Research Article

Univalent Functions in the Möbius Invariant Q_K Space

Fernando Pérez-González¹ and Jouni Rättyä²

¹ Departamento de Análisis Matemático, Universidad de La Laguna, Tenerife 38271 La Laguna, Spain

² Department of Physics and Mathematics, University of Eastern Finland, Campus of Joensuu, P.O. Box 111, 80101 Joensuu, Finland

Correspondence should be addressed to Fernando Pérez-González, fpergon@ull.es

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It is shown that a univalent function f belongs to Q_K if and only if $\sup_{a \in \mathbb{D}} \int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) K'(\log(1/r)) dr < \infty$, where $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$, provided K satisfies certain regularity conditions. It is also shown that under these conditions Q_K contains all univalent Bloch functions if and only if $\int_0^1 (\log((1+r)/(1-r)))^2 K'(\log(1/r)) dr < \infty$.

1. Introduction

The aim of this paper is to characterize univalent functions f in the Möbius invariant Q_K space in terms of the maximum modulus of $f \circ \varphi_a - f(a)$, where $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$, provided K satisfies certain regularity conditions. We will begin with a brief overview of characterizations of univalent functions in classical function spaces of the unit disc together with necessary definitions. Then, we will state the above-mentioned characterization of univalent functions in Q_K and its consequences. The proofs are postponed to the end of the paper and will be presented in the sequential order.

2. Notation, Background, and Results

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$. A function $f \in \mathcal{H}(\mathbb{D})$ is said to be *univalent* if it is one to one, and the class of all univalent functions is denoted by $\mathcal{U}$. For the theory of univalent functions, see [1–3].

For $0 < p \leq \infty$, the *Hardy space* H^p consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p} := \lim_{r \rightarrow 1^-} M_p(r, f) < \infty, \quad (2.1)$$

where

$$M_p(r, f) := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty, \quad (2.2)$$

are the standard L^p -means of the restriction of f to the circle of radius r centered at the origin, and

$$M_\infty(r, f) := \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})| \quad (2.3)$$

is the maximum modulus function. For the theory of Hardy spaces, see [4, 5].

Hardy and Littlewood [6], Pommerenke [7], and Prawitz [8] proved that if $f \in \mathcal{U}$, then $f \in H^p$, if and only if

$$\int_0^1 M_\infty^p(r, f) dr < \infty, \quad 0 < p < \infty. \quad (2.4)$$

For more information on univalent functions in Hardy spaces, see [2, 7–13]. A result due to Baernstein II [14] states that $f \in \text{BMOA}$, the space of analytic functions in the Hardy space H^1 with boundary values of *bounded mean oscillation*, if and only if $\sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^p} < \infty$, where $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is the automorphism of $\mathbb{D}$ which interchanges the origin and the point a . This, applied to (2.4), shows that $f \in \mathcal{U}$ belongs to BMOA if and only if

$$\sup_{a \in \mathbb{D}} \int_0^1 M_\infty^p(r, f \circ \varphi_a - f(a)) dr < \infty \quad (2.5)$$

for some (equivalently for all) $0 < p < \infty$. However, Pommerenke [15] has shown that BMOA and the Bloch space $\mathcal{B}$ contain the same univalent functions. The *Bloch space* $\mathcal{B}$ consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} := \sup_{z \in \mathbb{D}} |f'(z)| (1 - |z|^2) < \infty. \quad (2.6)$$

If, in addition, $f'(z)(1 - |z|^2) \rightarrow 0$, as $|z| \rightarrow 1^-$, then $f \in \mathcal{B}_0$. For the theory of Bloch spaces, see [3, 16]. If Ω is a simply connected proper subdomain of the complex plane and $f \in \mathcal{U}$ such that $f(\mathbb{D}) = \Omega$, then

$$d_\Omega(f(z)) \leq |f'(z)| (1 - |z|^2) \leq 4d_\Omega(f(z)), \quad z \in \mathbb{D}, \quad (2.7)$$

where $d_\Omega(w)$ stands for the Euclidean distance from w to the boundary of Ω , see, for example, [3]. Therefore, univalent functions in the Bloch space can be characterized by the following well-known geometric condition; $f \in \mathcal{B}$ if and only if $\sup_{w \in \Omega} d_\Omega(w) < \infty$, that is, if the image of $\mathbb{D}$ under f does not contain arbitrarily large discs. For other characterizations of univalent Bloch functions, see [11].

