Research Article

Strong Convergence Theorems for Equilibrium Problems and $k$-Strict Pseudocontractions in Hilbert Spaces

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We introduce a new iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed point of a finite family of $k$-strictly pseudo-contractive nonself-mappings. Strong convergence theorems are established in a real Hilbert space under some suitable conditions. Our theorems presented in this paper improve and extend the corresponding results announced by many others.

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $K$ be a nonempty closed convex subset of $H$. Let $F$ be a bifunction from $K \times K$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following problem: Find $x \in K$ such that

$$F(x, y) \geq 0, \quad \forall y \in K,$$

(1.1)

which is called equilibrium problem. We use $\text{EP}(F)$ to denote the set of solution of the problem (1.1). Given a mapping $T : K \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in K$. Then, $z \in \text{EP}(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in K$; that is, $z$ is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem (see, e.g., [1–3]).
Recall that a nonself-mapping $T : K \rightarrow H$ is called a $k$-strict pseudocontraction if there exists a constant $k \in [0, 1)$ such that
\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\| (I - T)x - (I - T)y \|^2, \quad \forall x, y \in K. \tag{1.2}
\]

We use $F(T)$ to denote the fixed point set of the mapping $T$, that is, $F(T) := \{x \in K : Tx = x\}$. As $k = 0$, $T$ is said to be nonexpansive, that is, $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$. $T$ is said to be pseudocontractive if $k = 1$ and is also said to be strongly pseudocontractive if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudocontractive. Clearly, the class of $k$-strict pseudocontractions falls into the one between classes of nonexpansive mappings and pseudocontractions. We remark also that the class of strongly pseudocontractive mappings is independent of the class of $k$-strict pseudocontractions (see, e.g., [4, 5]).

Iterative methods for equilibrium problem and nonexpansive mappings have been extensively investigated; see, for example, [1–18] and the references therein. However, iterative methods for strict pseudocontractions are far less developed than those for nonexpansive mappings though Browder and Petryshyn [5] initiated their work in 1967; the reason is probably that the second term appearing in the right-hand side of (1.2) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strict pseudocontraction $T$. On the other hand, strict pseudocontractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see, e.g., [6]). Therefore, it is interesting to develop the theory of iterative methods for equilibrium problem and strict pseudocontractions.

In 2007, Acedo and Xu [12] proposed the following parallel algorithm for a finite family of $k_i$-strict pseudocontractions $\{T_i\}_{i=1}^N$ in Hilbert space $H$:
\[
\forall x_0 \in K, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \sum_{i=1}^N \lambda_i T_i x_n, \tag{1.3}
\]
where $\{\alpha_n\} \subset (0, 1)$, and $\{\lambda_i\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i = 1$. They proved that the sequence $\{x_n\}$ defined by (1.3) converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$ under some appropriate conditions. Moreover, by applying additional projections, they further proved that algorithm can be modified to have strong convergence.

Recently, S. Takahashi and W. Takahashi [13] studied the equilibrium problem and fixed point of nonexpansive self-mappings $T$ in Hilbert spaces by a viscosity approximation methods for finding an element of $EP(F) \cap F(T)$. Very recently, by using the general approximation method, Qin et al. [14] obtained a strong convergence theorem for finding an element of $F(T)$. On the other hand, Ceng et al. [16] proposed an iterative scheme for finding an element of $EP(F) \cap F(T)$ and then obtained some weak and strong convergence theorems.

In this paper, inspired and motivated by research going in this area, we introduce a modified parallel iteration, which is defined in the following way:
\[
F(u_n, y) + \frac{1}{r}(y - u_n, u_n - x_n) \geq 0, \quad \forall y \in K,
\]
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\[ y_n = \alpha_n u_n + (1 - \alpha_n) \sum_{i=1}^{N} \eta_i^{(n)} T_i u_n, \]

\[ x_{n+1} = \beta_n u + y_n x_n + (1 - \beta_n - \gamma_n) y_n, \quad n \geq 0, \]

(1.4)

where \( u \in K \) is a given point, \( \{T_i\}_{i=1}^{N} : K \to H \) is a finite family of \( k_i \)-strictly pseudocontractive nonself-mappings, \( \{\eta_i^{(n)}\}_{i=1}^{N} \) is a finite sequences of positive numbers, \( \{\alpha_n\}, \{\beta_n\}, \) and \( \{\gamma_n\} \) are some sequences in \( (0,1) \).

