Research Article

On Value Distribution of Difference Polynomials of Meromorphic Functions

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We study the value distribution of the difference counterpart $\Delta f(z) - af(z)^n$ of $f'(z) - af(z)^n$ and obtain an almost direct difference analogue of results of Hayman.

1. Introduction and Results

Hayman proved the following Theorem A.

Theorem A (see [1]). *If $f(z)$ is a transcendental entire function, $n \geq 3$ is an integer, and $a (\neq 0)$ is a constant, then $f'(z) - af(z)^n$ assumes all finite values infinitely often.*

Recently, many papers (see [2–7]) have focused on complex differences, giving many difference analogues in value distribution theory of meromorphic functions.

It is well known that $\Delta f(z) = f(z+c) - f(z)$ (where $c \in \mathbb{C} \setminus \{0\}$ is a constant satisfying $f(z+c) - f(z) \not\equiv 0$) is regarded as the difference counterpart of $f'(z)$, so that $\Delta f(z) - af(z)^n$ is regarded as the difference counterpart of $f'(z) - af(z)^n$, where $a \in \mathbb{C} \setminus \{0\}$ is a constant.

Liu and Laine [7] obtain the following

Theorem B. *Let f be a transcendental entire function of finite order ρ , not of period c , where c is a nonzero complex constant, and let $s(z)$ be a nonzero function, small compared to f . Then the difference polynomial $f(z)^n + f(z+c) - f(z) - s(z)$ has infinitely many zeros in the complex plane, provided that $n \geq 3$.*

We use the basic notions of Nevanlinna's theory (see [8, 9]) and in addition use $\sigma(f)$ to denote the order of growth of the meromorphic $f(z)$ and $\lambda(f)$ to denote the exponent of convergence of the zeros of $f(z)$.

In this paper, we consider the difference counterpart of Theorem A. When $n \geq 3$ is an integer, we prove the following Theorem 1.1. Compared with Theorem B, Theorem 1.1 is an

almost direct difference analogue of Theorem A and gives an estimate of numbers of b -points, namely, $\lambda(\Psi_n(z) - b) = \sigma(f)$ for every $b \in \mathbb{C}$. Our method of the proof is also different from the method of the proof in Theorem B.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of finite order, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z + c) \neq f(z)$. Set $\Psi_n(z) = \Delta f(z) - af(z)^n$, where $\Delta f(z) = f(z + c) - f(z)$ and $n \geq 3$ is an integer. Then $\Psi_n(z)$ assumes all finite values infinitely often, and for every $b \in \mathbb{C}$ one has $\lambda(\Psi_n(z) - b) = \sigma(f)$.*

Example 1.2. For $f(z) = \exp\{z\}$, $c = \log 3$ and $a = 1$, we have

$$\Psi_3(z) = \Delta f(z) - af(z)^3 = -\exp\{z\}. \quad (1.1)$$

Here $\Psi_3(z) \neq 0$, which shows that Theorem 1.1 may fail for entire functions of infinite order.

Example 1.3. For $f(z) = \exp\{z\} + 1$, $c = \log 3$, $a = 1$, we have $\Psi_2(z) = \Delta f(z) - af(z)^2 = -\exp\{2z\} - 1$. Here $\Psi_2(z) \neq -1$, which shows that Theorem 1.1 may fail for $n = 2$ and that the condition $n \geq 3$ in Theorem 1.1 is sharp.

Example 1.4. For $f(z) = \exp\{z\}$, $c = \log 3$, $a = 1$, we have $\Psi_2(z) = \Delta f(z) - af(z)^2 = \exp\{z\}(2 - \exp\{z\})$, which assumes all finite values infinitely often.

What can we say about $\Psi_2(z)$ when $n = 2$? We consider this question and obtain the following Theorems 1.5 and 1.6.

Theorem 1.5. *Let $f(z)$ be a transcendental entire function of finite order with a Borel exceptional value 0, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z + c) \neq f(z)$. Then $\Psi_2(z)$ assumes all finite values infinitely often, and for every $b \in \mathbb{C}$ one has $\lambda(\Psi_2(z) - b) = \sigma(f)$.*

Theorem 1.6. *Let $f(z)$ be a transcendental entire function of finite order with a finite nonzero Borel exceptional value d , and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z + c) \neq f(z)$. Then for every $b \in \mathbb{C}$ with $b \neq -ad^2$, $\Psi_2(z)$ assumes the value b infinitely often, and $\lambda(\Psi_2(z) - b) = \sigma(f)$.*

Remark 1.7. From Theorems 1.5 and 1.6, we see that if $f(z)$ has the Borel exceptional value 0, then $\Psi_2(z)$ has not any finite Borel exceptional value, but if $f(z)$ has a nonzero Borel exceptional value, then $\Psi_2(z)$ may have a finite Borel exceptional value. From Theorem 1.6, this possible Borel exceptional value is $-ad^2$. Example 1.3 shows that this Borel exceptional value $-ad^2 (= -1)$ may arise, and thus the conclusion of Theorem 1.6 is sharp.

