Research Article

Existence of Homoclinic Orbits for a Class of Asymptotically p-Linear Difference Systems with p-Laplacian

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By applying a variant version of Mountain Pass Theorem in critical point theory, we prove the existence of homoclinic solutions for the following asymptotically p-linear difference system with p-Laplacian

\[ \Delta \left( |\Delta u(n-1)|^{p-2} \Delta u(n-1) \right) + \nabla [-K(n, u(n)) + W(n, u(n))] = 0, \quad n \in \mathbb{Z}, \]

where \( \Delta \) is the forward difference operator defined by \( \Delta u(n) = u(n + 1) - u(n) \), \( \Delta^2 u(n) = \Delta(\Delta u(n)) \), \( p \in (1, +\infty) \), \( n \in \mathbb{Z} \), \( u \in \mathbb{R}^N \), \( K : \mathbb{Z} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is not periodic in \( n \), and \( W \) is asymptotically p-linear at infinity.

1. Introduction

Consider the following p-Laplacian difference system:

\[ \Delta \left( |\Delta u(n-1)|^{p-2} \Delta u(n-1) \right) + \nabla [-K(n, u(n)) + W(n, u(n))] = 0, \quad n \in \mathbb{Z}, \quad (1.1) \]

where \( \Delta \) is the forward difference operator defined by \( \Delta u(n) = u(n + 1) - u(n) \), \( \Delta^2 u(n) = \Delta(\Delta u(n)) \), \( p \in (1, +\infty) \), \( n \in \mathbb{Z} \), \( u \in \mathbb{R}^N \), \( K : \mathbb{Z} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is not periodic in \( n \), \( W \) is asymptotically p-linear at infinity, and \( K \) and \( W \) are continuously differentiable in \( x \). As usual, we say that a solution \( u(n) \) of (1.1) is homoclinic (to 0) if \( u(n) \rightarrow 0 \) as \( n \rightarrow \pm \infty \). In addition, if \( u(n) \equiv 0 \), then \( u(n) \) is called a nontrivial homoclinic solution.

When \( p = 2 \), (1.1) can be regarded as a discrete analogue of the following second-order Hamiltonian system:

\[ \ddot{u}(t) + \nabla [-K(t, u(t)) + W(t, u(t))] = 0, \quad t \in \mathbb{R}. \quad (1.2) \]
The existence of homoclinic orbits for Hamiltonian systems is a classical problem and its importance in the study of the behavior of dynamical systems has been recognized by Poincaré [1]. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation and its perturbed system probably produces chaotic phenomenon. For the existence of homoclinic solutions of problem (1.2), one can refer to the papers [2–5].

Difference equations usually describe evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the \((n + 1)\)th generation \(x(n + 1)\) is a function of the \(n\)th generation \(x(n)\). In fact, difference equations provide a natural description of many discrete models in real world. Since discrete models exist in various fields of science and technology such as statistics, computer science, electrical circuit analysis, biology, neural network, and optimal control, it is of practical importance to investigate the solutions of difference equations. For more details about difference equations, we refer the readers to the books [6–8].

In some recent papers [9–20], the authors studied the existence of periodic solutions and subharmonic solutions of difference equations by applying critical point theory. These papers show that the critical point theory is an effective method to the study of periodic solutions for difference equations. Along this direction, several authors [21–28] used critical point theory to study the existence of homoclinic orbits for difference equations. Motivated by the above papers, we consider the existence of homoclinic orbits for problem (1.1) by using the variant version of Mountain Pass Theorem. Our result is new, which seems not to have been considered in the literature. Here is our main result.

**Theorem 1.1.** Suppose that \(K\) and \(W\) satisfy the following conditions.

(K1) There are two positive constants \(b_1\) and \(b_2\) such that

\[
b_1|x|^p \leq K(n, x) \leq b_2|x|^p, \quad \forall (n, x) \in \mathbb{Z} \times \mathbb{R}^N.
\]

(K2) There is a positive constant \(b_3\) such that

\[
b_3|x|^p \leq (\nabla K(n, x), x) \leq |\nabla K(n, x)||x| \leq pK(n, x), \quad \forall (n, x) \in \mathbb{Z} \times \mathbb{R}^N.
\]

(W1) \(W(n, 0) = 0, \nabla W(n, x) = o(|x|^{p-1})\) as \(|x| \to 0\) uniformly for \(n \in \mathbb{Z}\).