Our purpose is not only to modify the parallel algorithm (1.3) to the case of equilibrium problems and common fixed point for a finite family of \( k_i \)-strictly pseudocontractive nonself-mappings, but also to establish strong convergence theorems in a real Hilbert space under some different conditions. Our theorems presented in this paper improve and extend the main results of [9, 12–14, 16].

2. Preliminaries

Let \( K \) be a nonempty closed and convex subset of a Hilbert space \( H \). We use \( P_K \) to denote the metric or nearest point projection of \( H \) onto \( K \); that is, for \( x \in H \), \( P_K x \) is the only point in \( K \) such that \( \|x - P_K x\| = \inf\{\|x - z\| : z \in K\} \). We write \( x_n \rightarrow x \) and \( x_n \rightharpoonup x \) indicate that the sequence \( \{x_n\} \) convergence weakly and strongly to \( x \), respectively.

It is well known that Hilbert space \( H \) satisfies Opial’s condition [8], that is, for any sequence \( \{x_n\} \) with \( x_n \rightarrow x \) and every \( y \in H \) with \( y \neq x \), we have

\[ \liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|. \] (2.1)

To study the equilibrium problem (1.1), we may assume that the bifunction \( F \) of \( K \times K \) into \( \mathbb{R} \) satisfies the following conditions.

(A1) \( F(x, x) = 0 \) for all \( x \in K \).

(A2) \( F \) is monotone, that is, \( F(x, y) + F(y, x) \leq 0 \) for all \( x, y \in K \).

(A3) For each \( x, y, z \in K \), \( \lim_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y) \).

(A4) For each \( x \in K \), \( y \mapsto F(x, y) \) is convex and lower semi-continuous.

In order to prove our main results, we need the following Lemmas and Propositions.

**Lemma 2.1** (see [1, 3]). Let \( F \) be a bifunction from \( K \times K \) into \( \mathbb{R} \) satisfying (A1)–(A4). Then, for any \( r > 0 \) and \( x \in H \), there exists \( z \in K \) such that

\[ F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \forall y \in K. \] (2.2)

Further, if \( T_r x = \{z \in K : F(z, y) + (1/r)(y - z, z - x) \geq 0, \forall y \in K\} \), then the following holds.
(1) $T_r$ is single-valued.

(2) $T_r$ is firmly nonexpansive, that is, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$, for all $x, y \in H$.

(3) $F(T_r) = EP(F)$.

(4) $EP(F)$ is closed and convex.

Lemma 2.2 (see [7]). Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|ax + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta \|x - y\|^2 - \alpha \gamma \|x - z\|^2 - \beta \gamma \|y - z\|^2. \quad (2.3)$$

Lemma 2.3 (see [19]). Let $\{x_n\}$ and $\{z_n\}$ be bounded sequence in Banach space $E$, and let $\{\lambda_n\}$ be a sequence in $[0, 1]$ such that $0 < \lim \inf_{n \to \infty} \lambda_n \leq \lim \sup_{n \to \infty} \lambda_n < 1$. Suppose $x_{n+1} = \lambda_n x_n + (1 - \lambda_n)z_n$ and

$$\lim \sup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0, \quad \forall n \geq 0. \quad (2.4)$$

Then $\lim_{n \to \infty} \|z_n - x_n\| = 0$.

Lemma 2.4 (see [2, 10]). Let $T : K \to H$ be a $k$-strict pseudocontraction. For $\lambda \in [k, 1)$, define $S : K \to H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for each $x \in K$. Then, as $\lambda \in [k, 1)$, $S$ is a nonexpansive mapping such that $F(S) = F(T)$.

Lemma 2.5 (see [10]). If $T : K \to H$ is a $k$-strict pseudocontraction, then the fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined.

