Research Article

Some Identities on Bernstein Polynomials Associated with $q$-Euler Polynomials

A. Bayad, T. Kim, B. Lee, and S.-H. Rim

1 Département de Mathématiques, Université d’Evry Val d’Essonne, Bboulevard F. Mitterrand, 91025 Evry Cedex, France
2 Division of General Education-Mathematics, Kwang-Woon University, Seoul 139-701, Republic of Korea
3 Department of Wireless Communications Engineering, Kwangwoon University, Seoul 139-701, Republic of Korea
4 Department of Mathematics Education, Kyungpook National University, Taegu 702-701, Republic of Korea

Correspondence should be addressed to T. Kim, tkkim@kw.ac.kr

Received 11 January 2011; Accepted 11 February 2011

Academic Editor: John Rassias

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We investigate some interesting properties of the $q$-Euler polynomials. The purpose of this paper is to give some relationships between Bernstein and $q$-Euler polynomials, which are derived by the $p$-adic integral representation of the Bernstein polynomials associated with $q$-Euler polynomials.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic numbers, and the field of $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively (see [1–15]). Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The normalized $p$-adic absolute value is defined by $|p|_p = 1/p$. As an indeterminate, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$ I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} f(x) \mu_{-1}(x + p^N \mathbb{Z}_p) $$

$$ = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} f(x)(-1)^x, $$

(1.1)
(see [7–10]). For $n \in \mathbb{N}$, we can derive the following integral equation from (1.1):

$$I^{-1}(f_n) = (-1)^n \int_{\mathbb{R}^+} f(x) d\mu_{n-1}(x) + 2 \sum_{i=0}^{n-1} (-1)^{n-1-i} f(i),$$

(1.2)

where $f_n(x) = f(x + n)$ (see [7–11]). As well-known definition, the Euler polynomials are given by the generating function as follows:

$$\frac{2}{e^{tx} + 1} e^{tx} = e^{E_n(x) t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

(1.3)

(see [3, 5, 7–15]), with usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case $x = 0$, $E_n(0) = E_n$ are called the $n$th Euler numbers. From (1.3), we can derive the following recurrence formula for Euler numbers:

$$E_0 = 1, \quad (E + 1)^n + E_n = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

(1.4)

(see [12]), with usual convention about replacing $E^n$ by $E_n$. By the definitions of Euler numbers and polynomials, we get

$$E_n(x) = (E + x)^n = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l,$$

(1.5)

(see [3, 5, 7–15]). Let $C[0, 1]$ denote the set of continuous functions on $[0, 1]$. For $f \in C[0, 1]$, Bernstein introduced the following well-known linear positive operator in the field of real numbers $\mathbb{R}$:

$$\mathbb{B}_n(f \mid x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n}) B_{k,n}(x),$$

(1.6)

where $\binom{n}{k} = n(n-1) \cdots (n-k+1)/k! = n!/k!(n-k)!$ (see [1, 2, 7, 11, 12, 14]). Here, $\mathbb{B}_n(f \mid x)$ is called the Bernstein operator of order $n$ for $f$. For $k, n \in \mathbb{Z}_+$, the Bernstein polynomials of degree $n$ are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } x \in [0, 1].$$

(1.7)

In this paper, we study the properties of $q$-Euler numbers and polynomials. From these properties, we investigate some identities on the $q$-Euler numbers and polynomials. Finally, we give some relationships between Bernstein and $q$-Euler polynomials, which are derived by the $p$-adic integral representation of the Bernstein polynomials associated with $q$-Euler polynomials.
2. *q*-Euler Numbers and Polynomials

In this section, we assume that \( q \in \mathbb{C}_p \) with \( |1 - q|_p < 1 \). Let \( f(x) = q^x e^{xt} \). From (1.1) and (1.2), we have

\[
\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \frac{2}{qe^t + 1}.
\]  \hspace{1cm} (2.1)

Now, we define the \( q \)-Euler numbers as follows:

\[
\frac{2}{qe^t + 1} = e^{E_t} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!},
\]  \hspace{1cm} (2.2)

with the usual convention about replacing \( E_q^n \) by \( E_{n,q} \).