2. Proof of Theorem 1.1

We need the following lemmas.

Lemma 2.1 (see [3, 4]). *Let $f(z)$ be a meromorphic function of finite order, and let $c \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f), \quad (2.1)$$

where $S(r, f) = o\{T(r, f)\}$.

Lemma 2.2 (see [3]). Let $f(z)$ be a meromorphic function with order $\sigma(f) = \sigma < \infty$, and let c be a nonzero constant. Then, for each $\varepsilon > 0$, one has

$$T(r, f(z+c)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r). \quad (2.2)$$

Lemma 2.3. Suppose that $n, a, c, f(z), \Psi_n(z)$ satisfy the conditions of Theorem 1.1. If $b \in \mathbb{C}$, then $\Psi_n(z) - b$ is transcendental.

Proof. Suppose that $\Psi_n(z) - b = p(z)$, where $p(z)$ is a polynomial. Then

$$-af(z)^n = b - \Delta f(z) + p(z). \quad (2.3)$$

By Lemma 2.2, for each $\varepsilon > 0$, we have

$$T(r, \Delta f(z)) \leq 2T(r, f) + S(r, f) + O(r^{\sigma-1+\varepsilon}), \quad (2.4)$$

where $\sigma = \sigma(f)$. By an identity due to Valiron-Mohon'ko (see [10, 11]), we have

$$\begin{aligned} T(r, af(z)^n) &= nT(r, f) + S(r, f) \\ &= T(r, b - \Delta f(z) + p(z)) \\ &\leq 2T(r, f) + S(r, f) + O(r^{\sigma-1+\varepsilon}). \end{aligned} \quad (2.5)$$

This contradicts the fact that $n \geq 3$. Hence $\Psi_n(z) - b$ is transcendental. $\square$

Lemma 2.4. Suppose that $n, a, c, f(z), \Psi_n(z)$ satisfy the conditions of Theorem 1.1. Suppose also that $b \in \mathbb{C}$, $q(z) \not\equiv 0$ is a polynomial, and $p(z) \not\equiv 0$ is an entire function with $\sigma(p) < \sigma(f)$. If

$$\Psi_n(z) - b = p(z) \exp\{q(z)\}, \quad (2.6)$$

then

$$P(z, f) = np(z)f'(z) - (p'(z) + q'(z)p(z))f(z) \not\equiv 0. \quad (2.7)$$

Proof. Suppose that

$$np(z)f'(z) - (p'(z) + q'(z)p(z))f(z) \equiv 0. \quad (2.8)$$

Integrating (2.8) results in

$$f(z)^n = dp(z) \exp\{q(z)\}, \quad (2.9)$$

where $d(\neq 0)$ is a constant. Therefore, by (2.6), (2.9), and the definition of $\Psi_n(z)$, we obtain

$$\Psi_n(z) - b = f(z+c) - f(z) - af(z)^n - b = \frac{1}{d}f(z)^n, \quad (2.10)$$

and so

$$d(f(z+c) - f(z)) = (ad+1)f(z)^n + bd. \quad (2.11)$$

We must have $ad+1 \neq 0$. In fact, if $ad+1=0$, then by (2.11) and $d \neq 0$, we have that

$$f(z+c) - f(z) = b, \quad (2.12)$$

so $f'(z)$ is periodic. Then, write (2.8) as

$$f'(z) = R(z)f(z), \quad (2.13)$$

where

$$R(z) = \frac{p'(z) + q'(z)p(z)}{np(z)}. \quad (2.14)$$

Clearly, $R(z) \neq 0$ and $\sigma(R) \leq \sigma(p) < \sigma(f)$. We obtain from (2.12) and (2.13)

$$(R(z+c) - R(z))f(z) = -bR(z+c). \quad (2.15)$$

If $R(z+c) - R(z) \equiv 0$, then $b=0$ by (2.15) and $R(z) \neq 0$. Thus, by (2.12), we have $f(z+c) \equiv f(z)$, which contradicts our condition. If $R(z+c) - R(z) \neq 0$, then by (2.15), we have

$$\sigma(f) = \sigma\left(\frac{-bR(z+c)}{R(z+c) - R(z)}\right) \leq \sigma(R) < \sigma(f). \quad (2.16)$$

This is also a contradiction. Hence $ad+1 \neq 0$.