(W2) There exists a constant \(R > 0\) such that

\[
\frac{|\nabla W(n, x)|}{|x|^{p-1}} \leq R, \quad \forall n \in \mathbb{Z}, x \in \mathbb{R}^N.
\]

(W3) There exists a function \(V_\infty \in l^\infty(\mathbb{Z}, \mathbb{R}^+)\) such that

\[
\lim_{|x| \to \infty} \frac{|\nabla W(n, x) - V_\infty(n)| |x|^{p-2}x}{|x|^{p-1}} = 0 \quad \text{uniformly for } n \in \mathbb{Z},
\]

\[
\inf_{\mathbb{Z}} V_\infty(n) > \max\{1, pb_2\}.
\]
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(W4) \( \tilde{W}(n, x) = (\nabla W(n, x), x) - pW(n, x) \),

\[
\lim_{|x| \to \infty} \tilde{W}(n, x) = +\infty \quad \text{uniformly for } n \in \mathbb{Z},
\]

(1.7)

and for any fixed \( 0 < c_1 < c_2 < +\infty \),

\[
\inf_{n \in \mathbb{Z}, c_1 \leq |x| \leq c_2} \frac{\tilde{W}(n, x)}{|x|^p} > 0.
\]

(1.8)

Then problem (1.1) has at least one nontrivial homoclinic solution.

Remark 1.2. The function \( W(n, x) \) in this paper is asymptotically \( p \)-linear at infinity. The behavior of the gradient of \( W(n, x) \) at infinity is like that of a function \( V_\infty|n|^{p-2}x \), where \( V_\infty(n) \) is a real function but not a matrix function. To the best of our knowledge, similar results of this kind of \( p \)-Laplacian difference systems with asymptotically \( p \)-linear \( W(n, x) \) at infinity cannot be found in the literature. From this point, our result is new.

2. Preliminaries

Let

\[
S = \left\{ \{u(n)\}_{n \in \mathbb{Z}} : u(n) \in \mathbb{R}^N, n \in \mathbb{Z} \right\},
\]

\[
E = \left\{ u \in S : \sum_{n \in \mathbb{Z}} [||\Delta u(n-1)||^p + |u(n)|^p] < +\infty \right\},
\]

(2.1)

and for \( u \in E \), let

\[
\|u\| = \left\{ \sum_{n \in \mathbb{Z}} [||\Delta u(n-1)||^p + |u(n)|^p] < +\infty \right\}^{1/p}.
\]

(2.2)

Then \( E \) is a uniform convex Banach space with this norm. As usual, for \( 1 \leq p < +\infty \), let

\[
L^p(\mathbb{Z}, \mathbb{R}^N) = \left\{ u \in S : \sum_{n \in \mathbb{Z}} |u(n)|^p < +\infty \right\}, \quad L^\infty(\mathbb{Z}, \mathbb{R}^N) = \left\{ u \in S : \sup_{n \in \mathbb{Z}} |u(n)| < +\infty \right\},
\]

(2.3)

and their norms are given by

\[
\|u\|_p = \left( \sum_{n \in \mathbb{Z}} |u(n)|^p \right)^{1/p}, \quad \forall u \in L^p(\mathbb{Z}, \mathbb{R}^N),
\]

\[
\|u\|_\infty = \sup\{|u(n)| : n \in \mathbb{Z}\}, \quad \forall u \in L^\infty(\mathbb{Z}, \mathbb{R}^N),
\]

(2.4)

respectively.
For any $u \in E$, let
\[ \varphi(u) = \frac{1}{p} \sum_{n \in \mathbb{Z}} |\Delta u(n-1)|^p - \sum_{n \in \mathbb{Z}} [-K(n,u(n)) + W(n,u(n))]. \tag{2.5} \]

To prove our results, we need the following generalization of Lebesgue’s dominated convergence theorem.

**Lemma 2.1** (see [29]). Let $\{f_k(t)\}$ and $\{g_k(t)\}$ be two sequences of measurable functions on a measurable set $A$, and let
\[ |f_k(t)| \leq g_k(t), \quad \forall \text{a.e. } t \in A. \tag{2.6} \]

If
\[ \lim_{k \to \infty} f_k(t) = f(t), \quad \lim_{k \to \infty} g_k(t) = g(t), \quad \forall \text{a.e. } t \in A, \tag{2.7} \]
then
\[ \lim_{k \to \infty} \int_A g_k(t) dt = \int_A g(t) dt < +\infty, \]

\[ \lim_{k \to \infty} \int_A f_k(t) dt = \int_A f(t) dt. \tag{2.8} \]

**Lemma 2.2.** For $u \in E$,
\[ \|u\|_{\infty} \leq \|u\|_p \leq 2\|u\|. \tag{2.9} \]

**Proof.** Since $u \in E$, it follows that $\lim_{|n| \to \infty} |u(n)| = 0$. Hence, there exists $n^* \in \mathbb{Z}$ such that
\[ \|u\|_{\infty} = |u(n^*)| = \max_{n \in \mathbb{Z}} |u(n)|. \tag{2.10} \]