Lemma 2.6 (see [9]). Let $K$ be a nonempty bounded closed convex subset of $H$. Given $x \in H$ and $z \in K$, then $z = P_K x$ if and only if there holds the relation:

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in K. \quad (2.5)$$

Lemma 2.7 (see [20]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad n \geq 0, \quad (2.6)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim \sup_{n \to \infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

Proposition 2.8 (see, e.g., Acedo and Xu [12]). Let $K$ be a nonempty closed convex subset of Hilbert space $H$. Given an integer $N \geq 1$, assume that $\{T_i\}_{i=1}^{N} : K \to H$ is a finite family of $k_i$-strict pseudocontractions. Suppose that $\{\lambda_i\}_{i=1}^{N}$ is a positive sequence such that $\sum_{i=1}^{N} \lambda_i = 1$. Then $\sum_{i=1}^{N} \lambda_i T_i$ is a $k$-strict pseudocontraction with $k = \max \{k_i : 1 \leq i \leq N\}$.
**Proposition 2.9** (see, e.g., Acedo and Xu [12]). Let \( \{T_i\}_{i=1}^N \) and \( \{\lambda_i\}_{i=1}^N \) be given as in Proposition 2.8 above. Then \( F(\bigcap_{i=1}^N \lambda_i T_i) = \bigcap_{i=1}^N F(T_i) \).

### 3. Main Results

**Theorem 3.1.** Let \( K \) be a nonempty closed convex subset of Hilbert space \( H \), and let \( F \) be a bifunction from \( K \times K \) into \( \mathbb{R} \) satisfying (A1)–(A4). Let \( \{T_i\}_{i=1}^N : K \rightarrow H \) be a finite family of \( k_i \)-strict pseudocontractions such that \( k = \max\{k_i : 1 \leq i \leq N\} \) and \( \mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap \text{EP}(F) \neq \emptyset \). Assume \( \{\eta_i^{(n)}\}_{i=1}^N \) is a finite sequences of positive numbers such that \( \sum_{i=1}^N \eta_i^{(n)} = 1 \) for all \( n \geq 0 \). Given \( u \in K \) and \( x_0 \in K, \{\alpha_n\}, \{\beta_n\}, \) and \( \{\gamma_n\} \) are some sequences in \((0,1)\); the following control conditions are satisfied.

(i) \( k \leq \alpha_n \leq \lambda < 1 \) for all \( n \geq 0 \) and \( \lim_{n \rightarrow \infty} \alpha_n = \alpha \),

(ii) \( \lim_{n \rightarrow \infty} \beta_n = 0 \) and \( \sum_{n=0}^\infty \beta_n = \infty \),

(iii) \( 0 < \lim \inf_{n \rightarrow \infty} \gamma_n \leq \lim \sup_{n \rightarrow \infty} \gamma_n < 1 \),

(iv) \( \lim_{n \rightarrow \infty} |\eta_i^{(n+1)} - \eta_i^{(n)}| = 0 \).

Then the sequence \( \{x_n\} \) generated by (1.4) converges strongly to \( q \in \mathcal{F} \), where \( q = P_{\mathcal{F}}u \).

**Proof.** From Lemma 2.1, we see that \( \text{EP}(F) = F(T_i) \), and note that \( u_n \) can be rewritten as \( u_n = T_r x_n \). Putting \( A_n = \sum_{i=1}^N \eta_i^{(n)} T_i \), we have \( A_n : K \rightarrow H \) is a \( k \)-strict pseudocontraction and \( F(A_n) = \bigcap_{i=1}^N F(T_i) \) by Propositions 2.8 and 2.9, where \( k = \max\{k_i : 1 \leq i \leq N\} \).