Differentiating (2.11), and then dividing by $f'(z)$ result in

$$d\left(\frac{f'(z+c)}{f'(z)} - 1\right) = n(ad+1)f(z)^{n-1}. \quad (2.17)$$

Therefore, by Lemma 2.1, we get that

$$(n-1)T(r, f) = (n-1)m(r, f) = S(r, f') = S(r, f), \quad (2.18)$$

a contradiction for $n \geq 2$. Hence $P(z, f) \neq 0$. □

Halburd and Korhonen obtained the following difference analogue of the Clunie lemma [4, Corollary 3.3].

Lemma 2.5. *Let $f(z)$ be a nonconstant, finite order meromorphic solution of*

$$f^n P_1(z, f) = Q_1(z, f), \quad (2.19)$$

where $P_1(z, f)$, $Q_1(z, f)$ are difference polynomials in $f(z)$ with small meromorphic coefficients, and let $\delta < 1$. If the degree of $Q_1(r, f)$ as a polynomial in $f(z)$ and its shifts is at most n , then

$$m(r, P_1(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f)) \quad (2.20)$$

for all r outside an exceptional set of finite logarithmic measure.

Remark 2.6. If coefficients of P_1 , Q_1 are $a_j(z)$ ($j = 1, \dots, s$) satisfying $\sigma(a_j) < \sigma(f)$, then using the same method as in the proof of Lemma 2.5 (see [4]), we have

$$m(r, P_1(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f)) + O\left(\sum_{j=1}^s m(r, a_j)\right) \quad (2.21)$$

for all r outside an exceptional set of finite logarithmic measure.

We are now able to prove Theorem 1.1. We only prove the case $\sigma(f) > 0$. For the case $\sigma(f) = 0$, we can use the same method in the proof. Suppose that $b \in \mathbb{C}$ and $\lambda(\Psi_n(z) - b) < \sigma(f)$. Then, by Lemma 2.3, we see that $\Psi_n(z) - b$ is transcendental. Thus, $\Psi_n(z) - b$ can be written as

$$\Psi_n(z) - b = p(z) \exp\{q(z)\}, \quad (2.22)$$

where $q(z) \not\equiv 0$ is a polynomial, $p(z) \not\equiv 0$ is an entire function with $\sigma(p) < \sigma(f)$.

Differentiating (2.22) and eliminating $\exp\{q(z)\}$, we obtain

$$f(z)^{n-1} P(z, f) = Q(z, f), \quad (2.23)$$

where

$$P(z, f) = a n p(z) f'(z) - a(p'(z) + q'(z)p(z))f(z)$$

$$Q(z, f) = p(z)f'(z) - p(z)f'(z+c) + \Delta f(z)(p'(z) + q'(z)p(z)) - b(p'(z) + q'(z)p(z)). \quad (2.24)$$

By Lemma 2.4, we see that $P(z, f) \not\equiv 0$. Since $n \geq 3$ and the total degree of $Q(z, f)$ as a polynomial in $f(z)$ and its shifts, $\deg_f Q(z, f) = 1$, by (2.23), Lemma 2.5, and Remark 2.6, we obtain that for $\delta < 1$

$$\begin{aligned} T(r, P(z, f)) &= m(r, P(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f)) + O(m(r, p)), \\ T(r, fP(z, f)) &= m(r, fP(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f)) + O(m(r, p)) \end{aligned} \quad (2.25)$$

for all r outside of an exceptional set of finite logarithmic measure.

Thus, (2.25) give that

$$T(r, f) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f)) + O(m(r, p)) \quad (2.26)$$

for all r outside of an exceptional set of finite logarithmic measure. This is a contradiction. Hence $\Psi_n(z) - b$ has infinitely many zeros and $\lambda(\Psi_n(z) - b) = \sigma(f)$, which proves Theorem 1.1.

3. Proof of Theorem 1.5

We need the following lemma.