Hence, we have
\[ \|u\|_{\infty} \leq \left( \sum_{n \in \mathbb{Z}} |u(n)|^p \right)^{1/p} = \|u\|_p = \left( \sum_{n \in \mathbb{Z}} |u(n) - u(n-1) + u(n-1)|^p \right)^{1/p} \]
\[ \leq \left( \sum_{n \in \mathbb{Z}} (|u(n) - u(n-1)| + |u(n-1)|)^p \right)^{1/p} \]
\[ \leq \left( 2^p \sum_{n \in \mathbb{Z}} (|u(n) - u(n-1)|^p + |u(n-1)|^p) \right)^{1/p}. \]
Suppose that (K1), (K2), and (W2) hold. If \( u_k \to u \) in \( E \), then \( \nabla K(n, u_k) \to \nabla K(n, u) \) and \( \nabla W(n, u_k) \to \nabla W(n, u) \) in \( l^p'(\mathbb{R}, \mathbb{R}^N) \), where \( p' > 1 \) satisfies \( 1/p + 1/p' = 1 \).

**Proof.** From (K1) and (K2), we have

\[
|\nabla K(n, x)| \leq pb_2|x|^{p-1}, \quad \forall (n, x) \in \mathbb{Z} \times \mathbb{R}^N.
\]

Hence, from (2.12), we have

\[
|\nabla K(n, u_k(n)) - \nabla K(n, u(n))|^{p'} \leq \left[ pb_2 \left( |u_k(n)|^{p-1} + |u(n)|^{p-1} \right) \right]^{p'}
\]

\[
\leq \left[ pb_2 2^{p-1} |u_k(n) - u(n)|^{p-1} + pb_2 \left( 1 + 2^{p-1} \right) |u(n)|^{p-1} \right]^{p'}
\]

\[
\leq 2^{p'} \left( pb_2 \right)^{p'} |u_k(n) - u(n)|^{p} + 2^{p'} \left( pb_2 \right)^{p'} \left( 1 + 2^{p-1} \right) |u(n)|^{p'}
\]

\[
:= g_k(n).
\]  

Moreover, since \( u_k \to u \) in \( l^p(\mathbb{Z}, \mathbb{R}^N) \) and \( u_k(n) \to u(n) \) for almost every \( n \in \mathbb{Z} \), hence,

\[
\lim_{k \to \infty} g_k(n) = 2^{p'} \left( pb_2 \right)^{p'} \left( 1 + 2^{p-1} \right)^{p'} |u(n)|^{p'} := g(n), \quad \forall a.e. \ n \in \mathbb{Z},
\]

\[
\lim_{k \to \infty} \sum_{n \in \mathbb{Z}} g_k(n) = \lim_{k \to \infty} \sum_{n \in \mathbb{Z}} \left[ 2^{p'} \left( pb_2 \right)^{p'} |u_k(n) - u(n)|^{p} + 2^{p'} \left( pb_2 \right)^{p'} \left( 1 + 2^{p-1} \right) |u(n)|^{p'} \right]
\]

\[
= 2^{p'} \left( pb_2 \right)^{p'} \lim_{k \to \infty} \sum_{n \in \mathbb{Z}} |u_k(n) - u(n)|^{p} + 2^{p'} \left( pb_2 \right)^{p'} \left( 1 + 2^{p-1} \right) \sum_{n \in \mathbb{Z}} |u(n)|^{p'}
\]

\[
= 2^{p'} \left( pb_2 \right)^{p'} \left( 1 + 2^{p-1} \right)^p \sum_{n \in \mathbb{Z}} |u(n)|^{p'}
\]

\[
= \sum_{n \in \mathbb{Z}} g(n) < +\infty.
\]
It follows from Lemma 2.1, (2.13), and the previous equations that

$$
\lim_{k \to \infty} \sum_{n \in \mathbb{Z}} |\nabla K(n, u_k(n)) - \nabla K(n, u(n))|^p = 0. 
$$

(2.15)

This shows that $\nabla K(n, u_k) \to \nabla K(n, u)$ in $l^p(\mathbb{Z}, \mathbb{R}^N)$. By a similar proof, we can prove that $\nabla W(n, u_k) \to \nabla W(n, u)$ in $l^p(\mathbb{Z}, \mathbb{R}^N)$. The proof is complete.

**Lemma 2.4.** Under the conditions of Theorem 1.1, one has

$$
\langle \varphi'(u), v \rangle = \sum_{n \in \mathbb{Z}} \left[ |\Delta u(n-1)|^{p-2}(\Delta u(n-1), \Delta v(n-1)) + (\nabla K(n, u(n)) - \nabla W(n, u(n)), v(n)) \right]
$$

for $u, v \in E$, which yields that

$$
\langle \varphi'(u), u \rangle = \sum_{n \in \mathbb{Z}} \left[ |\Delta u(n-1)|^p + (\nabla K(n, u(n)), u(n)) - (\nabla W(n, u(n)), u(n)) \right].
$$

(2.17)

Moreover, $\varphi$ is continuously Fréchet-differential defined on $E$; that is, $\varphi \in C^1(E, \mathbb{R})$ and any critical point $u$ of $\varphi$ on $E$ is classical solution of (1.1) with $u(\pm \infty) = 0$.