From (1.4), condition (i), and Lemma 2.2, taking a point \( p \in \mathcal{F} \), we have

\[
\|y_n - p\|^2 = \|\alpha_n (u_n - p) + (1 - \alpha_n) (A_n u_n - p)\|^2 \\
= \alpha_n \|u_n - p\|^2 + (1 - \alpha_n) \|A_n u_n - p\|^2 - \alpha_n (1 - \alpha_n) \|u_n - A_n u_n\|^2 \\
\leq \alpha_n \|u_n - p\|^2 + (1 - \alpha_n) \|u_n - p\|^2 + k \|u_n - A_n u_n\|^2 \\
- \alpha_n (1 - \alpha_n) \|u_n - A_n u_n\|^2
\]

(3.1)

Furthermore, we have

\[
\|y_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|.
\]

(3.2)
It follows from (1.4) and (3.2) that

\[
\|x_{n+1} - p\| = \|\beta_n u + \gamma_n x_n + (1 - \beta_n - \gamma_n) y_n - p\|
\leq \beta_n \|u - p\| + \gamma_n \|x_n - p\| + (1 - \beta_n - \gamma_n) \|y_n - p\|
\leq \beta_n \|u - p\| + (1 - \beta_n) \|x_n - p\|
\leq \max\{\|u - p\|, \|x_0 - p\|\}.
\]  

(3.3)

Consequently, sequence \(\{x_n\}\) is bounded and so are \(\{u_n\}\) and \(\{y_n\}\).

Define a mapping \(T_n x := \alpha_n x + (1 - \alpha_n) A_n x\) for each \(x \in K\). Then \(T_n : K \to H\) is nonexpansive. Indeed, by using (1.1), condition (i), and Lemma 2.2, we have for all \(x, y \in K\) that

\[
\|T_n x - T_n y\|^2 = \alpha_n \|x - y\|^2 + (1 - \alpha_n) \|A_n x - A_n y\|^2
- \alpha_n (1 - \alpha_n) \|x - A_n x - (y - A_n y)\|^2
\leq \alpha_n \|x - y\|^2 + (1 - \alpha_n) \|x - A_n x - (y - A_n y)\|^2
\leq \|x - y\|^2,
\]

which shows that \(T_n : K \to H\) is nonexpansive.

Next we show that \(\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0\). Setting \(x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n\), we have

\[
z_{n+1} - z_n = \frac{x_{n+2} - \gamma_{n+1} x_{n+1}}{1 - \gamma_{n+1}} - \frac{x_{n+1} - \gamma_n x_n}{1 - \gamma_n}
= \frac{\beta_{n+1} u + (1 - \beta_{n+1} - \gamma_{n+1}) y_{n+1}}{1 - \gamma_{n+1}} - \frac{\beta_n u + (1 - \beta_n - \gamma_n) y_n}{1 - \gamma_n}
\leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}} (u - y_{n+1}) + (y_{n+1} - y_n) - \frac{\beta_n}{1 - \gamma_n} (u - y_n).
\]

(3.5)

It follows that

\[
\|z_{n+1} - z_n\| \leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}} \|u - y_{n+1}\| + \|y_{n+1} - y_n\| + \frac{\beta_n}{1 - \gamma_n} \|u - y_n\|.
\]

(3.6)
From (1.4), we have $y_n = T_n u_n$ and

$$
\| y_{n+1} - y_n \| \leq \| T_{n+1} u_{n+1} - T_{n+1} u_n \| + \| T_{n+1} u_n - T_n u_n \|
$$

$$
\leq \| u_{n+1} - u_n \| + \| \alpha_{n+1} u_n + (1 - \alpha_{n+1}) A_{n+1} u_n - [\alpha_n u_n + (1 - \alpha_n) A_n u_n] \|
$$

$$
\leq \| u_{n+1} - u_n \| + |\alpha_{n+1} - \alpha_n| \| u_n - A_n u_n \| + (1 - \alpha_{n+1}) \| A_{n+1} u_n - A_n u_n \|
$$

$$
\leq \| u_{n+1} - u_n \| + |\alpha_{n+1} - \alpha_n| \| u_n - A_n u_n \| + (1 - \alpha_{n+1}) \sum_{i=1}^{N} |\eta_i^{(n+1)} - \eta_i^{(n)}| \| T_i u_n \|,
$$

(3.7)

