Research Article

Construction of Compactly Supported Refinable Componentwise Polynomial Functions in $\mathbb{R}^2$

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We provide a sufficient condition for constructing a class of compactly supported refinable functions with componentwise polynomial property in $\mathbb{R}^2$. An iteration algorithm is developed to compute the polynomial on each component of the functions' support. Finally, two examples for constructing the symmetric refinable componentwise polynomial functions are given.

1. Introduction

Refinable functions are among the most important functions; they form the foundation of wavelet theory and subdivision scheme theory. The details can be found in [1–3]. Other areas in which refinable functions play important roles are fractal geometry and self-affine tilings (see [4–6]).

Splines, as well as refinable functions, have been widely used in numerical solutions of differential and integral equations, digital signal processing, image compression, and many others. Particularly, the refinable splines such as B -splines in $\mathbb{R}$ and the box splines in $\mathbb{R}^n$ play a key role in approximation theory and in computer-aided geometric design. In recent years, some researches on refinable splines have made great progress (see [7–15]).

As we know, a refinable function with an analytic expression is of great importance, especially in those requiring high-precision domains. However, refinable functions are solutions of refinement equations and are usually defined by some cascade algorithms. Due to the iterative process, most of refinable functions do not have explicit analytic expression. In [9, 11], the authors have proved that there are no other piecewise smooth compactly supported refinable functions with explicit analytic expression than B -splines. Then much

attention has been given in the literature to the componentwise polynomial since it is a polynomial on each connected component of the function's support, and there are some preliminary researches on this field (see [16–20]). What is a spline in $\mathbb{R}^n$? And what is a componentwise polynomial in $\mathbb{R}^n$?

Definition 1.1 (see [7]). A compactly supported function $f(x)$ in $\mathbb{R}^n$ with $\text{supp}(f) = \Omega$ is called a *spline* if there exists a partition $\Omega = \bigcup_{j=1}^N R_j$ into simplices $\{R_j\}$ in $\mathbb{R}^n$ such that f is a polynomial on each $\{R_j\}$.

Definition 1.2 (see [16]). A compactly supported function $f : \mathbb{R}^n \mapsto \mathbb{C}$ is a *componentwise polynomial* if there exists an open set G such that the Lebesgue measure of $\mathbb{R}^n \setminus G$ is zero and the restriction of f on any connected open component of G coincides with some polynomial.

Generally, the componentwise polynomial is not a spline; it is much broader than a spline. The difference between the componentwise polynomial and the spline is that the former's support consists of infinitely many simplices, and the latter's is just a finitely union of some simplices. In one dimension, there have been some good results on the componentwise polynomials (see [17–20]). The concept of componentwise polynomials in $\mathbb{R}$ is first introduced in [17, 18] under the name of local polynomials, and the componentwise polynomial property of a one-dimensional refinable function has been well studied and demonstrated to be useful there. In particular, an iteration algorithm is given to compute the polynomial on each component (see [18]). In [19], a few examples of compactly supported refinable componentwise constant functions, which satisfy either orthogonality or interpolation property and which are continuous and symmetric, are given. Additional examples of refinable componentwise linear functions that are differentiable and symmetric are also given in [19]. In [20], the authors present another example of a componentwise constant function that is not locally integrable, and they further assert that any refinable componentwise polynomial function with the dilation factor 2 must be a finite linear combination of the integer shifts of some B -spline. In two dimensions, the authors study a class of compactly supported refinable componentwise constant functions and give a sufficient condition of the refinable function being a componentwise constant function, which is much harder than that in one dimension (see [16]). How about the refinable componentwise polynomial functions in $\mathbb{R}^2$?

In this paper, we present a sufficient condition of the compactly supported refinable functions being componentwise polynomials in $\mathbb{R}^2$. By using a reasonable partition of the functions' support, we provide an iteration algorithm to compute the polynomial on each connected component of the functions' support. Compared with [16], the algorithm has been improved broader than that in [16] since the technical condition (7) in [16] can be removed.

2. Preliminaries and Main Results

Let $f(x)$ be a compactly supported function in $\mathbb{R}^2$, and let $A \in \mathbf{M}_2(\mathbb{Z})$ be an expanding matrix; that is, the modules of all eigenvalues are greater than 1. For all $h(\mathbf{k}) \in \mathbb{R}$ with only finitely many $h(\mathbf{k}) \neq 0$ and $\sum_{\mathbf{k} \in \mathbb{Z}^2} h(\mathbf{k}) = |\det A|$, we say that a measurable function $f(x) : \mathbb{R}^2 \rightarrow \mathbb{C}$ is a *measurable function solution* of the refinement equation

$$f(x) = \sum_{\mathbf{k} \in \mathbb{Z}^2} h(\mathbf{k}) f(Ax - \mathbf{k}), \quad \text{a.e. } x \in \mathbb{R}^2, \quad (2.1)$$

if $f(\mathbf{x})$ satisfies the refinement equation (2.1) in the sense of almost everywhere, where $H(\xi) = (1/|A|)\sum_{k \in \mathbb{Z}^2} h(k)e^{-ik^T \xi}$ is called the *mask symbol* of the refinable function f .

Theorem 2.1. Let $g(\mathbf{x}) \in L_1(\mathbb{R}^2)$ be a compactly supported function satisfying the refinement equation

$$g(\mathbf{x}) = \sum_{k \in \mathbb{Z}^2} c(k)g(B\mathbf{x} - k), \quad a.e. \mathbf{x} \in \mathbb{R}^2, \quad (2.2)$$

where $B = \text{diag}(\lambda_1, \lambda_2)$. If the mask symbol of $g(\mathbf{x})$ is given by

$$H_1(\xi) = \left[\frac{1 - e^{-i\lambda_1 \xi_1}}{\lambda_1(1 - e^{-i\xi_1})} \right]^{N_1} \left[\frac{1 - e^{-i\lambda_2 \xi_2}}{\lambda_2(1 - e^{-i\xi_2})} \right]^{N_2} Q_{(r_1, r_2)}(\xi), \quad (2.3)$$

where $Q_{(r_1, r_2)}(\xi) := \sum_{s=(0,0)}^{(r_1, r_2)} q(s)e^{-is^T \xi}$, $Q_{(r_1, r_2)}(0) = 1$, $\mathbf{s} = (s_1, s_2)^T$, $\xi = (\xi_1, \xi_2)^T$, and $0 < r_1 < \lambda_1 - 1$, $0 < r_2 < \lambda_2 - 1$, $\lambda_1, \lambda_2 \geq 3$, $r_1, r_2, s_1, s_2, \lambda_1, \lambda_2, N_1, N_2 \in \mathbb{Z}$. Then $g(\mathbf{x})$ is a refinable componentwise polynomial function supported on $(0, N_1 + r_1/(\lambda_1 - 1)) \times (0, N_2 + r_2/(\lambda_2 - 1))$.