**Proof.** Firstly, we show that $\varphi : E \to \mathbb{R}$. Let $u \in E$, by (2.9) and (K1), we have

$$
\sum_{n \in \mathbb{Z}} K(n, u(n)) \leq \sum_{n \in \mathbb{Z}} b_2 |u(n)|^p \leq b_2 2^p \|u\|^p.
$$

(2.18)

By (W2), we get

$$
|W(n, x)| = \left| \int_0^{1} (\nabla W(n, sx), x) ds \right| \leq R|x|^p, \quad \forall (n, x) \in \mathbb{Z} \times \mathbb{R}^N.
$$

(2.19)

Hence, from (2.9) and (2.19), we have

$$
\left| \sum_{n \in \mathbb{Z}} W(n, u(n)) \right| \leq \sum_{n \in \mathbb{Z}} |W(n, u(n))| \leq \sum_{n \in \mathbb{Z}} R|u(n)|^p \leq R2^p \|u\|^p.
$$

(2.20)

It follows from (2.5), (2.18), and (2.20) that $\varphi : E \to \mathbb{R}$. Next we prove that $\varphi \in C^1(E, \mathbb{R})$. Rewrite $\varphi$ as follows:

$$
\varphi(u) = \varphi_1(u) + \varphi_2(u) - \varphi_3(u),
$$

(2.21)
where

\[
\varphi_1(u) := \frac{1}{p} \sum_{n \in \mathbb{Z}} |\Delta u(n-1)|^p, \quad \varphi_2(u) := \sum_{n \in \mathbb{Z}} K(n, u(n)), \quad \varphi_3(u) := \sum_{n \in \mathbb{Z}} W(n, u(n)).
\] (2.22)

It is easy to check that \( \varphi_1 \in C^1(E, \mathbb{R}) \) and

\[
\langle \varphi'_1(u), v \rangle = \sum_{n \in \mathbb{Z}} |\Delta u(n-1)|^{p-2} (\Delta u(n-1), \Delta v(n-1)), \quad \forall u, v \in E.
\] (2.23)

Next, we prove that \( \varphi_i \in C^1(E, \mathbb{R}) \), \( i = 2, 3 \), and

\[
\langle \varphi'_2(u), v \rangle = \sum_{n \in \mathbb{Z}} (\nabla K(n, u(n)), v(n)), \quad \forall u, v \in E,
\] (2.24)

\[
\langle \varphi'_3(u), v \rangle = \sum_{n \in \mathbb{Z}} (\nabla W(n, u(n)), v(n)), \quad \forall u, v \in E.
\] (2.25)

For any \( u, v \in E \) and for any function \( \theta : \mathbb{R} \to (0, 1) \), by (K2), we have

\[
\sum_{n \in \mathbb{Z}} \max_{h \in [0, 1]} |(\nabla K(n, u(n) + \theta(t)h\nu(n)), v(n))| \leq pb_2 \sum_{n \in \mathbb{Z}} \max_{h \in [0, 1]} |u(n) + \theta(t)h\nu(n)|^{p-1} |v(n)|
\]

\[
\leq 2^{p-1} pb_2 \sum_{n \in \mathbb{Z}} \left( |u(n)|^{p-1} + |v(n)|^{p-1} \right) |v(n)|
\]

\[
\leq 2^{p-1} pb_2 \left[ \|u\|_{p}^{p-1} \|v\|_{p} + \|v\|_{p}^{p} \right]
< +\infty.
\] (2.26)

Then by the previous equations and Lebesgue’s dominated convergence theorem, we have

\[
\langle \varphi'_2(u), v \rangle = \lim_{h \to 0^+} \frac{\varphi_2(u + h\nu) - \varphi_2(u)}{h}
= \lim_{h \to 0^+} \frac{1}{h} \sum_{n \in \mathbb{Z}} [K(n, u(n) + h\nu(n)) - K(n, u(n))]
\]

\[
= \lim_{h \to 0^+} \sum_{n \in \mathbb{Z}} (\nabla K(t, u(n) + \theta(t)h\nu(n)), v(n))
\]

\[
= \sum_{n \in \mathbb{Z}} (\nabla K(n, u(n)), v(n)), \quad \forall u, v \in E.
\] (2.27)
Step 1. From (2.25) holds by using (W2) instead of (K2). Finally, we prove that \( \varphi_i \in C^1(E, \mathbb{R}) \), \( i = 2, 3 \). Let \( u_k \to u \) in \( E \); then by Lemma 2.3, we have

\[
\left| \langle \varphi_2'(u_k) - \varphi_2'(u), v \rangle \right| = \left| \sum_{n \in \mathbb{Z}} (\nabla K(n, u_k(n)) - \nabla K(n, u(n)), v(n)) \right| \leq \sum_{n \in \mathbb{Z}} |\nabla K(n, u_k(n)) - \nabla K(n, u(n))||v(n)| \leq ||v|| \left[ \sum_{n \in \mathbb{Z}} |\nabla K(n, u_k(n)) - \nabla K(n, u(n))|^{p'} \right]^{1/p'} \to 0, \quad k \to \infty, \quad \forall v \in E.
\]