By Lemma 2.1, $u_n = T_r x_n$ and $u_{n+1} = T_r x_{n+1}$, we have

$$
F(u_n, y) + \frac{1}{r}(y - u_n, u_n - x_n) \geq 0, \quad \forall y \in K,
$$

(3.8)

$$
F(u_{n+1}, y) + \frac{1}{r}(y - u_{n+1}, u_{n+1} - x_{n+1}) \geq 0, \quad \forall y \in K.
$$

(3.9)

Putting $y = u_{n+1}$ in (3.8) and $y = u_n$ in (3.9), we obtain

$$
F(u_n, u_{n+1}) + \frac{1}{r}(u_{n+1} - u_n, u_n - x_n) \geq 0,
$$

(3.10)

$$
F(u_{n+1}, u_n) + \frac{1}{r}(u_n - u_{n+1}, u_{n+1} - x_{n+1}) \geq 0.
$$

So, from (A2) and $r > 0$, we have

$$
\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - (u_{n+1} - x_{n+1}) \rangle \geq 0,
$$

(3.11)

and hence

$$
\| u_{n+1} - u_n \|^2 \leq \langle u_{n+1} - u_n, x_{n+1} - x_n \rangle,
$$

(3.12)

which implies that

$$
\| u_{n+1} - u_n \| \leq \| x_{n+1} - x_n \|.
$$

(3.13)

Combining (3.6), (3.7), and (3.13), we have

$$
\| z_{n+1} - z_n \| \leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}} \| u - y_{n+1} \| + \| x_{n+1} - x_n \| + \frac{\beta_n}{1 - \gamma_n} \| u - y_n \|
$$

$$
+ |\alpha_{n+1} - \alpha_n| \| u_n - A_n u_n \| + (1 - \alpha_{n+1}) \sum_{i=1}^{N} |\eta_i^{(n+1)} - \eta_i^{(n)}| \| T_i u_n \|.
$$

(3.14)
This together with (i), (ii) and (iv) imply that
\[
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\] (3.15)

Hence, by Lemma 2.3, we obtain
\[
\lim_{n \to \infty} \|z_n - x_n\| = 0.
\] (3.16)

Consequently,
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \gamma_n)\|z_n - x_n\| = 0.
\] (3.17)

On the other hand, by (1.4) and (iii), we have
\[
\|x_{n+1} - y_n\| \leq \beta_n \|u - y_n\| + \gamma_n \|x_n - x_{n+1}\| + \gamma_n \|x_{n+1} - y_n\|.
\] (3.18)

which implies that
\[
\|x_{n+1} - y_n\| \leq \frac{\beta_n}{1 - \gamma_n} \|u - y_n\| + \frac{\gamma_n}{1 - \gamma_n} \|x_n - x_{n+1}\|.
\] (3.19)

Combining (ii), (3.17), and (3.19), we have
\[
\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0.
\] (3.20)

Note that
\[
\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.
\] (3.21)

which together with (3.17) and (3.20) implies
\[
\lim_{n \to \infty} \|x_n - y_n\| = 0.
\] (3.22)

Moreover, for \(p \in \mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap EP(F)\), we have
\[
\|u_n - p\|^2 = \|T_r x_n - T_r p\|^2 \leq (x_n - p, u_n - p)
\]
\[
= \frac{1}{2} \left( \|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \right),
\] (3.23)

and hence
\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2.
\] (3.24)
From Lemma 2.2, (3.2) and (3.24), we have

\[
\|x_{n+1} - p\|^2 = \|\beta_n u + \gamma_n x_n + (1 - \beta_n - \gamma_n)y_n - p\|^2 \\
\leq \beta_n\|u - p\|^2 + \gamma_n\|x_n - p\|^2 + (1 - \beta_n - \gamma_n)\|y_n - p\|^2 \\
\leq \beta_n\|u - p\|^2 + \gamma_n\|x_n - p\|^2 + (1 - \beta_n - \gamma_n)\left(\|x_n - p\|^2 - \|x_n - u\|^2\right) \\
\leq \beta_n\|u - p\|^2 + \|x_n - p\|^2 - (1 - \beta_n - \gamma_n)\|x_n - u\|^2,
\]

and hence

\[
(1 - \beta_n - \gamma_n)\|x_n - u\|^2 \leq \beta_n\|u - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
\leq \beta_n\|u - p\|^2 + \|x_n - x_{n+1}\|\left(\|x_n - p\| + \|x_{n+1} - p\|\right).
\]