In what follows we need Lemma 2.2 formulated below.

Lemma 2.2. Let $g(\mathbf{x})$ be defined as Theorem 2.1 and $\mathbf{x} = (x_1, x_2)^T$. Denote $G(x_1, x_2) := (g(x_1, x_2), \dots, g(x_1, x_2 + N_2 - 1), \dots, g(x_1 + N_1 - 1, x_2), \dots, g(x_1 + N_1 - 1, x_2 + N_2 - 1))^T$. Then $g(\mathbf{x})$ is a polynomial at most $N_1 + N_2 - 2$ degree on $\bigcup_{j_1=0}^{N_1-1} \bigcup_{j_2=0}^{N_2-1} (j_1 + (r_1/(\lambda_1 - 1), 1)) \times (j_2 + (r_2/(\lambda_2 - 1), 1))$.

Proof. Denote $\Lambda := (0, N_1 + r_1/(\lambda_1 - 1)) \times (0, N_2 + r_2/(\lambda_2 - 1))$. It is easy to check that

$$\text{supp}(g) \subseteq \Lambda. \quad (2.4)$$

Since $g(\mathbf{x}) \in L_1(\mathbb{R}^2)$, it follows from (2.3) that $(\partial^{j+s} \widehat{g} / \partial \xi_1^j \partial \xi_2^s)(2n_1\pi, 2n_2\pi) = 0$ holds for all $(n_1, n_2) \in \mathbb{Z}^2 - (0, 0)$, $0 \leq j \leq N_1 - 1$, $0 \leq s \leq N_2 - 1$. Then by the following Poisson summation formula in $\mathbb{R}^2$:

$$\sum_{k_1, k_2} (x_1 + k_1)^j (x_2 + k_2)^s g(x_1 + k_1, x_2 + k_2) = \sum_{n_1, n_2} i^{j+s} \frac{\partial^{j+s} \widehat{g}}{\partial \xi_1^j \partial \xi_2^s} (2n_1\pi, 2n_2\pi) e^{i2\pi(n_1 x_1 + n_2 x_2)}, \quad (2.5)$$

we get that $\sum_{k_1, k_2} (x_1 + k_1)^j (x_2 + k_2)^s g(x_1 + k_1, x_2 + k_2) = m_{js}$, $0 \leq j \leq N_1 - 1$, $0 \leq s \leq N_2 - 1$. Denote $A_i(t) := t^i((x_2 + j)^k)_{0 \leq j, k \leq N_2-1}$ and $M(x_1, x_2) := (A_i(x_1 + j))_{0 \leq i, j \leq N_1-1}$. Then according to (2.4), we get that

$$M(x_1, x_2)G(x_1, x_2) = (m_{00}, \dots, m_{0, N_2-1}, \dots, m_{N_1-1, 0}, \dots, m_{N_1-1, N_2-1})^T \quad (2.6)$$

when $(x_1, x_2) \in (r_1/(\lambda_1 - 1), 1) \times (r_2/(\lambda_2 - 1), 1)$. By calculating, it is not difficult to get that

$$\det M(x_1, x_2) = \begin{vmatrix} 1 & 2 & \cdots & N_2 - 1 \\ 1 & 2^2 & \cdots & (N_2 - 1)^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{N_2-1} & \cdots & (N_2 - 1)^{N_2-1} \end{vmatrix}^{N_1} \cdot (2! \cdot 3! \cdot \cdots \cdot (N_1 - 1)!)^{N_2} \neq 0. \quad (2.7)$$

Then according to (2.6) and (2.7), $g(\mathbf{x})$ is a polynomial at most $N_1 + N_2 - 2$ degree on $\bigcup_{j_1=0}^{N_1-1} \bigcup_{j_2=0}^{N_2-1} (j_1 + (r_1/(\lambda_1 - 1), 1)) \times (j_2 + (r_2/(\lambda_2 - 1), 1))$. $\square$

To accomplish Theorem 2.1, an iteration method is carried out to the proof. For convenience, we use $g \in P$ to denote that g is a (componentwise) polynomial. We first form a reasonable partition of the region Λ which can be divided into $(2N_1 + 1)(2N_2 + 1)$ pairwise disjoint components, and then we prove that $g \in P$ in all these components by Steps 1–4.

Proof of Theorem 2.1. Let us divide the interval $(0, N_1 + r_1/(\lambda_1 - 1))$ into $2N_1 + 1$ parts with points of division $0 = u_1 < u_2 < \cdots < u_{2N_1+1} < u_{2N_1+2} = N_1 + r_1/(\lambda_1 - 1)$, where $u_{2k+1} = k$, $u_{2k+2} = k + r_1/(\lambda_1 - 1)$, $k = 0, 1, \dots, N_1$. Let us also divide the interval $(0, N_2 + r_2/(\lambda_2 - 1))$ into $2N_2 + 1$ parts with points of division $0 = v_1 < v_2 < \cdots < v_{2N_2+1} < v_{2N_2+2} = N_2 + r_2/(\lambda_2 - 1)$, where $v_{2k+1} = k$, $v_{2k+2} = k + r_2/(\lambda_2 - 1)$, $k = 0, 1, \dots, N_2$. This results in the partition of the region Λ into $(2N_1 + 1)(2N_2 + 1)$ components. For convenience, we give for these components some notations as follows:

$$\begin{aligned} \mathcal{A}'_0 &:= \bigcup_{j_1=0}^{N_1-1} \bigcup_{j_2=0}^{N_2-1} \left(j_1 + \left(0, \frac{r_1}{\lambda_1 - 1} \right) \right) \times \left(j_2 + \left(0, \frac{r_2}{\lambda_2 - 1} \right) \right), \\ \mathcal{B}'_0 &:= \bigcup_{j_1=0}^{N_1-1} \bigcup_{j_2=0}^{N_2-1} \left(j_1 + \left(\frac{r_1}{\lambda_1 - 1}, 1 \right) \right) \times \left(j_2 + \left(0, \frac{r_2}{\lambda_2 - 1} \right) \right), \\ \mathcal{C}'_0 &:= \bigcup_{j_1=0}^{N_1-1} \bigcup_{j_2=0}^{N_2-1} \left(j_1 + \left(0, \frac{r_1}{\lambda_1 - 1} \right) \right) \times \left(j_2 + \left(\frac{r_2}{\lambda_2 - 1}, 1 \right) \right), \\ \mathcal{D}'_0 &:= \bigcup_{j_1=0}^{N_1-1} \bigcup_{j_2=0}^{N_2-1} \left(j_1 + \left(\frac{r_1}{\lambda_1 - 1}, 1 \right) \right) \times \left(j_2 + \left(\frac{r_2}{\lambda_2 - 1}, 1 \right) \right), \\ \mathcal{E}'_0 &:= \bigcup_{j_2=0}^{N_2-1} \left(N_1 + \left(0, \frac{r_1}{\lambda_1 - 1} \right) \right) \times \left(j_2 + \left(0, \frac{r_2}{\lambda_2 - 1} \right) \right), \\ \mathcal{F}'_0 &:= \bigcup_{j_2=0}^{N_2-1} \left(N_1 + \left(0, \frac{r_1}{\lambda_1 - 1} \right) \right) \times \left(j_2 + \left(\frac{r_2}{\lambda_2 - 1}, 1 \right) \right), \\ \mathcal{G}'_0 &:= \bigcup_{j_1=0}^{N_1-1} \left(j_1 + \left(0, \frac{r_1}{\lambda_1 - 1} \right) \right) \times \left(N_2 + \left(0, \frac{r_2}{\lambda_2 - 1} \right) \right), \end{aligned}$$

$$\begin{aligned}\mathcal{H}'_0 &:= \bigcup_{j_1=0}^{N_1-1} \left(j_1 + \left(\frac{r_1}{\lambda_1-1}, 1 \right) \right) \times \left(N_2 + \left(0, \frac{r_2}{\lambda_2-1} \right) \right), \\ \mathcal{J}'_0 &:= \left(N_1 + \left(0, \frac{r_1}{\lambda_1-1} \right) \right) \times \left(N_2 + \left(0, \frac{r_2}{\lambda_2-1} \right) \right).\end{aligned}\tag{2.8}$$

Denote

$$\begin{aligned}\Omega(\epsilon_1, \dots, \epsilon_k) &:= \left(\sum_{i=1}^k \frac{\epsilon_i}{\lambda_1^i} + \frac{r_1}{(\lambda_1-1)\lambda_1^k}, \sum_{i=1}^k \frac{\epsilon_i}{\lambda_1^i} + \frac{1}{\lambda_1^k} \right), \\ \Omega'(\epsilon'_1, \dots, \epsilon'_s) &:= \left(\sum_{j=1}^s \frac{\epsilon'_j}{\lambda_2^j} + \frac{r_2}{(\lambda_2-1)\lambda_2^s}, \sum_{j=1}^s \frac{\epsilon'_j}{\lambda_2^j} + \frac{1}{\lambda_2^s} \right),\end{aligned}\tag{2.9}$$

where $\epsilon_i \in \{0, 1, \dots, r_1\}$, $\epsilon'_j \in \{0, 1, \dots, r_2\}$, $\epsilon_k \neq r_1$, $\epsilon'_s \neq r_2$, and $1 \leq i \leq k$, $1 \leq j \leq s$,

$$\begin{aligned}\Omega_\infty &:= \bigcup_{k=1}^{\infty} \bigcup_{(\epsilon_1, \dots, \epsilon_{k-1}) \in \{0, 1, \dots, r_1\}^{k-1}} \bigcup_{\epsilon_k \in \{0, 1, \dots, r_1-1\}} \Omega(\epsilon_1, \dots, \epsilon_k), \\ \Omega'_\infty &:= \bigcup_{s=1}^{\infty} \bigcup_{(\epsilon'_1, \dots, \epsilon'_{s-1}) \in \{0, 1, \dots, r_2\}^{s-1}} \bigcup_{\epsilon'_s \in \{0, 1, \dots, r_2-1\}} \Omega'(\epsilon'_1, \dots, \epsilon'_s).\end{aligned}\tag{2.10}$$

Applying Lemma 2.2, it follows that $g(\mathbf{x}) \in P$ in $\mathfrak{D}'_0$. In the following, we mainly study the function $g(\mathbf{x})$ in $\mathcal{A}'_0$, $\mathcal{B}'_0$, and $\mathcal{C}'_0$ for the case $j_1 = 0$, $j_2 = 0$. In such case, we denote the three regions as $\mathcal{A}_0$, $\mathcal{B}_0$, and $\mathcal{C}_0$, respectively.

Divide $\mathcal{A}_0$ into $\mathcal{A} := \bigcup_{k=1}^{\infty} \bigcup_{s=1}^{\infty} \Omega(\epsilon_1, \dots, \epsilon_k) \times \Omega'(\epsilon'_1, \dots, \epsilon'_s)$.

Divide $\mathcal{B}_0$ into $\mathcal{B} := \bigcup_{i=r_1}^{\lambda_1-2} \bigcup_{s=1}^{\infty} [(r_1/\lambda_1(\lambda_1-1) + i/\lambda_1, (i+1)/\lambda_1) \cup ((i+1)/\lambda_1, (i+1)/\lambda_1 + r_1/\lambda_1(\lambda_1-1)) \cup (r_1/\lambda_1(\lambda_1-1) + (\lambda_1-1)/\lambda_1, 1)] \times \Omega'(\epsilon'_1, \dots, \epsilon'_s)$.

Divide $\mathcal{C}_0$ into $\mathcal{C} := \bigcup_{k=1}^{\infty} \bigcup_{j=r_2}^{\lambda_2-2} \Omega(\epsilon_1, \dots, \epsilon_k) \times [(r_2/\lambda_2(\lambda_2-1) + j/\lambda_2, (j+1)/\lambda_2) \cup ((j+1)/\lambda_2, (j+1)/\lambda_2 + r_2/\lambda_2(\lambda_2-1)) \cup (r_2/\lambda_2(\lambda_2-1) + (\lambda_2-1)/\lambda_2, 1)]$.

From [16, 17], the above division can ensure that all the components of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are pairwise disjoint, respectively, and the areas of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy the equality $\mathbb{S}_{\mathcal{A}} = \mathbb{S}_{\mathcal{A}'_0}$, $\mathbb{S}_{\mathcal{B}} = \mathbb{S}_{\mathcal{B}'_0}$ and $\mathbb{S}_{\mathcal{C}} = \mathbb{S}_{\mathcal{C}'_0}$. Therefore, we prove that $g \in P$ in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively.