Lemma 3.1 (see [12, page 69–70], [13, page 79–80], or [14]). *Suppose that $n \geq 2$, and let $f_j(z)$, $j = 1, \dots, n$, be meromorphic functions and $g_j(z)$, $j = 1, \dots, n$, entire functions such that*

- (i) $\sum_{j=1}^n f_j(z) \exp\{g_j(z)\} \equiv 0$;
- (ii) when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not constant;
- (iii) when $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\} \quad (r \rightarrow \infty, r \notin E), \quad (3.1)$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j(z) \equiv 0$, $j = 1, \dots, n$.

To prove Theorem 1.5, note first that $f(z)$ has a Borel exceptional value 0, we can write $f(z)$ as

$$f(z) = g(z) \exp\{\alpha z^k\}, \quad f(z + c) = g(z + c)g_1(z) \exp\{\alpha z^k\}, \quad (3.2)$$

where $\alpha (\neq 0)$ is a constant, $k (\geq 1)$ is an integer satisfying $\sigma(f) = k$, and $g(z), g_1(z)$ are entire functions such that $g(z)g_1(z) \not\equiv 0$, $\sigma(g) < k$, $\sigma(g_1) = k - 1$.

First, we prove $\Psi_2(z) - b = \Delta f(z) - af(z)^2 - b$ is transcendental. If $\Psi_2(z) - b = p(z)$, where $p(z)$ is a polynomial, then

$$af(z)^2 = \Delta f(z) - b + p(z). \quad (3.3)$$

Thus by Lemma 2.1 and an identity due to Valiron-Mohon'ko (see [10, 11]), we have

$$\begin{aligned} T(r, af^2) &= 2T(r, f) + S(r, f) \\ T(r, \Delta f(z) - b + p(z)) &= m(r, \Delta f(z) - b + p(z)) \\ &\leq m(r, f) + m\left(r, \frac{f(z+c)}{f(z)} - 1\right) + O(\log r) \\ &\leq T(r, f) + S(r, f). \end{aligned} \quad (3.4)$$

By (3.4), we see that (3.3) is a contradiction.

Secondly, we prove $\sigma(\Psi_2 - b) = \sigma(f) = k \geq 1$. By the expression of $\Psi_2(z)$, we have $\sigma(\Psi_2 - b) \leq k$. Set $G(z) = \Psi_2(z) - b$. If $\sigma(G) = k_1 < k$, then by (3.2), we have

$$(g(z+c)g_1(z) - g(z)) \exp\{\alpha z^k\} - ag(z)^2 \exp\{2\alpha z^k\} - (b + G(z)) = 0. \quad (3.5)$$

Since $\sigma(G) = k_1 < k$ and $\sigma(g) < k$, we see that the left hand side of (3.5) is of order $= k$ by applying the general form of the Valiron-Mohon'ko lemma in [10], a contradiction. So, $\sigma(G) = k$.

Thirdly, we prove $\lambda(G) = k (\geq 1)$. If $\lambda(G) < k$, then $G(z)$ can be written as

$$G(z) = g^*(z) \exp\{\beta z^k\}, \quad (3.6)$$

where $\beta (\neq 0)$ is a constant, $g^*(z) (\neq 0)$ is an entire function satisfying $\sigma(g^*) < k$. Thus by (3.2), (3.6), and $G(z) = \Psi_2(z) - b$, we have

$$(g(z+c)g_1(z) - g(z)) \exp\{\alpha z^k\} - ag(z)^2 \exp\{2\alpha z^k\} - b - g^* \exp\{\beta z^k\} = 0. \quad (3.7)$$

In (3.7), there are three cases for β :

- (i) $\beta \neq \alpha$ and $\beta \neq 2\alpha$;
- (ii) $\beta = \alpha$;
- (iii) $\beta = 2\alpha$.

Applying Lemma 3.1 to (3.7), we have in case (i)

$$g(z+c)g_1(z) - g(z) \equiv ag(z)^2 \equiv g^*(z) \equiv 0; \quad (3.8)$$

in case (ii), we have

$$ag(z)^2 \equiv 0; \quad (3.9)$$

in case (iii), we have

$$g(z+c)g_1(z) - g(z) \equiv 0. \quad (3.10)$$

Since $f(z+c) - f(z) \not\equiv 0$ and $f(z)$ is transcendental, we see that any one of (3.8)–(3.10) is a contradiction. Hence $\lambda(\Psi_2 - b) = \sigma(f)$.