By (ii) and (3.17), we obtain

\[
\lim_{n \to \infty} \|x_n - u_n\| = 0. \tag{3.27}
\]

It follows from (3.22) and (3.27) that

\[
\lim_{n \to \infty} \|y_n - u_n\| = 0. \tag{3.28}
\]

Define \(S_n : K \to H\) by \(S_n x = \alpha x + (1-\alpha)A_n x\). Then, \(S_n\) is a nonexpansive with \(F(S_n) = F(A_n)\) by Lemma 2.4. Note that \(\lim_{n \to \infty} a_n = \alpha \in [k,1]\) by condition (i) and

\[
\|u_n - S_n u_n\| \leq \|u_n - y_n\| + \|y_n - S_n u_n\| \\
\leq \|u_n - y_n\| + \|\alpha_n u_n + (1 - \alpha_n)A_n u_n - [\alpha u_n + (1 - \alpha)A_n u_n]\| \\
\leq \|u_n - y_n\| + |\alpha_n - \alpha|\|u_n - A_n u_n\|, \tag{3.29}
\]

which combines with condition (i) and (3.28) yielding that

\[
\lim_{n \to \infty} \|u_n - S_n u_n\| = 0. \tag{3.30}
\]

We now show that \(\limsup_{n \to \infty} \langle u - q, x_n - q \rangle \leq 0\), where \(q = P_q u\). To see this, we choose a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) such that

\[
\limsup_{n \to \infty} \langle u - q, x_n - q \rangle = \lim_{i \to \infty} \langle u - q, x_{n_i} - q \rangle. \tag{3.31}
\]

Since \(\{u_{n_i}\}\) is bounded, there exists a subsequence \(\{u_{n_{i_j}}\}\) of \(\{u_{n_i}\}\) converging weakly to \(u^*\). Without loss of generality, we assume that \(u_{n_i} \rightharpoonup u^*\) as \(i \to \infty\). Form (3.27), we obtain \(x_{n_i} \rightharpoonup u^*\).
as \( i \to \infty \). Since \( K \) is closed and convex, \( K \) is weakly closed. So, we have \( u^* \in K \) and \( u^* \in F(S_n) \). Otherwise, from \( u^* \notin S_n u^* \) and Opial’s condition, we obtain

\[
\liminf_{i \to \infty} \|u_i - u^*\| < \liminf_{i \to \infty} \|u_i - S_n u^*\| \\
\leq \liminf_{i \to \infty} (\|u_i - S_n u_n\| + \|S_n u_n - S_n u^*\|) \\
\leq \liminf_{i \to \infty} \|u_i - u^*\|.
\]

This is a contradiction. Hence, we get \( u^* \in F(S_n) = F(A_n) \). Moreover, by \( u_n = T_n x_n \), we have

\[
F(u_n, y) + \frac{1}{r} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in K. \tag{3.33}
\]

It follows from (A2) that

\[
\frac{1}{r} \langle y - u_n, u_n - x_n \rangle \geq F(y, u_n). \tag{3.34}
\]

Replacing \( n \) by \( n_i \), we have

\[
\frac{1}{r} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq F(y, u_{n_i}). \tag{3.35}
\]

Since \( u_{n_i} - x_{n_i} \to 0 \) and \( u_{n_i} \rightharpoonup u^* \), it follows from (A4) that \( F(y, u^*) \leq 0 \) for all \( y \in K \). Put \( z_t = ty + (1-t) u^* \) for all \( t \in (0, 1] \) and \( y \in K \). Then, we have \( z_t \in K \), and hence, \( F(z_t, u^*) \leq 0 \). By (A1) and (A4), we have

\[
0 = F(z_t, z_t) \leq t F(z_t, y) + (1-t) F(z_t, u^*) \leq t F(z_t, y), \tag{3.36}
\]

which implies \( F(z_t, y) \geq 0 \). From (A3), we have \( F(u^*, y) \geq 0 \) for all \( y \in K \), and hence, \( u^* \in EP(F) \). Therefore, \( u^* \in F(S_n) \cap EP(F) \). From Lemma 2.6, we know that

\[
\langle u - P_q u, u^* - P_q u \rangle \leq 0. \tag{3.37}
\]