Aulaskari et al. [17] improved the result by Pommerenke by showing that $\mathcal{B} \cap \mathcal{U} = Q_p \cap \mathcal{U}$ for any $p > 0$. Recall that Q_p is a Möbius invariant subspace of $\mathcal{B}$ and consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{Q_p}^2 := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z, a) dA(z) < \infty, \quad (2.8)$$

where $dA(z)$ is the Euclidean area element on $\mathbb{D}$. In particular, $Q_1 = \text{BMOA}$ and $Q_p = \mathcal{B}$ for all $p > 1$. Moreover, Q_0 is the classical *Dirichlet space* $\mathfrak{D}$ which consists of all $f \in \mathcal{H}(\mathbb{D})$ with finite area of image counting multiplicities. The Dirichlet space $\mathfrak{D}$ is a special case of classical *Besov spaces*. A geometric characterization of univalent functions in Besov spaces was found by Walsh [18], see also the related results by Donaire et al. [19]. For the theory of Q_p spaces, see [20, 21].

For $K : [0, \infty) \rightarrow [0, \infty)$, the Möbius invariant space Q_K consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{Q_K}^2 := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) dA(z) < \infty. \quad (2.9)$$

Moreover, if the integral above tends to zero as a approaches to the boundary of $\mathbb{D}$, then $f \in Q_{K,0}$. If $K(t) = t^p$, then $Q_K = Q_p$, and therefore Q_K can be viewed as a generalized Q_p space. For results on Q_K , see [22–24] and the references therein. From now on, the weight K is assumed to admit the following basic properties:

- (1) $K(0) = 0$;
- (2) $\int_0^{1/e} K(\log(1/r)) r \, dr < \infty$;
- (3) $K'(t) \geq 0$ for all $t \in (0, 1)$;
- (4) $K''(t) \leq 0$ for all $t \in (0, 1)$.

Requirements (1) and (2) are standard; the first one ensures that K indeed plays an essential role in the definition, and the second one guarantees the nontriviality of Q_K as well as the inclusions $Q_K \subset \mathcal{B}$ and $Q_{K,0} \subset \mathcal{B}_0$. Conditions (3) and (4) are, of course, restrictions, yet, for example, $K(t) = t^p$ satisfies both of them for all $0 < p \leq 1$.

Before proceeding further, we give an example related to Q_K spaces in order to illustrate the variety of spaces induced by different choices of K .

Example 2.1. For $0 < p < \infty$ and $q \in \mathbb{R}$, consider the weight

$$K(t) = K(p, q)(t) \begin{cases} := \begin{cases} t^p \left(\log \frac{1}{t} \right)^q, & 0 < t < e^{-q/p}, \\ e^{-q} \left(\frac{q}{p} \right)^q, & e^{-q/p} \leq t < 1, \end{cases} & 0 \leq q < \infty, \\ \\ := \begin{cases} t^p \left(\log \frac{1}{t} \right)^q, & 0 < t < \frac{1}{2}, \\ 2^{-p} (\log 2)^q, & \frac{1}{2} \leq t < 1, \end{cases} & -\infty < q < 0. \end{cases} \quad (2.10)$$

Obviously, $Q_{K(p,0)} = Q_p$ for all $0 < p < \infty$. Since

$$K'(t) = t^{p-1} \left(\log \frac{1}{t} \right)^{q-1} \left(p \log \frac{1}{t} - q \right) \quad (2.11)$$

for either $0 < t < e^{-q/p}$ (the case $q \geq 0$) or $0 < t < 1/2$ (the case $q < 0$), the weight K is increasing. If $0 < p \leq 1$, then the integral

$$\int_0^1 K \left(\log \frac{1}{r} \right) (1-r^2)^{-2} r \, dr \quad (2.12)$$

diverges for all $q \in \mathbb{R}$, and therefore $Q_K \subsetneq \mathcal{B}$ by [22, Theorem 2.3]. Moreover, [22, Theorem 2.6] shows that

$$Q_{p_1} \subsetneq Q_{K(p_1, -q_1)} \subsetneq Q_{K(p_1, -q_2)} \subsetneq Q_{K(p_2, q_2)} \subsetneq Q_{K(p_2, q_1)} \subsetneq Q_{p_2} \quad (2.13)$$

for all $0 < p_1 < p_2 \leq 1$ and $0 < q_1 < q_2 < \infty$.