By (2.2), we easily get

\[
E_{0,q} = \frac{2}{q + 1}, \quad q(E_q + 1)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}
\]  \hspace{1cm} (2.3)

with usual convention about replacing \( E_q^n \) by \( E_{n,q} \).

We note that

\[
\frac{2}{qe^t + 1} = \frac{2}{e^t + q^{-1}} \cdot \frac{2}{1 + q} = \frac{2}{1 + q} \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!},
\]  \hspace{1cm} (2.4)

where \( H_n(-q^{-1}) \) is the \( n \)th Frobenius-Euler numbers.

From (1.2), (2.1), and (2.4), we have

\[
\int_{\mathbb{Z}_p} q^x e^{xt} d\mu_1(x) = E_{n,q} = \frac{2}{1 + q} H_n(-q^{-1}), \quad \text{for } n \in \mathbb{Z}_+.
\]  \hspace{1cm} (2.5)

Now, we consider the \( q \)-Euler polynomials as follows:

\[
\frac{2}{qe^t + 1} e^{xt} = e^{E_q(x)t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!},
\]  \hspace{1cm} (2.6)

with the usual convention \( E_q^n(x) \) by \( E_{n,q}(x) \).

From (1.2), (2.1), and (2.6), we get

\[
\int_{\mathbb{Z}_p} q^x e^{(x+y)t} d\mu_1(y) = \frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.
\]  \hspace{1cm} (2.7)
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By comparing the coefficients on the both sides of (2.6) and (2.7), we get the following Witt’s formula for the \( q \)-Euler polynomials as follows:

\[
\int q^y (x + y)^n d\mu_1(y) = E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l,q}.
\]  

(2.8)

From (2.6) and (2.8), we can derive the following equation:

\[
\frac{2q}{qe^{t} + 1} e^{(1-x)t} = \frac{2}{1 + q^{-1} e^{-t}} e^{-xt} = \sum_{n=0}^{\infty} E_{n,q^{-1}}(x)(-1)^n \frac{t^n}{n!}.
\]

(2.9)

By (2.6) and (2.9), we obtain the following reflection symmetric property for the \( q \)-Euler polynomials.

Theorem 2.1. For \( n \in \mathbb{Z}_+ \), one has

\[
(-1)^n E_{n,q^{-1}}(x) = qE_{n,q}(1 - x).
\]

(2.10)

From (2.5), (2.6), (2.7), and (2.8), we can derive the following equation: for \( n \in \mathbb{N} \),

\[
E_{n,q}(2) = (E_{q} + 1 + 1)^n = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}(1)
\]

\[
= E_{0,q} + \frac{2}{q} \sum_{l=1}^{n} \binom{n}{l} E_{l,q}(1) = \frac{2}{1 + q} - \frac{1}{q} \sum_{l=1}^{n} \binom{n}{l} E_{l,q}
\]

(2.11)

\[
= \frac{2}{1 + q} + \frac{2}{q(1 + q)} - \frac{1}{q} \sum_{l=0}^{n} \binom{n}{l} E_{l,q}
\]

\[
= \frac{2}{q} \frac{1}{1 - q^2} qE_{n,q}(1) = \frac{2}{q} + \frac{1}{q^2} E_{n,q},
\]

by using recurrence formula (2.3). Therefore, we obtain the following theorem.

Theorem 2.2. For \( n \in \mathbb{N} \), one has

\[
qE_{n,q}(2) = 2 + \frac{1}{q} E_{n,q}.
\]

(2.12)
By using (2.5) and (2.8), we get

\[
\int_{\mathbb{Z}_p} q^{-x}(1-x)^n d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} q^{-x}(x-1)^n d\mu_{-1}(x)
\]

\[
= (-1)^n E_{n,q^{-1}}(-1) = q \int_{\mathbb{Z}_p} (x+2)^n d\mu_{-1}(x) = q \left( \frac{2}{q} + \frac{1}{q^2} E_{n,q} \right)
\]

\[
= 2 + \frac{1}{q} E_{n,q} = 2 + \frac{1}{q} \int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x), \quad \text{for } n > 0.
\]

Therefore, we obtain the following theorem.