It follows from (3.31) and (3.37) that

\[
\limsup_{n \to \infty} \langle u - q, x_n - q \rangle = \lim_{i \to \infty} \langle u - q, x_i - q \rangle = \langle u - q, u^* - q \rangle \leq 0. \tag{3.38}
\]
Finally, we prove that \( x_n \to q = P_\mathcal{F}u \) as \( n \to \infty \). From (1.4) again, we have

\[
\|x_{n+1} - q\|^2 = \langle \beta_n u + \gamma_n x_n + (1 - \beta_n - \gamma_n) y_n - q, x_{n+1} - q \rangle \\
= \beta_n \langle u - q, x_{n+1} - q \rangle + \gamma_n \langle x_n - q, x_{n+1} - q \rangle + (1 - \beta_n - \gamma_n) \langle y_n - q, x_{n+1} - q \rangle \\
\leq \beta_n \langle u - q, x_{n+1} - q \rangle + \gamma_n \|x_n - q\| \|x_{n+1} - q\| + (1 - \beta_n - \gamma_n) \|y_n - q\| \|x_{n+1} - q\| \\
\leq (1 - \beta_n) \|x_n - q\| \|x_{n+1} - q\| + \beta_n \langle u - q, x_{n+1} - q \rangle \\
\leq \frac{1 - \beta_n}{2} \left( \|x_n - q\|^2 + \|x_{n+1} - q\|^2 \right) + \beta_n \langle u - q, x_{n+1} - q \rangle \\
\leq \frac{1 - \beta_n}{2} \|x_n - q\|^2 + \frac{1}{2} \|x_{n+1} - q\|^2 + \beta_n \langle u - q, x_{n+1} - q \rangle,
\]

(3.39)

which implies that

\[
\|x_{n+1} - q\|^2 \leq (1 - \beta_n) \|x_n - q\|^2 + 2\beta_n \langle u - q, x_{n+1} - q \rangle.
\]

(3.40)

It follows from (3.38), (3.40), and Lemma 2.7 that \( \lim_{n \to \infty} \|x_n - q\| = 0 \). This completes the proof. \( \square \)

As \( N = 1 \), that is, \( A_n = T \) and \( \eta^{(n)}_i \equiv 1 \) in Theorem 3.1, we have the following results immediately.

**Theorem 3.2.** Let \( K \) be a nonempty closed convex subset of Hilbert space \( H \), and let \( F \) be a bifunction from \( K \times K \) into \( \mathbb{R} \) satisfying (A1)–(A4). Let \( T : K \to H \) be a \( k \)-strict pseudocontractions such that \( \mathcal{F} = F(T) \cap \text{EP}(F) \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated in the following manner:

\[
F(u_n, y) + \frac{1}{r} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in K, \\
y_n = \alpha_n u_n + (1 - \alpha_n) T u_n, \\
x_{n+1} = \beta_n u + \gamma_n x_n + (1 - \beta_n - \gamma_n) y_n, \quad n \geq 0,
\]

(3.41)

where \( u \in K \) and \( x_0 \in K \), \( \{\alpha_n\} \), \( \{\beta_n\} \), and \( \{\gamma_n\} \) are some sequences in (0,1). If the following control conditions are satisfied:

(i) \( k \leq \alpha_n \leq \lambda < 1 \) for all \( n \geq 0 \) and \( \lim_{n \to \infty} \alpha_n = \alpha \),

(ii) \( \lim_{n \to \infty} \beta_n = 0 \) and \( \sum_{n=0}^{\infty} \beta_n = \infty \),

(iii) \( 0 < \lim \inf_{n \to \infty} \gamma_n \leq \lim \sup_{n \to \infty} \gamma_n < 1 \),

then \( \{x_n\} \) converges strongly to \( q \in \mathcal{F} \), where \( q = P_\mathcal{F}u \).
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References


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