Step 1. Let $b_s^r := (r_1/\lambda_1(\lambda_1-1) + r_1/\lambda_1, 1) \times \Omega'(\epsilon'_1, \dots, \epsilon'_s)$ be the s th row block of $\mathcal{B}$. Let $b_{2i+1}^c := (r_1/\lambda_1(\lambda_1-1) + (r_1+i)/\lambda_1, (r_1+i+1)/\lambda_1) \times \Omega'_\infty$, $i = 0, 1, \dots, \lambda_1 - r_1 - 1$, be the $(2i+1)$ th column block of $\mathcal{B}$; let $b_{2i+2}^c := ((r_1+i+1)/\lambda_1, r_1/\lambda_1(\lambda_1-1) + (r_1+i+1)/\lambda_1) \times \Omega'_\infty$, $i = 0, 1, \dots, \lambda_1 - r_1 - 2$, be the $(2i+2)$ th column block of $\mathcal{B}$. And b_{ij} is the i th row and the j th column block of $\mathcal{B}$ according to the defined rules of b_s^r , b_{2i+1}^c , and b_{2i+2}^c . According to Lemma 2.2, $g(\mathbf{x})$ is

a polynomial in $(r_1/(\lambda_1 - 1), 1) \times (r_2/(\lambda_2 - 1), 1)$. Then it follows from the refinement equation (2.2) that

$$\forall \left(\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2} \right) \in b_{1,2i+1} = \left(\frac{r_1}{\lambda_1(\lambda_1 - 1)} + \frac{r_1 + i}{\lambda_1}, \frac{r_1 + i + 1}{\lambda_1} \right) \times \Omega'(\epsilon'_1), \quad g\left(\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2} \right) \in P, \quad (2.11)$$

where $i = 0, 1, \dots, \lambda_1 - r_1 - 1$; $\epsilon'_1 \in \{0, 1, \dots, r_2 - 1\}$.

As in [16], by (2.2), it can be shown that the components where $g(x_1/\lambda_1, x_2/\lambda_2) \in P$ in $b_{k,2i+1}$ are from those in b_{k-1}^r . And it is easy to check that the distribution of the components where $g(\mathbf{x}) \in P$ in $b_{k,2i+1}$ is the same as that in b_{k-1}^r . For simplicity, we call the components where $g(\mathbf{x})$ are unknown as the blanks.

In the following, we first consider the case $i = 0$ since the components where $g(x_1/\lambda_1, x_2/\lambda_2) \in P$ in $b_{k,2i+1}$ can be gotten from the translation of those in b_{k1} . Then we get the components where $g(x_1/\lambda_1, x_2/\lambda_2) \in P$ in b_1^c :

$$\forall \left(\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2} \right) \in \begin{cases} b_{11} : \left[\left(\frac{r_1}{\lambda_1(\lambda_1 - 1)}, \frac{1}{\lambda_1} \right) + \frac{r_1}{\lambda_1} \right] \times \Omega'(\epsilon'_1); \\ b'_{21} : \left[\left(\frac{r_1}{\lambda_1^2(\lambda_1 - 1)}, \frac{1}{\lambda_1^2} \right) + \frac{r_1}{\lambda_1} + \frac{s_2}{\lambda_1^2} \right] \times \Omega'(\epsilon'_1, \epsilon'_2) \subset b_{21}; \\ \vdots \\ b'_{k1} : \left[\left(\frac{r_1}{\lambda_1^k(\lambda_1 - 1)}, \frac{1}{\lambda_1^k} \right) + \frac{r_1}{\lambda_1} + \frac{s_2}{\lambda_1^2} + \dots + \frac{s_k}{\lambda_1^k} \right] \times \Omega'(\epsilon'_1, \dots, \epsilon'_k) \subset b_{k1}, \end{cases} \quad (2.12)$$

where $r_1 \leq s_2, \dots, s_{k-1} \leq \lambda_1 - 1$ and $s_i \in \mathbb{Z}$, ϵ'_j is defined as above, and all the components of $b_{11}, b'_{21}, \dots, b'_{k1}$ are pairwise disjoint, respectively. The above formula (2.12) is different from formula (15) in [16] since we cannot get that $g(x_1/\lambda_1, x_2/\lambda_2) \in P$ in $b_{1,2i+2}$, $i = 0, 1, \dots, \lambda_1 - r_1 - 2$. In fact, the coefficients of mask symbol in (2.3) do not satisfy condition (7) in [16]; then, we just get $g(x_1/\lambda_1, x_2/\lambda_2) \in P$ in $b_{1,2i+1}$, $i = 0, 1, \dots, \lambda_1 - r_1 - 1$. Therefore, the blanks in b_1^r which is $b_{1,2i+2}$, $i = 0, 1, \dots, \lambda_1 - r_1 - 2$, lead to the blanks in b_{21} . Then $b'_{21} \subset b_{21}$. Step by step, the blanks in b_{k-1}^r lead to the blanks in b_{k1} and $b'_{k1} \subset b_{k1}$. For simplicity, we denote the area of the components where $g(\mathbf{x}) \in P$ in $\mathcal{X}$ by $S_{\mathcal{X}}$. Denoting the above total area in (2.12) by $S_{b_1^c}$, we obtain

$$\begin{aligned} S_{b_1^c} &= \lim_{k \rightarrow \infty} (S_{b_{11}} + S_{b'_{21}} + \dots + S_{b'_{k1}}) \\ &= \frac{\lambda_1 - r_1 - 1}{\lambda_1 - 1} \cdot \frac{\lambda_2 - r_2 - 1}{\lambda_2 - 1} \cdot \frac{r_2}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)}. \end{aligned} \quad (2.13)$$

Let $c_k^c := \Omega(\epsilon_1, \dots, \epsilon_k) \times (r_2/\lambda_2(\lambda_2 - 1) + r_2/\lambda_2, 1)$ be the k th column block of $\mathcal{C}$. Let $c_{2j+1}^r := \Omega_{\infty} \times (r_2/\lambda_2(\lambda_2 - 1) + (r_2 + j)/\lambda_2, (r_2 + j + 1)/\lambda_2)$, $j = 0, 1, \dots, \lambda_2 - r_2 - 1$, be the $(2j + 1)$ th row block of $\mathcal{C}$; let $c_{2j+2}^r := \Omega_{\infty} \times ((r_2 + j + 1)/\lambda_2, r_2/\lambda_2(\lambda_2 - 1) + (r_2 + j + 1)/\lambda_2)$, $j = 0, 1, \dots, \lambda_2 - r_2 - 2$, be the $(2j + 2)$ th row block of $\mathcal{C}$. And c_{ij} is the i th row and the j th

column block of $\mathcal{C}$ according to the defined rules of c_k^c , c_{2j+1}^r , and c_{2j+2}^r . Symmetrically, we can get the total area of the components where $g(\mathbf{x}) \in P$ in c_1^c :

$$\begin{aligned} S_{c_1^r} &= \lim_{k \rightarrow \infty} (S_{c_{11}} + S_{c_{12}} + \cdots + S_{c_{1k}}) \\ &= \frac{\lambda_1 - r_1 - 1}{\lambda_1 - 1} \cdot \frac{\lambda_2 - r_2 - 1}{\lambda_2 - 1} \cdot \frac{r_1}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)}, \end{aligned} \quad (2.14)$$

where $c'_{12}, \dots, c'_{1k}$ are the components where $g(\mathbf{x}) \in P$ in $c_{12}, \dots, c_{1k}$, respectively.