This shows that \( \varphi_2 \in C^1(E, \mathbb{R}) \). Similarly, we can prove that \( \varphi_3 \in C^1(E, \mathbb{R}) \). Furthermore, by a standard argument, it is easy to show that the critical points of \( \varphi \) in \( E \) are classical solutions of (1.1) with \( u(\pm \infty) = 0 \). The proof is complete. \( \square \)

**Lemma 2.5** (see [30]). Let \( E \) be a real Banach space with its dual space \( E^* \) and suppose that \( \varphi \in C^1(E, \mathbb{R}) \) satisfies

\[
\max\{\varphi(0), \varphi(e)\} \leq \eta_0 < \eta \leq \inf_{\|u\|_p = 1} \varphi(u),
\]

for some \( \eta_0 < \eta, \rho > 0 \), and \( e \in E \) with \( \|e\| > \rho \). Let \( c \geq \eta \) be characterized by

\[
c = \inf \max \varphi(Y(\tau)),
\]

where \( \Gamma = \{ Y \in C([0, 1], E) \colon Y(0) = 0, Y(1) = e \} \) is the set of continuous paths joining 0 to \( e \); then there exists \( \{u_k\}_{k \in \mathbb{N}} \subset E \) such that

\[
\varphi(u_k) \to c, \quad (1 + \|u_k\|)\|\varphi'(u_k)\|_{E^*} \to 0 \quad \text{as} \quad k \to \infty.
\]

**3. Proof of Theorem 1.1**

**Proof of Theorem 1.1.** We divide the proof of Theorem 1.1 into three steps.

**Step 1.** From (W1), there exists \( \rho_0 > 0 \) such that

\[
\nabla W(n, x) \leq \frac{C_1}{2^p} |x|^{p-1}, \quad \forall n \in \mathbb{Z}, \; |x| \leq \rho_0.
\]
where \( C_1 = \min\{1/p, b_1\} \). From (3.1), we have

\[
W(n, x) = \int_0^1 (\nabla W(n, sx), x) ds \\
\leq \int_0^1 \frac{C_1}{2^p} |x|^p s^{p-1} ds = \frac{C_1}{p2^p} |x|^p, \quad \forall n \in \mathbb{Z}, \ |x| \leq \rho_0.
\]

Let \( \rho = \rho_0/2 \) and \( S = \{ u \in E \mid \| u \| = \rho \} \); then from (2.9), we obtain

\[
\| u \|_\infty \leq \rho_0, \quad \| u \|_p \leq 2\rho, \quad \forall u \in S,
\]

which together with (2.9), (3.2), and (K1) implies that

\[
\varphi(u) = \frac{1}{p} \sum_{n \in \mathbb{Z}} |\Delta u(n-1)|^p - \sum_{n \in \mathbb{Z}} [-K(n, u(n)) + W(n, u(n))] \\
\geq \frac{1}{p} \sum_{n \in \mathbb{Z}} |\Delta u(n-1)|^p + b_1 \sum_{n \in \mathbb{Z}} |u(n)|^p - \sum_{n \in \mathbb{Z}} \frac{C_1}{p2^p} |u(n)|^p \\
\geq \min\left\{ \frac{1}{p}, b_1 \right\} \| u \|_p^p - \frac{C_1}{p2^p} \| u \|_p^p \\
\geq C_1 \| u \|_p^p - \frac{C_1}{p} \| u \|_p^p = \frac{(p-1)C_1}{p} \| u \|_p^p = \alpha_1 > 0, \quad u \in S.
\]

**Step 2.** From (K1), we have

\[
\varphi(u) = \frac{1}{p} \sum_{n \in \mathbb{Z}} |\Delta u(n-1)|^p - \sum_{n \in \mathbb{Z}} [-K(n, u(n)) + W(n, u(n))] \\
\leq \frac{1}{p} \sum_{n \in \mathbb{Z}} |\Delta u(n-1)|^p + b_2 \sum_{n \in \mathbb{Z}} |u(n)|^p - \sum_{n \in \mathbb{Z}} W(n, u(n)) \\
\leq \max\left\{ \frac{1}{p}, b_2 \right\} \| u \|_p^p - \sum_{n \in \mathbb{Z}} W(n, u(n)) \\
\equiv C_2 \| u \|_p^p - \sum_{n \in \mathbb{Z}} W(n, u(n)).
\]