4. Proof of Theorem 1.6

Since $f(z)$ has a nonzero Borel exceptional value d , we can write $f(z)$ as

$$f(z) = d + \varphi(z) \exp\{\alpha z^k\}, \quad f(z+c) = d + \varphi(z+c)\varphi_1(z) \exp\{\alpha z^k\}, \quad (4.1)$$

where $\alpha (\neq 0)$ is a constant, $k (\geq 1)$ is an integer satisfying $\sigma(f) = k$, and $\varphi(z), \varphi_1(z)$ are entire functions such that $\varphi(z)\varphi_1(z) \not\equiv 0$, $\sigma(\varphi) < k$, $\sigma(\varphi_1) = k - 1$.

Using the same method as in the proof of Theorem 1.5, we can show that

$$\Psi_2(z) - b = \Delta f(z) - af(z)^2 - b \quad (4.2)$$

is transcendental and $\sigma(\Psi_2 - b) = \sigma(f) = k \geq 1$.

Now we show that $\lambda(\Psi_2(z) - b) = k (\geq 1)$. Set $G(z) = \Psi_2(z) - b$. If $\lambda(G) < k$, then G can be written as

$$G(z) = \varphi^*(z) \exp\{sz^k\}, \quad (4.3)$$

where $s (\neq 0)$ is a constant and $\varphi^*(z) (\not\equiv 0)$ is an entire function satisfying $\sigma(\varphi^*) < k$. Thus by (4.1) and (4.3), we have

$$\begin{aligned} & (\varphi(z+c)\varphi_1(z) - \varphi(z) - 2ad\varphi(z)) \exp\{\alpha z^k\} - a\varphi(z)^2 \exp\{2\alpha z^k\} - \varphi^*(z) \exp\{sz^k\} \\ & - (ad^2 + b) = 0. \end{aligned} \quad (4.4)$$

In (4.4), there are three cases for s :

- (i) $s \neq \alpha$ and $s \neq 2\alpha$;
- (ii) $s = \alpha$;
- (iii) $s = 2\alpha$.

Applying the same method as in the proof of Theorem 1.5 to these three cases, we obtain

$$ad^2 + b = 0, \quad (4.5)$$

which contradicts our supposition that $b \neq -ad^2$. Hence $\lambda(\Psi_2 - b) = \sigma(f)$.

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References

- [1] W. K. Hayman, "Picard values of meromorphic functions and their derivatives," *Annals of Mathematics*, vol. 70, pp. 9–42, 1959.
- [2] Z.-X. Chen and K. H. Shon, "On zeros and fixed points of differences of meromorphic functions," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 373–383, 2008.
- [3] Y.-M. Chiang and S.-J. Feng, "On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane," *Ramanujan Journal*, vol. 16, no. 1, pp. 105–129, 2008.
- [4] R. G. Halburd and R. J. Korhonen, "Difference analogue of the lemma on the logarithmic derivative with applications to difference equations," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 477–487, 2006.
- [5] R. G. Halburd and R. J. Korhonen, "Nevanlinna theory for the difference operator," *Annales Academiæ Scientiarum Fennicæ. Mathematica*, vol. 31, no. 2, pp. 463–478, 2006.
- [6] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, and J. Zhang, "Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity," *Journal of Mathematical Analysis and Applications*, vol. 355, no. 1, pp. 352–363, 2009.
- [7] K. Liu and I. Laine, "A note on value distribution of difference polynomials," *Bulletin of the Australian Mathematical Society*, vol. 81, no. 3, pp. 353–360, 2010.
- [8] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, UK, 1964.
- [9] L. Yang, *Value Distribution Theory*, Springer, Berlin, Germany, 1993.
- [10] A. Z. Mohon'ko, "The Nevanlinna characteristics of certain meromorphic functions," *Teorija Funkcij, Funkcional'nyĭ Analiz i ih Prilozhenija*, no. 14, pp. 83–87, 1971.
- [11] G. Valiron, "Sur la dérivée des fonctions algébroides," *Bulletin de la Société Mathématique de France*, vol. 59, pp. 17–39, 1931.
- [12] F. Gross, *Factorization of Meromorphic Functions*, U.S. Government Printing Office, Washinton, DC, USA, 1972.
- [13] C.-C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*, vol. 557 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.
- [14] J. Wang and I. Laine, "Growth of solutions of nonhomogeneous linear differential equations," *Abstract and Applied Analysis*, vol. 2009, Article ID 363927, 11 pages, 2009.