Step 2. Let $a_s^r := \Omega_\infty \times \Omega'(e'_1, \dots, e'_s)$ be the s th row block of $\mathcal{A}$. Let $a_k^c := \Omega(e_1, \dots, e_k) \times \Omega'_\infty$ be the k th column block of $\mathcal{A}$. And a_{ij} is the i th row and the j th column block of $\mathcal{A}$ according to the defined rules of a_s^r and a_k^c . Applying the same method in [16], we get the first incremental area of the components where $g(\mathbf{x}) \in P$ in $\mathcal{A}$: $\Delta^1 S_{\mathcal{A}} = (r_2 S_{c_1^r} + r_1 S_{b_1^c} - S_{a_{11}}) / (1 - q)$, where $q = (r_1 + 1)(r_2 + 1) / \lambda_1 \lambda_2$, $S_{a_{11}} = ((\lambda_1 - r_1 - 1) / \lambda_1 (\lambda_1 - 1)) \cdot ((\lambda_2 - r_2 - 1) / \lambda_2 (\lambda_2 - 1)) \cdot r_1 r_2$. And it is easy to obtain that the first incremental area of the components where $g(\mathbf{x}) \in P$ in $\mathcal{C}$ is $\Delta^1 S_{\mathcal{C}} = (\lambda_2 - r_2) S_{c_1^r}$.

Step 3. Applying the refinement equation (2.2), it can be shown that the components where $g(\mathbf{x}) \in P$ in b_{12} can be got from those in $\mathcal{C}$, and $S_{b_{12}} / S_{\mathcal{C}} = r_2 / \lambda_1 \lambda_2$. Then the first incremental area of the components where $g(\mathbf{x}) \in P$ in b_{12} is $\Delta^1 S_{b_{12}} = (r_2 / \lambda_1 \lambda_2) \Delta^1 S_{\mathcal{C}} = (r_2 / \lambda_1 \lambda_2) (\lambda_2 - r_2) S_{c_1^r}$. Similarly, $\Delta^1 S_{b_{1,2i+2}} = \Delta^1 S_{b_{12}}$, $i = 0, 1, \dots, \lambda_1 - r_1 - 2$. Furthermore, the components where $g(\mathbf{x}) \in P$ in b_{j2} can be got from those in a_{j-1}^r , $j = 2, \dots, k$, respectively. Since $S_{b_{j2}} / S_{a_{j-1}^r} = (r_2 + 1) / \lambda_1 \lambda_2$, we get the first incremental area of the components where $g(\mathbf{x}) \in P$ in b_{j2} : $\Delta^1 S_{b_{j2}} = ((r_2 + 1) / \lambda_1 \lambda_2) S_{a_{j-1}^r}$. Similarly, $\Delta^1 S_{b_{j,2i+2}} = \Delta^1 S_{b_{j2}}$, $i = 0, 1, \dots, \lambda_1 - r_1 - 2$. Therefore, we obtain the first incremental area of the components where $g(\mathbf{x}) \in P$ in b_2^c :

$$\begin{aligned} \Delta^1 S_{b_2^c} &= \lim_{k \rightarrow \infty} (\Delta^1 S_{b_{12}} + \Delta^1 S_{b_{22}} + \cdots + \Delta^1 S_{b_{k2}}) \\ &= \frac{r_2}{\lambda_1 \lambda_2} (\lambda_2 - r_2) S_{c_1^r} + \frac{r_2 + 1}{\lambda_1 \lambda_2} \Delta^1 S_{\mathcal{A}}. \end{aligned} \quad (2.15)$$

Symmetrically, we deduce that

$$\Delta^1 S_{c_2^r} = \frac{r_1}{\lambda_1 \lambda_2} (\lambda_1 - r_1) S_{b_1^c} + \frac{r_1 + 1}{\lambda_1 \lambda_2} \Delta^1 S_{\mathcal{A}}. \quad (2.16)$$

Since the components where $g(\mathbf{x}) \in P$ in b_{j1} and the blanks in b_{j1} are one-to-one correspondence with those in b_{j-1}^r , $j = 2, \dots, k$, respectively, then the parts of blanks in b_{j1} can be filled with $\Delta^1_{b_{j-1,2i+2}}$, $i = 0, 1, \dots, \lambda_1 - r_1 - 2$, correspondingly, and the incremental components $\Delta^1 b_{j1}$ where $g(\mathbf{x}) \in P$ in b_{j1} from $\Delta^1_{b_{j-1,2i+2}}$ are disjoint with b'_{j1} . Therefore, by (2.12), it follows that

$$\frac{\Delta^1 S_{b_{21}}}{\Delta^1 S_{b_{12}}} = \frac{\Delta^1 S_{b_{31}}}{\Delta^1 S_{b_{22}}} = \cdots = \frac{\Delta^1 S_{b_{k1}}}{\Delta^1 S_{b_{k-1,2}}} = \frac{(r_2 + 1)(\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2}. \quad (2.17)$$

Similarly,

$$\frac{\Delta^1 S_{b_{31}}}{\Delta^1 S_{b_{21}}} = \frac{\Delta^1 S_{b_{41}}}{\Delta^1 S_{b_{31}}} = \dots = \frac{\Delta^1 S_{b_{k1}}}{\Delta^1 S_{b_{k-1,1}}} = \frac{(r_2 + 1)(\lambda_1 - r_1)}{\lambda_1 \lambda_2}. \quad (2.18)$$