Let us turn back to univalent functions. A special case of [11, Theorem 4] shows that a univalent function f belongs to Q_p if and only if

$$\sup_{a \in \mathbb{D}} \int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) \left(\log \frac{1}{r} \right)^{p-1} dr < \infty, \quad 0 < p \leq 1. \quad (2.14)$$

Of course, this is not a natural way to state the result because $\mathcal{U} \cap Q_p = \mathcal{U} \cap \mathcal{B}$ for all p , but it appears to be useful for our purposes. The case $p = 1$ of (2.14), which corresponds to BMOA, reduces to (2.5) with $p = 2$. An appropriate interpretation of (2.14) allows us to conclude what happens with Q_K . Namely, since $p(\log(1/r))^{p-1} = K'(\log(1/r))$ for $K(t) = t^p$, condition (2.14), with $(\log(1/r))^{p-1}$ being replaced by $K'(\log(1/r))$, gives a candidate for a characterization of

univalent functions in Q_K . The main result of this paper is Theorem 2.2 which shows that this is indeed the case, provided

$$-\frac{(K''(t) + K'(t))K(t)}{(K'(t))^2} \leq C, \quad 0 < t < \infty, \quad (2.15)$$

for some positive constant C .

Theorem 2.2. *Let $f \in \mathcal{U}$ and assume that K satisfies condition (2.15). Then, $f \in Q_K$ if and only if*

$$\sup_{a \in \mathbb{D}} \int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) K' \left(\log \frac{1}{r} \right) dr < \infty. \quad (2.16)$$

Moreover, $f \in Q_{K,0}$ if and only if

$$\lim_{|a| \rightarrow 1^-} \int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) K' \left(\log \frac{1}{r} \right) dr = 0. \quad (2.17)$$

It is an immediate consequence of the proof that the assertions in Theorem 2.2 remain valid for areally mean q -valent functions. In addition to standard techniques of univalent functions, the proof of Theorem 2.2 uses a result by Pavlović and Peláez [25] on weighted integrals of analytic functions and their derivatives. An application of this result yields the requirement (2.15). We will analyze the importance of (2.15) after discussing consequences of Theorem 2.2 and its proof.

The first part of the proof of Theorem 2.2 shows that $f \in \mathcal{H}(\mathbb{D})$ belongs to Q_K if and only if

$$\sup_{a \in \mathbb{D}} \int_0^1 A(f(\Delta(a, r))) K' \left(\log \frac{1}{r} \right) dr < \infty, \quad (2.18)$$

where $A(f(\Delta(a, r)))$ denotes the area of image of the pseudohyperbolic disc $\Delta(a, r) := \{z : |\varphi_a(z)| < r\}$ under f counting multiplicities. This, together with Theorem 2.2, shows that for $f \in Q_K \cap \mathcal{U}$, the quantities $A(f(\Delta(a, r)))$ and $M_\infty^2(r, f \circ \varphi_a - f(a))$ are of the same growth (uniformly in a) when measured in terms of $L^1(K'(\log(1/r))dr)$.

Since $\mathcal{B} \cap \mathcal{U} = Q_p \cap \mathcal{U}$ for any $p > 0$ by a result due to Aulaskari et al. [17], it is natural to ask when does the identity $\mathcal{B} \cap \mathcal{U} = Q_K \cap \mathcal{U}$ hold. An application of Theorem 2.2 yields Corollary 2.3 which answers this question provided K satisfies (2.15).

Corollary 2.3. *Assume that K satisfies condition (2.15). Then, $Q_K \cap \mathcal{U} = \mathcal{B} \cap \mathcal{U}$ if and only if*

$$\int_0^1 \left(\log \frac{1+r}{1-r} \right)^2 K' \left(\log \frac{1}{r} \right) dr < \infty. \quad (2.19)$$

Moreover, if (2.19) is satisfied, then $Q_{K,0} \cap \mathcal{U} = \mathcal{B}_0 \cap \mathcal{U}$.

