in \text{Tr}(\mathbb{R}_{-\infty, \alpha}) \). This means \( z \in \ell^{1, (0, \alpha)} \). \( \square \)
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Table 1

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