Then we get the first incremental area of the components where $g(\mathbf{x}) \in P$ in b_1^c :

$$\begin{aligned} \Delta^1 S_{b_1^c} &= \lim_{k \rightarrow \infty} \left(\Delta^1 S_{b_{21}} + \Delta^1 S_{b_{31}} + \dots + \Delta^1 S_{b_{k1}} \right) \\ &= \lim_{k \rightarrow \infty} \Delta^1 S_{b_2^c} \cdot \frac{(r_2 + 1)(\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2} \left[1 + \frac{(r_2 + 1)(\lambda_1 - r_1)}{\lambda_1 \lambda_2} + \dots + \left(\frac{(r_2 + 1)(\lambda_1 - r_1)}{\lambda_1 \lambda_2} \right)^{k-3} \right] \\ &= \left[\frac{r_2}{\lambda_1 \lambda_2} (\lambda_2 - r_2) S_{c_1^r} + \frac{r_2 + 1}{\lambda_1 \lambda_2} \Delta^1 S_{\mathcal{A}} \right] \cdot \frac{(r_2 + 1)(\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)}. \end{aligned} \quad (2.19)$$

Symmetrically, we deduce that

$$\Delta^1 S_{c_1^r} = \left[\frac{r_1}{\lambda_1 \lambda_2} (\lambda_1 - r_1) S_{b_1^c} + \frac{r_1 + 1}{\lambda_1 \lambda_2} \Delta^1 S_{\mathcal{A}} \right] \cdot \frac{(r_1 + 1)(\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)}. \quad (2.20)$$

It follows from Step 2 that the second incremental areas of the components where $g(\mathbf{x}) \in P$ in $\mathcal{A}$ and $\mathcal{C}$, respectively, are

$$\begin{aligned} \Delta^2 S_{\mathcal{A}} &= \frac{r_2 \Delta^1 S_{c_1^r} + r_1 \Delta^1 S_{b_1^c}}{1 - q}, \\ \Delta^2 S_{\mathcal{C}} &= (\lambda_2 - r_2) \Delta^1 S_{c_1^r} + (\lambda_2 - r_2 - 1) \Delta^1 S_{c_2^r} \\ &= (\lambda_2 - r_2) \cdot \frac{(r_1 + 1)(\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)} \Delta^1 S_{c_2^r} + (\lambda_2 - r_2 - 1) \Delta^1 S_{c_2^r} \\ &= \Delta^1 S_{c_2^r} \cdot \frac{\lambda_1 \lambda_2 (\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)}. \end{aligned} \quad (2.21)$$

Symmetrically, we deduce that

$$\Delta^2 S_{\mathcal{B}} = \Delta^1 S_{b_2^c} \cdot \frac{\lambda_1 \lambda_2 (\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)}. \quad (2.22)$$

Step 4. Repeating Step 3, we get the second incremental areas of the components where $g(\mathbf{x}) \in P$ in b_2^c and c_2^r , respectively:

$$\begin{aligned} \Delta^2 S_{b_2^c} &= \frac{r_2}{\lambda_1 \lambda_2} \Delta^2 S_{\mathcal{C}} + \frac{r_2 + 1}{\lambda_1 \lambda_2} \Delta^2 S_{\mathcal{A}}, \\ \Delta^2 S_{c_2^r} &= \frac{r_1}{\lambda_1 \lambda_2} \Delta^2 S_{\mathcal{B}} + \frac{r_1 + 1}{\lambda_1 \lambda_2} \Delta^2 S_{\mathcal{A}}. \end{aligned} \quad (2.23)$$

Then

$$\begin{aligned}\Delta^2 S_{b_1^c} &= \Delta^2 S_{b_2^c} \cdot \frac{(r_2 + 1)(\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)}, \\ \Delta^2 S_{c_1^c} &= \Delta^2 S_{c_2^c} \cdot \frac{(r_1 + 1)(\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)}.\end{aligned}\quad (2.24)$$

Therefore, we get the third incremental areas of the components where $g(\mathbf{x}) \in P$ in $\mathcal{A}$ and $\mathcal{C}$, respectively:

$$\begin{aligned}\Delta^3 S_{\mathcal{A}} &= \frac{r_2 \Delta^2 S_{c_1^c} + r_1 \Delta^2 S_{b_1^c}}{1 - q} \\ &= \frac{r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1) \left[(r_2 + 1) \Delta^1 S_{c_2^c} + (r_1 + 1) \Delta^1 S_{b_2^c} \right]}{(1 - q) [\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)] [\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)]} \\ &\quad + \frac{r_2 \Delta^1 S_{c_1^c} + r_1 \Delta^1 S_{b_1^c}}{\lambda_1 \lambda_2 (1 - q)^2} \left[\frac{r_1 (r_2 + 1)^2 (\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)} + \frac{r_2 (r_1 + 1)^2 (\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)} \right], \\ \Delta^3 S_{\mathcal{C}} &= (\lambda_2 - r_2) \Delta^2 S_{c_1^c} + (\lambda_2 - r_2 - 1) \Delta^2 S_{c_2^c} \\ &= \left[\frac{r_1}{\lambda_1 \lambda_2} \cdot \Delta^1 S_{b_2^c} \cdot \frac{\lambda_1 \lambda_2 (\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)} + \frac{r_1 + 1}{\lambda_1 \lambda_2} \cdot \frac{r_2 \Delta^1 S_{c_1^c} + r_1 \Delta^1 S_{b_1^c}}{1 - q} \right] \\ &\quad \times \frac{\lambda_1 \lambda_2 (\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)}.\end{aligned}\quad (2.25)$$

Furthermore, by calculating, we can obtain that the fourth incremental area of the components where $g(\mathbf{x}) \in P$ in $\mathcal{A}$ is

$$\begin{aligned}\Delta^4 S_{\mathcal{A}} &= \frac{r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1) \left(r_2 \Delta^1 S_{c_1^c} + r_1 \Delta^1 S_{b_1^c} \right)}{(1 - q) [\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)] [\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)]} \\ &\quad + \frac{r_2 \Delta^1 S_{c_1^c} + r_1 \Delta^1 S_{b_1^c}}{\lambda_1 \lambda_2 (1 - q)^2} \cdot \frac{2 r_1 r_2 (r_1 + 1)(r_2 + 1)(\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1)}{[\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)] [\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)]} \\ &\quad + \frac{r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1) \left[(r_2 + 1) \Delta^1 S_{c_2^c} + (r_1 + 1) \Delta^1 S_{b_2^c} \right]}{\lambda_1 \lambda_2 (1 - q)^2 [\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)] [\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)]} \\ &\quad \times \left[\frac{r_1 (r_2 + 1)^2 (\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)} + \frac{r_2 (r_1 + 1)^2 (\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)} \right] \\ &\quad + \frac{r_2 \Delta^1 S_{c_1^c} + r_1 \Delta^1 S_{b_1^c}}{(\lambda_1 \lambda_2)^2 (1 - q)^3} \cdot \left[\frac{r_1 (r_2 + 1)^2 (\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)} + \frac{r_2 (r_1 + 1)^2 (\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)} \right]^2.\end{aligned}\quad (2.26)$$