By (W2) and (W3), we get

\[
\lim_{|x| \to \infty} \frac{pW(n, x)}{|x|^p} = V_\infty(n) \quad \text{uniformly for } n \in \mathbb{Z}.
\]
Let \( \overline{W}(n, x) = pW(n, x) - V_\infty(n)|x|^p \); it follows from (W2), (W3), (2.19), and (3.6) that

\[
\overline{W}(n, x) \leq \left( pR + \sup_{z} V_\infty(n) \right) |x|^p, \quad \forall x \in \mathbb{R}^N, \quad \lim_{|x| \to \infty} \overline{W}(n, x) = 0. \tag{3.7}
\]

Define \( E_1 := \{u(n) = xe^{-n}: x \in \mathbb{R}^N, n \in \mathbb{Z}\} \subset E \) with

\[
\inf_{z} V_\infty(n) > \max\{1, \rho b_2\} \left( 1 + \left| 1 - e^{n[-|n-1|]} \right|^p \right). \tag{3.8}
\]

By an easy calculation, we have

\[
\|u\|^p = \left( 1 + \left| 1 - e^{n[-|n-1|]} \right|^p \right)\|u\|^p. \tag{3.9}
\]

In what follows, we prove that for some \( u \in E_1 \) with \( \|u\| = 1 \), \( \varphi(su) \to -\infty \) as \( s \to \infty \). Otherwise, there exist a sequence \( \{s_k\} \) with \( s_k \to \infty \) as \( k \to \infty \) and a positive constant \( C_3 \) such that \( \varphi(s_k u) \geq -C_3 \) for all \( k \). From (3.5), we obtain

\[
- \frac{C_3}{s_k^p} \leq \frac{\varphi(s_k u)}{s_k^p} \leq C_2 - \frac{1}{p} \sum_{n \in \mathbb{Z}} \overline{W}(n, s_k u(n)) - \frac{1}{p} \sum_{n \in \mathbb{Z}} V_\infty(n) |u(n)|^p \\
\leq C_2 - \frac{1}{p} \sum_{n \in \mathbb{Z}} \overline{W}(n, s_k u(n)) - \frac{1}{p} \inf_{z} V_\infty(n) \|u\|^p. \tag{3.10}
\]

It follows from (3.7) that

\[
\overline{W}(n, s_k u(n)) \leq \left( pR + \sup_{z} V_\infty(n) \right) |u(n)|^p, \quad \overline{W}(n, s_k u(n)) \left/ \frac{|s_k|^p}{p} \right. \to 0 \quad \text{as } k \to \infty. \tag{3.11}
\]

Hence, from Lebesgue's dominated theorem and (3.11), we have

\[
\sum_{n \in \mathbb{Z}} \overline{W}(n, s_n u(n)) \left/ \frac{|s_k|^p}{p} \right. \to 0 \quad \text{as } k \to \infty. \tag{3.12}
\]

It follows from (3.8), (3.9), (3.10), and (3.12) that

\[
0 \leq - \frac{C_3}{s_k^p} \leq C_2 - \frac{1}{p} \left( 1 + \left| 1 - e^{n[-|n-1|]} \right|^p \right) \inf_{z} V_\infty(n) \left/ \frac{|s_k|^p}{p} \right. < 0 \quad \text{as } k \to \infty, \tag{3.13}
\]

which is a contradiction. Hence, there exists \( e \in E \) with \( \|e\| > \rho \) such that \( \varphi(e) \leq 0 \).
Step 3. From Step 1, Step 2, and Lemma 2.5, we know that there is a sequence \( \{u_k\}_{k \in \mathbb{N}} \subset E \) such that
\[
\varphi(u_k) \to c, \quad (1 + \|u_k\|)\|\varphi'(u_k)\|_{E^*} \to 0 \quad \text{as } k \to \infty, \tag{3.14}
\]
where \( E^* \) is the dual space of \( E \). In the following, we will prove that \( \{u_k\}_{k \in \mathbb{N}} \) is bounded in \( E \). Otherwise, assume that \( \|u_k\| \to \infty \) as \( k \to \infty \). Let \( z_k = u_k/\|u_k\| \); we have \( \|z_k\| = 1 \). It follows from (2.5), (2.16), (3.14), and (K2) that
\[
C_4 \geq p\varphi(u_k) - \langle \varphi'(u_k), u_k \rangle
\]
\[
= \sum_{n \in \mathbb{Z}} \left[ (\nabla W(n, u_k(n)), u_k(n)) - pW(n, u_k(n)) \right]
+ \sum_{n \in \mathbb{Z}} [pK(n, u_k(n)) - (\nabla K(n, u_k(n)), u_k(n))]
\geq \sum_{n \in \mathbb{Z}} \left[ (\nabla W(n, u_k(n)), u_k(n)) - pW(t, u_k(n)) \right] := \sum_{n \in \mathbb{Z}} \tilde{W}(n, u_k(n)).
\tag{3.15}
\]