Wulan [26] showed that every areally mean q -valent Bloch function belongs to Q_K whenever K satisfies

$$\int_0^1 \left(\log \frac{1}{1-r} \right)^2 \frac{K(\log(1/r))}{1-r} r \, dr < \infty. \quad (2.20)$$

This result is more general than Corollary 2.3 because the crucial condition (2.15) is not needed. However, K is concave whenever (4) is satisfied, and in that case, $K'(t) \leq K(t)/t$, and then (2.20) implies (2.19).

We next analyze the necessity of condition (2.15). This is needed in two consecutive steps in the proof of Theorem 2.2. These steps together establish the asymptotic inequality

$$\int_0^1 M_\infty^2(r, f) \frac{1}{r} K' \left(\log \frac{1}{r} \right) dr \lesssim \int_0^1 M_\infty^2(r, f') \left(\int_r^1 K \left(\log \frac{1}{s} \right) ds \right) dr \quad (2.21)$$

for all $f \in \mathcal{H}(\mathbb{D})$. As usual, we write $A \lesssim B$, if there exists $C > 0$ such that $A \leq CB$, and the notation $A \gtrsim B$ is understood in an analogous manner. In particular, if $A \lesssim B$ and $A \gtrsim B$, then we write $A \asymp B$. Set now $v(r) = \int_r^1 K(\log(1/s)) \, ds$, so that (2.21) becomes

$$\int_0^1 M_\infty^2(r, f) v''(r) dr \lesssim \int_0^1 M_\infty^2(r, f') v(r) dr. \quad (2.22)$$

We will show next that (2.22) does not necessarily remain true even for univalent functions f , if K , that induces v , does not satisfy (2.15). To see this, take

$$v(r) = \frac{1-r}{\log(1/(1-r))}, \quad v'(r) \asymp -\frac{1}{\log(1/(1-r))}, \quad v''(r) \asymp \frac{1}{(1-r)(\log(1/(1-r)))^2}, \quad r \rightarrow 1^-, \quad (2.23)$$

and $f \in \mathcal{U}$ with $f^2(z) = 1 - \log(1-z)$, so that

$$M_\infty^2(r, f) \asymp \log \frac{1}{1-r}, \quad M_\infty^2(r, f') \asymp \frac{1}{(1-r)^2 \log(1/(1-r))}, \quad r \rightarrow 1^-. \quad (2.24)$$

For these choices, the right-hand side of (2.22) is finite, whereas the left-hand side is not. If $K(t) = -1/\log t$, then $v(r) = \int_r^1 K(\log(1/s)) \, ds \asymp (1-r)(\log(1/(1-r)))^{-1}$, but (2.15) for this K is equivalent to $-2 - \log t + t \log t \leq C$, which clearly fails as $t \rightarrow 0^+$. Moreover, $f \in \mathcal{U} \cap \mathcal{B}$, but (2.19) fails for $K(t) = -1/\log t$. This justifies the assumption (2.15) in Theorem 2.2 and Corollary 2.3.

3. Proof of Theorem 2.2

We will prove the first assertion, the second one, then immediately follows by the proof.

3.1. Sufficiency of (2.16)

Let $f \in \mathcal{U}$. Recall first that $M_2(r, h)$ is nondecreasing for any $h \in \mathcal{H}(\mathbb{D})$. Therefore we may use conditions (1) and (3) together with Fubini's theorem to obtain

$$\begin{aligned}
 I(f, K, a) &:= \int_{\mathbb{D}} \left| (f \circ \varphi_a)'(z) \right|^2 K\left(\log \frac{1}{|z|}\right) dA(z) \\
 &\asymp \frac{1}{2\pi} \int_{\mathbb{D}} \left| (f \circ \varphi_a)'(z) \right|^2 |z| K\left(\log \frac{1}{|z|}\right) dA(z) \\
 &= \int_0^1 M_2^2\left(r, (f \circ \varphi_a - f(a))'\right) \int_r^1 \frac{1}{s} K'\left(\log \frac{1}{s}\right) ds r^2 dr \\
 &\leq \int_0^1 M_2^2\left(r, (f \circ \varphi_a - f(a))'\right) \int_r^1 K'\left(\log \frac{1}{s}\right) ds r dr \\
 &= \int_0^1 \int_{\Delta(0,s)} \left| (f \circ \varphi_a - f(a))'(z) \right|^2 dA(z) K'\left(\log \frac{1}{s}\right) ds.
 \end{aligned} \tag{3.1}$$