Step by step, applying mathematical induction, it can be shown that

$$\begin{aligned}\Delta^n S_{\mathcal{A}} &= \frac{r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1)}{[\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)][\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)]} \left[1 + \frac{2(r_1 + 1)(r_2 + 1)}{\lambda_1 \lambda_2 (1 - q)} \right] \Delta^{n-2} S_{\mathcal{A}} \\ &\quad + \frac{1}{\lambda_1 \lambda_2 (1 - q)} \left[\frac{r_1 (r_2 + 1)^2 (\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)} + \frac{r_2 (r_1 + 1)^2 (\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)} \right] \Delta^{n-1} S_{\mathcal{A}} \quad (n \geq 4).\end{aligned}\tag{2.27}$$

Since we always use the form of filling in the blank to calculate the incremental areas, there is no repeated area in $\Delta^1 S_{\mathcal{A}}, \Delta^2 S_{\mathcal{A}}, \dots, \Delta^n S_{\mathcal{A}}$. Then $\lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta^k S_{\mathcal{A}} \leq \mathbb{S}_{\mathcal{A}}$. So we let $S = \lim_{n \rightarrow \infty} \sum_{k=2}^n \Delta^k S_{\mathcal{A}}$. According to the above recursive formula, it follows that

$$\begin{aligned}S - (\Delta^2 S_{\mathcal{A}} + \Delta^3 S_{\mathcal{A}}) &= \frac{r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1)}{[\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)][\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)]} \\ &\quad \times \left[1 + \frac{2(r_1 + 1)(r_2 + 1)}{\lambda_1 \lambda_2 (1 - q)} \right] S + \frac{1}{\lambda_1 \lambda_2 (1 - q)} \\ &\quad \times \left[\frac{r_1 (r_2 + 1)^2 (\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)} + \frac{r_2 (r_1 + 1)^2 (\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)} \right] (S - \Delta^2 S_{\mathcal{A}}).\end{aligned}\tag{2.28}$$

Substituting the values of q , $\Delta^2 S_{\mathcal{A}}$, and $\Delta^3 S_{\mathcal{A}}$ into the above formula and simplifying, we get

$$\begin{aligned}&\left\{ 1 - \frac{\lambda_1 \lambda_2 + (r_1 + 1)(r_2 + 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(r_2 + 1)} \cdot \frac{r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1)}{[\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)][\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)]} \right. \\ &\quad \left. - \frac{1}{\lambda_1 \lambda_2 - (r_1 + 1)(r_2 + 1)} \cdot \left[\frac{r_1 (r_2 + 1)^2 (\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)} + \frac{r_2 (r_1 + 1)^2 (\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)} \right] \right\} S \\ &= \frac{\lambda_1 \lambda_2}{\lambda_1 \lambda_2 - (r_1 + 1)(r_2 + 1)} \\ &\quad \times \left\{ r_2 \Delta^1 S_{c_1^c} + r_1 \Delta^1 S_{b_1^c} + \frac{r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1) \left[(r_2 + 1) \Delta^1 S_{c_2^c} + (r_1 + 1) \Delta^1 S_{b_2^c} \right]}{[\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)][\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)]} \right\}.\end{aligned}\tag{2.29}$$

Again substituting the values of $\Delta^1 S_{c_1^c}$, $\Delta^1 S_{b_1^c}$, $\Delta^1 S_{c_2^c}$, and $\Delta^1 S_{b_2^c}$ into the above formula and simplifying, we conclude that

$$S = \frac{r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1)}{(\lambda_1 - 1)(\lambda_2 - 1)} \cdot \frac{m_1}{m_2},\tag{2.30}$$

where

$$\begin{aligned}
m_1 &= r_1(r_2 + 1)(\lambda_2 - r_2)(\lambda_1 - r_1 - 1) + r_2(r_1 + 1)(\lambda_1 - r_1)(\lambda_2 - r_2 - 1) \\
&\quad + r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1) \left[\frac{(r_1 + 1)(\lambda_2 - r_2)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)} + \frac{(r_2 + 1)(\lambda_1 - r_1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)} \right] \\
&\quad + \frac{1}{\lambda_1 \lambda_2 - (r_1 + 1)(r_2 + 1)} \{ \lambda_1 \lambda_2 [2\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2) - (r_2 + 1)(\lambda_1 - r_1)] \\
&\quad \quad - [\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)][\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)] \} \\
&\quad \times \left\{ \frac{r_1(r_2 + 1)^2(\lambda_1 - r_1 - 1)}{\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)} + \frac{r_2(r_1 + 1)^2(\lambda_2 - r_2 - 1)}{\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)} \right. \\
&\quad \quad \left. + \frac{2r_1 r_2 (r_1 + 1)(r_2 + 1)(\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1)}{[\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)][\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)]} \right\}, \\
m_2 &= [\lambda_1 \lambda_2 - (r_1 + 1)(r_2 + 1)][\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)][\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)] \\
&\quad - r_1 r_2 (\lambda_1 - r_1 - 1)(\lambda_2 - r_2 - 1)[\lambda_1 \lambda_2 + (r_1 + 1)(r_2 + 1)] \\
&\quad - r_1(r_2 + 1)^2(\lambda_1 - r_1 - 1)[\lambda_1 \lambda_2 - (r_1 + 1)(\lambda_2 - r_2)] \\
&\quad - r_2(r_1 + 1)^2(\lambda_2 - r_2 - 1)[\lambda_1 \lambda_2 - (r_2 + 1)(\lambda_1 - r_1)].
\end{aligned} \tag{2.31}$$

Finally, we obtain all the incremental areas of the components where $g(\mathbf{x}) \in P$ in $\mathcal{A}$:

$$\begin{aligned}
\Theta S_{\mathcal{A}} &= \lim_{n \rightarrow \infty} (\Delta^1 S_{\mathcal{A}} + \Delta^2 S_{\mathcal{A}} + \cdots + \Delta^n S_{\mathcal{A}}) \\
&= \Delta^1 S_{\mathcal{A}} + S \\
&= \frac{r_2 S_{c_1^r} + r_1 S_{b_1^c} - S_{a_{11}}}{1 - q} + S.
\end{aligned} \tag{2.32}$$

Substituting the values of q , $S_{c_1^r}$, $S_{b_1^c}$, $S_{a_{11}}$, and S into the above formula, it follows that $\Theta S_{\mathcal{A}} = (r_1/(\lambda_1 - 1)) \cdot (r_2/(\lambda_2 - 1)) = \mathbb{S}_{\mathcal{A}}$. This implies that all the blanks in $\mathcal{A}$ can be filled by the above iteration method. And there is no repeated area in $\Delta^1 S_{\mathcal{A}}, \Delta^2 S_{\mathcal{A}}, \dots, \Delta^n S_{\mathcal{A}}$.