Set \( \Omega_k(\alpha, \beta) = \{n \in \mathbb{Z} : \alpha \leq |u_k(n)| \leq \beta\} \) for \( 0 < \alpha < \beta \). Then from (3.15), we have
\[
C_4 \geq \sum_{n \in \Omega_k(0, \alpha)} \tilde{W}(n, u_k(n)) + \sum_{n \in \Omega_k(\alpha, \beta)} \tilde{W}(n, u_k(n)) + \sum_{n \in \Omega_k(\beta, +\infty)} \tilde{W}(n, u_k(n)). \tag{3.16}
\]

From (K1), (K2), and (3.14), we get
\[
o(1) = \langle \varphi'(u_k), u_k \rangle
= \sum_{n \in \mathbb{Z}} [\|\Delta u_k(n-1)\| + (\nabla K(n, u_k(n)) - \nabla W(n, u_k(n)), u_k(n))]
\geq \sum_{n \in \mathbb{Z}} [\|\Delta u_k(n-1)\| + b_3|u_k(n)|^p - (\nabla W(n, u_k(n)), u_k(n))]
\geq \min\{1, b_3\} \|u_k\|^p - \sum_{n \in \mathbb{Z}} (\nabla W(n, u_k(n)), u_k(n))
\geq \min\{1, b_3\} \|u_k\|^p - \sum_{n \in \Omega_k(\beta, +\infty)} (\nabla W(n, u_k(n)), u_k(n))
:= C_5 \|u_k\|^p - \sum_{n \in \mathbb{Z}} (\nabla W(n, u_k(n)), u_k(n))
\geq \|u_k\|^p \left( C_5 - \sum_{n \in \mathbb{Z}} (\nabla W(n, u_k(n)), u_k(n)) / \|u_k\|^p \right),
\tag{3.17}
\]

which implies that
\[
\limsup_{k \to \infty} \sum_{n \in \mathbb{Z}} \frac{(\nabla W(n, u_k(n)), u_k(n))}{|u_k(n)|^p} |z_k(n)|^p = \limsup_{k \to \infty} \sum_{n \in \mathbb{Z}} \frac{(\nabla W(n, u_k(n)), u_k(n))}{\|u_k\|^p} \geq C_5. \tag{3.18}
\]
Let $0 < \varepsilon < C_5/3$. From (W1), there exists $\alpha_\varepsilon > 0$ such that
\[
|\nabla W(n, x)| \leq \frac{\varepsilon}{2p} |x|^{p-1} \quad \text{for } |x| \leq \alpha_\varepsilon \text{ uniformly for } n \in \mathbb{Z}.
\] (3.19)

Since $\|z_k\| = 1$, it follows from (2.9) and (3.19) that
\[
\sum_{n \in \Omega_k (\alpha_\varepsilon, \beta_\varepsilon)} \frac{|\nabla W(n, u_k(n))|}{|u_k(n)|^{p-1}} |z_k(n)|^p \leq \sum_{n \in \Omega_k (\alpha_\varepsilon, \beta_\varepsilon)} \frac{\varepsilon}{2p} |z_k(n)|^p \leq \varepsilon, \quad \forall k \in \mathbb{N}.
\] (3.20)

For $s > 0$, let
\[
h(s) := \inf \left\{ W(n, x) \mid n \in \mathbb{Z}, \ x \in \mathbb{R}^N \text{ with } |x| \geq s \right\}.
\] (3.21)

Thus, from (W4), we have $h(s) \to +\infty$ as $s \to +\infty$, which together with (3.16) implies that
\[
\text{meas} (\Omega_k (\beta_\varepsilon, +\infty)) \leq \frac{C_6}{h(\beta_\varepsilon)} \to 0, \quad \text{as } \beta_\varepsilon \to +\infty.
\] (3.22)

Hence, we can take $\beta_\varepsilon$ sufficiently large such that
\[
\sum_{n \in \Omega_k (\beta_\varepsilon, +\infty)} |z_k(n)|^p < \frac{\varepsilon}{R}.
\] (3.23)

The previous inequality and (W2) imply that
\[
\sum_{n \in \Omega_k (\beta_\varepsilon, +\infty)} \frac{|\nabla W(n, u_k(n))|}{|u_k(n)|^{p-1}} |z_k(n)|^p \leq R \sum_{n \in \Omega_k (\beta_\varepsilon, +\infty)} |z_k(n)|^p < \varepsilon, \quad \forall k \in \mathbb{N}.
\] (3.24)