**Theorem 2.3.** For \( n \in \mathbb{N} \), one has

\[
\int_{\mathbb{Z}_p} q^{-x}(1-x)^n d\mu_{-1}(x) = 2 + \frac{1}{q} \int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x).
\]

By using Theorem 2.3, we will study for the \( p \)-adic integral representation on \( \mathbb{Z}_p \) of the Bernstein polynomials associated with \( q \)-Euler polynomials in Section 3.

### 3. Bernstein Polynomials Associated with \( q \)-Euler Numbers and Polynomials

Now, we take the \( p \)-adic integral on \( \mathbb{Z}_p \) for the Bernstein polynomials in (1.7) as follows:

\[
\int_{\mathbb{Z}_p} B_{k,n}(x) q^x d\mu_{-1}(x) = \int_{\mathbb{Z}_p} \binom{n}{k} x^k (1-x)^{n-k} q^x d\mu_{-1}(x)
\]

\[
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} \int_{\mathbb{Z}_p} x^{n-j} q^x d\mu_{-1}(x)
\]

\[
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} E_{n-j,q}
\]

\[
= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{k+j,q}, \quad \text{where } n, k \in \mathbb{Z}_+.
\]

By the definition of Bernstein polynomials, we see that

\[
B_{k,n}(x) = B_{n-k,n}(1-x), \quad \text{where } n, k \in \mathbb{Z}_+.
\]
Let $n, k \in \mathbb{Z}_+$ with $n > k$. Then, by (3.2), we get

$$
\int_{\mathbb{Z}_+} q^x B_{k,n}(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_+} q^x B_{n-k,n}(1-x) d\mu_{-1}(x)
$$

$$
= \binom{n}{n-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \int_{\mathbb{Z}_+} (1-x)^{n-j} q^x d\mu_{-1}(x)
$$

$$
= \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left( 2 + q \int_{\mathbb{Z}_+} x^{n-j} q^x d\mu_{-1}(x) \right)
$$

$$
= \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left( 2 + q E_{n-j,q} \right)
$$

$$
= \begin{cases} 
2 + q E_{n,q} & \text{if } k = 0, \\
\binom{n}{k} q \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E_{n-j,q} & \text{if } k > 0.
\end{cases}
$$

Thus, we obtain the following theorem.

**Theorem 3.1.** For $n, k \in \mathbb{Z}_+$ with $n > k$, one has

$$
\int_{\mathbb{Z}_+} q^{1-x} B_{k,n}(x) d\mu_{-1}(x) = \begin{cases} 
2q + E_{n,q} & \text{if } k = 0, \\
\binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E_{n-j,q} & \text{if } k > 0.
\end{cases}
$$

By (3.1) and Theorem 3.1, we get the following corollary.

**Corollary 3.2.** For $n, k \in \mathbb{Z}_+$ with $n > k$, one has

$$
\sum_{j=0}^{k-1} \binom{n-k}{j} (-1)^j E_{k+j,q} = \begin{cases} 
2 + \frac{1}{q} E_{n,q} & \text{if } k = 0, \\
\binom{k}{0} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \frac{1}{q} E_{n-j,q} & \text{if } k > 0.
\end{cases}
$$
For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$. Then, we get

\[
\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)q^{-x}d\mu_1(x)
\]

\[
= \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \int_{\mathbb{Z}_p} q^{-x} (1 - x)^{n+m-j}d\mu_1(x)
\]

\[
= \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} q \int_{\mathbb{Z}_p} (x + 2)^{n+m-j}q^x d\mu_1(x)
\]

\[
= \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \left( \frac{2}{q} + \frac{1}{q^2} E_{n+m-j,q} \right)
\]

\[
= \begin{cases} 
2 + \frac{1}{q} E_{n+m,q} & \text{if } k = 0, \\
\binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \frac{1}{q} E_{n+m-j,q} & \text{if } k > 0.
\end{cases}
\]

Therefore, we obtain the following theorem.