Hence, $g(\mathbf{x}) \in P$ in $\mathcal{A}$, and from the above steps, it can be shown that $g(\mathbf{x}) \in P$ in $\mathcal{B}$ and $\mathcal{C}$, respectively. It is obvious that $\mathcal{A}'_0$, $\mathcal{B}'_0$, and $\mathcal{C}'_0$ can be obtained by j_1 units of right translation and j_2 units of up translation of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively, where $j_1 = 0, \dots, N_1 - 1$, $j_2 = 0, \dots, N_2 - 1$. Then applying the refinement equation (2.2), $g(\mathbf{x}) \in P$ in $\mathcal{A}'_0$, $\mathcal{B}'_0$, and $\mathcal{C}'_0$, respectively. Similarly, $g(\mathbf{x}) \in P$ in $\mathcal{E}'_0$, $\mathcal{F}'_0$, $\mathcal{G}'_0$, $\mathcal{H}'_0$, and $\mathcal{J}'_0$, respectively. Therefore, $g(\mathbf{x})$ is a componentwise polynomial function supported on $(0, N_1 + r_1/(\lambda_1 - 1)) \times (0, N_2 + r_2/(\lambda_2 - 1))$. $\square$

To make it easier to understand, the above algorithm is briefly summarized as follows.

Step 1. Applying the refinement equation (2.2), we get the areas of the components where $g \in P$ in b_1^c and c_1^r : $S_{b_1^c}$ and $S_{c_1^r}$.

Step 2. From $S_{b_1^c}$ and $S_{c_1^r}$, we can get the first incremental areas of the components where $g \in P$ in $\mathcal{A}$ and $\mathcal{C}$: $\Delta^1 S_{\mathcal{A}}$ and $\Delta^1 S_{\mathcal{C}}$.

Step 3. From $\Delta^1 S_{\mathcal{A}}$ and $\Delta^1 S_{\mathcal{C}}$, we can get the first incremental area of the components where $g \in P$ in b_2^c : $\Delta^1 S_{b_2^c}$. Then again applying (2.2), we obtain the first incremental areas of the components where $g \in P$ in b_1^c and c_1^r : $\Delta^1 S_{b_1^c}$ and $\Delta^1 S_{c_1^r}$. Repeating Step 2, we get the second incremental areas of the components where $g \in P$ in $\mathcal{A}$ and $\mathcal{C}$: $\Delta^2 S_{\mathcal{A}}$ and $\Delta^2 S_{\mathcal{C}}$.

Step 4. Repeating Step 3, we get the third, the fourth, ..., the n th incremental areas of the components where $g \in P$ in $\mathcal{A}$: $\Delta^3 S_{\mathcal{A}}, \Delta^4 S_{\mathcal{A}}, \dots, \Delta^n S_{\mathcal{A}}$, and there is no repeated area in $\Delta^1 S_{\mathcal{A}}, \Delta^2 S_{\mathcal{A}}, \dots, \Delta^n S_{\mathcal{A}}$. Finally, we can obtain all the incremental areas of the components where $g \in P$ in $\mathcal{A}$ and prove that the total area of the components is equal to the area of the region Λ .

Applying Lemma 1 in [16] and Theorem 2.1, we conclude the following.

Corollary 2.3. *Let $f(\mathbf{x}) \in L_1(\mathbb{R}^2)$ be a compactly supported function satisfying the refinement equation (2.1). If there exists a matrix P such that $P^{-1}AP = B$, and $f(\mathbf{x}) := g(P^{-1}\mathbf{x})$, where B and $g(\mathbf{x})$ are defined as Theorem 2.1. Then $f(\mathbf{x})$ is a compactly supported refinable componentwise polynomial function.*

3. Examples

Example 3.1. Let $f(\mathbf{x}) \in L_1(\mathbb{R}^2)$ be the refinable function with dilation matrix $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, and its mask symbol is given by

$$H(\xi_1, \xi_2) = \left[\frac{1 - e^{-3i\xi_1}}{3(1 - e^{-i\xi_1})} \right]^2 \left[\frac{1 - e^{-4i\xi_2}}{4(1 - e^{-i\xi_2})} \right] \times \frac{1}{6} \left(1 + e^{-i\xi_1} + e^{-i\xi_2} + e^{-i\xi_1} e^{-i\xi_2} + e^{-2i\xi_2} + e^{-i\xi_1} e^{-2i\xi_2} \right). \quad (3.1)$$

Applying Lemma 2.2, it is easy to check that $f(\mathbf{x}) \in P$ in $\bigcup_{j_1=0}^1 (j_1 + (1/2, 1)) \times (2/3, 1)$. Then, according to Theorem 2.1, $f(\mathbf{x})$ is a componentwise polynomial function supported on $(0, 5/2) \times (0, 5/3)$. The refinable function $f(\mathbf{x})$ is given in Figure 1.

Example 3.2. Let $f(\mathbf{x}) \in L_1(\mathbb{R}^2)$ be the refinable function with dilation matrix $3I_2$, where I_2 is the identity matrix of order 2. And its mask symbol is given by

$$H(\xi_1, \xi_2) = \left[\frac{1 - e^{-3i\xi_1}}{3(1 - e^{-i\xi_1})} \right]^2 \left[\frac{1 - e^{-3i\xi_2}}{3(1 - e^{-i\xi_2})} \right]^2 \left(\frac{1}{4} + \frac{1}{4} e^{-i\xi_1} + \frac{1}{4} e^{-i\xi_2} + \frac{1}{4} e^{-i\xi_1} e^{-i\xi_2} \right). \quad (3.2)$$

Figure 1: The symmetric refinable componentwise polynomial function f .

Figure 2: The symmetric refinable componentwise polynomial function f .

Applying Lemma 2.2, it is easy to check that $f(\mathbf{x}) \in P$ in $\bigcup_{j_1=0}^1(j_1+(1/2,1)) \times \bigcup_{j_2=0}^1(j_2+(1/2,1))$. Then, according to Theorem 2.1, $f(\mathbf{x})$ is a componentwise polynomial function supported on $(0,5/2) \times (0,5/2)$. The refinable function $f(\mathbf{x})$ is given in Figure 2.

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