If $h \in \mathcal{U}$, then

$$\begin{aligned}
 \frac{1}{\pi} \int_{\Delta(0,s)} |h'(z)|^2 dA(z) &\leq \frac{1}{\pi} \int_{|w| \leq M_\infty(s,h)} dA(w) \\
 &= 2 \int_0^{M_\infty(s,h)} t dt \\
 &= M_\infty^2(s, h),
 \end{aligned} \tag{3.2}$$

and an application of this inequality to $h = f \circ \varphi_a - f(a)$, together, with (3.1), gives

$$I(f, K, a) \lesssim \int_0^1 M_\infty^2(s, f \circ \varphi_a - f(a)) K'\left(\log \frac{1}{s}\right) ds. \tag{3.3}$$

Therefore a univalent function f belongs to Q_K if (2.16) is satisfied. Note that this part of the proof uses the univalence of f , but does not rely on (2.15). Moreover, if f is areally mean q -valent, then the reasoning in (3.2) remains valid as soon as the right-hand side of the inequality is multiplied by q .

3.2. Necessity of (2.16)

To prove this, we will need a special case of a result due to Pavlović and Peláez [25], which states that

$$\int_0^1 M_\infty^2(r, h) \omega(r) dr \asymp |h(0)|^2 + \int_0^1 M_\infty^2(r, h') \frac{\left(\int_r^1 \omega(s) ds\right)^2}{\omega(r)} dr, \quad h \in \mathcal{H}(\mathbb{D}), \quad (3.4)$$

provided $\omega : [0, 1) \rightarrow (0, \infty)$ is differentiable and satisfies

$$\frac{\omega'(r)}{\omega(r)^2} \int_r^1 \omega(x) dx \leq C, \quad 0 < r < 1, \quad (3.5)$$

for some constant $C > 0$. Assumption (2.15) is equivalent to (3.5) for $\omega(r) := 1/rK'(\log 1/r)$. Therefore, (3.4) yields

$$\begin{aligned} & \int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) \frac{1}{r} K' \left(\log \frac{1}{r} \right) dr \\ & \lesssim \int_0^1 M_\infty^2(r, (f \circ \varphi_a)') r \frac{(K(\log(1/r)))^2}{K'(\log(1/r))} dr. \end{aligned} \quad (3.6)$$

We next show that

$$r \frac{(K(\log(1/r)))^2}{K'(\log(1/r))} \leq (C+2) \int_r^1 K \left(\log \frac{1}{s} \right) ds, \quad (3.7)$$

where C is the constant in (2.15). To prove (3.7), consider the function

$$g(r) := (C+2) \int_r^1 K \left(\log \frac{1}{s} \right) ds - r \frac{(K(\log(1/r)))^2}{K'(\log(1/r))}. \quad (3.8)$$

By the assumptions (1), (3), and (4) on K , $g(r) \rightarrow 0$ as $r \rightarrow 1^-$. Moreover, (2.15) yields

$$g'(r) = -CK \left(\log \frac{1}{r} \right) - \frac{(K'(\log(1/r)) + (K''(\log(1/r))))(K(\log(1/r)))^2}{(K'(\log(1/r)))^2} \leq 0, \quad (3.9)$$

and thus g is decreasing, and therefore (3.7) holds. This, together with Fubini's theorem, shows that

$$\begin{aligned} & \int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) \frac{1}{r} K' \left(\log \frac{1}{r} \right) dr \\ & \lesssim \int_0^1 M_\infty^2(r, (f \circ \varphi_a)') \int_r^1 K \left(\log \frac{1}{s} \right) ds dr \\ & = \int_0^1 K \left(\log \frac{1}{s} \right) \int_0^s M_\infty^2(r, (f \circ \varphi_a)') dr ds, \end{aligned} \quad (3.10)$$

from which the inequality $\int_0^s M_\infty^2(r, h) dr \leq \pi M_2^2(s, h)$, valid for all $h \in \mathcal{H}(\mathbb{D})$ (see [9, page 841] or [7, Hilfssatz 1]), yields

$$\begin{aligned} \int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) K' \left(\log \frac{1}{r} \right) dr & \lesssim \int_0^1 M_2^2(s, (f \circ \varphi_a)') K \left(\log \frac{1}{s} \right) ds \\ & \asymp I(f, K, a) \leq \|f\|_{Q_K}^2. \end{aligned} \quad (3.11)$$

Thus, (2.16) is satisfied if $f \in Q_K$.