**Theorem 3.3.** For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$, one has

\[
\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)q^{-x}d\mu_1(x) = \begin{cases} 
2q + E_{n+m,q} & \text{if } k = 0, \\
\binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j,q} & \text{if } k > 0.
\end{cases}
\]

(3.7)

By using binomial theorem, for $m, n, k \in \mathbb{Z}_+$, we get

\[
\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)q^{-x}d\mu_1(x) = \begin{cases} 
\sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (n + m - 2k)^j \int_{\mathbb{Z}_p} x^{j+2k}q^{-x}d\mu_1(x)
\end{cases}
\]

(3.9)

\[
= q \binom{n}{k} \binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^{j+2k} E_{j+2k,q}. 
\]

By comparing the coefficients on the both sides of (3.8) and Theorem 3.3, we obtain the following corollary.
Corollary 3.4. Let \( m, n, k \in \mathbb{Z}_+ \) with \( m + n > 2k \). Then, we get

\[
\sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k, q-1} = \begin{cases} 
2 \frac{1}{q} E_{n+m, q} & \text{if } k = 0, \\
\frac{1}{q} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j, q} & \text{if } k > 0.
\end{cases}
\] (3.10)

For \( s \in \mathbb{N} \), let \( n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 + \cdots + n_s > sk \). By induction, we get

\[
\int_{\mathbb{Z}_p} B_{k, n_1}(x) \cdots B_{k, n_s}(x) q^{-x} d\mu_{-1}(x)
\]

\[
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \int_{\mathbb{Z}_p} x^{sk} (1-x)^{n_1+\cdots+n_s-sk} q^{-x} d\mu_{-1}(x)
\]

\[
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} \int_{\mathbb{Z}_p} (1-x)^{n_1+\cdots+n_s-j} q^{-x} d\mu_{-1}(x)
\]

\[
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} \int_{\mathbb{Z}_p} (x+2)^{n_1+\cdots+n_s-j} q^{-x} d\mu_{-1}(x)
\]

\[
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} \left( \frac{2}{q} + \frac{1}{q^2} E_{n_1+\cdots+n_s-j, q} \right)
\]

\[
= \begin{cases} 
2 \frac{1}{q} E_{n_1+\cdots+n_s, q} & \text{if } k = 0, \\
\left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \frac{1}{q} \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+\cdots+n_s-j, q} & \text{if } k > 0.
\end{cases}
\] (3.11)

Therefore, we obtain the following theorem.

Theorem 3.5. Let \( s \in \mathbb{N} \). For \( n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 + \cdots + n_s > sk \), one has

\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k, n_i}(x) \right) q^{-x} d\mu_{-1}(x) = \begin{cases} 
2q + E_{n_1+n_2+\cdots+n_s, q} & \text{if } k = 0, \\
\left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+n_2+\cdots+n_s-j, q} & \text{if } k > 0.
\end{cases}
\] (3.12)
For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ by binomial theorem, we get

$$
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x) \right) q^{-x} d\mu_{-1}(x)
$$

$$
= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j \int_{\mathbb{Z}_p} x^{j+sk} q^{-x} d\mu_{-1}(x) (3.13)
$$

$$
= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j E_{j+sk,q}^{-1}.
$$

By using (3.13) and Theorem 3.5, we obtain the following corollary.

**Corollary 3.6.** Let $s \in \mathbb{N}$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$, one has

$$
\sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j E_{j+sk,q}^{-1} = \begin{cases}
2 + \frac{1}{q} E_{n_1+n_2+\cdots+n_j,q} & \text{if } k = 0,
\frac{1}{q} \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+n_2+\cdots+n_j,q} & \text{if } k > 0.
\end{cases}
$$

(3.14)

**References**