4. Proof of Corollary 2.3

4.1. Sufficiency of (2.19)

Since $Q_K \subset \mathcal{B}$, it suffices to show that $\mathcal{U} \cap \mathcal{B} \subset Q_K$. To see this, let $f \in \mathcal{B}$. An application of the inequality

$$M_\infty(r, g) \leq \|g\|_{\mathcal{B}} \frac{1}{2} \log \frac{1+r}{1-r} + |g(0)|, \quad g \in \mathcal{B}, \quad (4.1)$$

to the function $g = f \circ \varphi_a - f(a) \in \mathcal{B}$ gives

$$M_\infty(r, f \circ \varphi_a - f(a)) \leq \|f \circ \varphi_a - f(a)\|_{\mathcal{B}} \frac{1}{2} \log \frac{1+r}{1-r} = \|f\|_{\mathcal{B}} \frac{1}{2} \log \frac{1+r}{1-r} \quad (4.2)$$

for all $a \in \mathbb{D}$. This yields

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) K' \left(\log \frac{1}{r} \right) dr \\ & \leq \frac{1}{4} \|f\|_{\mathcal{B}}^2 \int_0^1 \left(\log \frac{1+r}{1-r} \right)^2 K' \left(\log \frac{1}{r} \right) dr < \infty, \end{aligned} \quad (4.3)$$

and thus $f \in Q_K$ by Theorem 2.2.

4.2. Necessity of (2.19)

If K satisfies condition (2.15) and $Q_K \cap \mathcal{U} = \mathcal{B} \cap \mathcal{U}$, then the univalent Bloch function $\log((1+z)/(1-z))$, together with Theorem 2.2, shows that (2.19) holds.

4.3. The Case $Q_{K,0} \cap \mathcal{U} = \mathcal{B}_0 \cap \mathcal{U}$

Assume that K satisfies (2.19) and let $\varepsilon > 0$. Since $Q_{K,0} \subset \mathcal{B}_0$, it suffices to show that $\mathcal{U} \cap \mathcal{B}_0 \subset Q_{K,0}$. To see this, let $f \in \mathcal{B}_0$. By applying (4.2) we obtain

$$\begin{aligned} & \int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) K' \left(\log \frac{1}{r} \right) dr \\ & \leq \int_0^R M_\infty^2(r, f \circ \varphi_a - f(a)) K' \left(\log \frac{1}{r} \right) dr \\ & \quad + \|f\|_{\mathcal{B}}^2 \frac{1}{4} \int_R^1 \left(\log \frac{1+r}{1-r} \right)^2 K' \left(\log \frac{1}{r} \right) dr = I_1(R) + I_2(R). \end{aligned} \quad (4.4)$$

Since $f \in \mathcal{B}_0 \subset \mathcal{B}$, the assumption (2.19) implies that $I_2(R) < \varepsilon/2$ for all R sufficiently large. Fix such an R . The image of the circle of radius r centered at the origin under a univalent function $h - h(0)$ is a Jordan curve with zero in its inner domain. The length of this image is $2\pi r M_1(r, h')$, and hence $M_\infty(r, h) \leq \pi r M_1(r, h') + |f(0)|$. This estimate, applied to $h = f \circ \varphi_a - f(a)$, together with the assumption $f \in \mathcal{B}_0$, yields

$$\begin{aligned} I_1(R) & \leq \pi \int_0^R \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f'(\varphi_a(re^{i\theta})) \varphi'_a(re^{i\theta}) \right| d\theta \right)^2 K' \left(\log \frac{1}{r} \right) r dr \\ & = \frac{1}{4\pi} \int_0^R \left(\int_0^{2\pi} \left| f'(\varphi_a(re^{i\theta})) \right| \left(1 - |\varphi_a(re^{i\theta})|^2 \right) d\theta \right)^2 \frac{K'(\log(1/r))}{(1-r^2)^2} r dr < \frac{\varepsilon}{2} \end{aligned} \quad (4.5)$$

for all sufficiently large $|a|$. Therefore,

$$\int_0^1 M_\infty^2(r, f \circ \varphi_a - f(a)) K' \left(\log \frac{1}{r} \right) dr \longrightarrow 0, \quad |a| \longrightarrow 1^-, \quad (4.6)$$

and hence $f \in Q_{K,0}$ by Theorem 2.2.

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