Next, for the previous $0 < \alpha_\varepsilon < \beta_\varepsilon$, let
\[
c_\varepsilon := \inf \left\{ \frac{W(n, x)}{|x|^p} \mid n \in \mathbb{Z}, \ x \in \mathbb{R}^N \text{ with } \alpha_\varepsilon \leq |x| \leq \beta_\varepsilon \right\},
\] (3.25)
\[
d_\varepsilon := \max \left\{ \frac{|\nabla W(n, x)|}{|x|^{p-1}} \mid n \in \mathbb{Z}, \ x \in \mathbb{R}^N \text{ with } \alpha_\varepsilon \leq |x| \leq \beta_\varepsilon \right\}.
\]

From (W4), we have $c_\varepsilon > 0$ and
\[
W(n, u_k(n)) \geq c_\varepsilon |u_k(n)|^p, \quad \forall n \in \Omega_k (\alpha_\varepsilon, \beta_\varepsilon).
\] (3.26)
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From (3.15) and (3.26), we get

\[
\sum_{n \in \Omega_k(a, \beta_k)} |z_k(n)|^p = \frac{1}{\|u_k\|^p} \sum_{n \in \Omega_k(a, \beta_k)} |u_k(n)|^p \\
\leq \frac{1}{\|u_k\|^p} \sum_{n \in \Omega_k(a, \beta_k)} \frac{1}{c_\epsilon} \hat{W}(n, u_k(n)) \\
\leq \frac{C_4}{c_\epsilon \|u_k\|^p} \to 0 \quad \text{as} \quad k \to \infty,
\]

which implies that

\[
\sum_{n \in \Omega_k(a, \beta_k)} \frac{|\nabla W(n, u_k(n))|}{|u_k(n)|^{p-1}} |z_k(n)|^p \leq d_\epsilon \sum_{n \in \Omega_k(a, \beta_k)} |z_k(n)|^p \to 0 \quad \text{as} \quad k \to \infty.
\]

Therefore, there exists \( k_0 > 0 \) such that

\[
\sum_{n \in \Omega_k(a, \beta_k)} \frac{|\nabla W(n, u_k(n))|}{|u_k(n)|^{p-1}} |z_k(n)|^p \leq \epsilon, \quad \forall k \geq k_0.
\]

It follows from (3.20), (3.24), and (3.29) that

\[
\sum_{n \in \mathbb{Z}} \frac{(|\nabla W(n, u_k(n))|, u_k(n))}{|u_k(n)|^p} |z_k(n)|^p \leq \sum_{n \in \mathbb{Z}} \frac{|\nabla W(t, u_k(n))|}{|u_k(n)|^{p-1}} |z_k(n)|^p < 3\epsilon < C_5, \quad \forall k \geq k_0,
\]

which implies that

\[
\limsup_{n \to -\infty} \sum_{n \in \mathbb{Z}} \frac{(|\nabla W(n, u_k(n))|, u_k(n))}{|u_k(n)|^p} |z_k(n)|^p < C_5,
\]

but this contradicts to (3.18). Hence, \( \|u_k\| \) is bounded in \( E \).

Going to a subsequence if necessary, we may assume that there exists \( u \in E \) such that \( u_k \to u \) as \( k \to \infty \). In order to prove our theorem, it is sufficient to show that \( \varphi'(u) = 0 \). For any \( a \in \mathbb{Z} \) with \( a > 0 \), let \( \chi_a(t) = 1 \) for \( t \in \mathbb{Z}[-a, a] \) and let \( \chi_a(t) = 0 \) for \( t \in \mathbb{Z}(-\infty, -a) \cup \mathbb{Z}(a, \infty) \). Then from (2.16), we have

\[
\langle \varphi'(u_k) - \varphi'(u), \chi_a(u_k - u) \rangle \\
= \sum_{n \in \mathbb{Z}[-a, a]} |\Delta u_k(n-1)|^{p-2} (\Delta u_k(n-1), \Delta u_k(n-1) - \Delta u(n-1)) \\
- \sum_{n \in \mathbb{Z}[-a, a]} |\Delta u(n-1)|^{p-2} (\Delta u(n-1), \Delta u_k(n-1) - \Delta u(n-1)) \\
+ \sum_{n \in \mathbb{Z}[-a, a]} (\nabla K(n, u_k(n)) - \nabla K(n, u(n)), u_k(n) - u(n))
\]
\[
- \sum_{n \in \mathbb{Z}[-a,a]} (\nabla W(n, u_k(n)) - \nabla W(n, u(n)), u_k(n) - u(n))
\]
\[
\geq \|\Delta u_k\|_{p\mathbb{Z}[-a,a]}^p + \|\Delta u\|_{p\mathbb{Z}[-a,a]}^p - \sum_{n \in \mathbb{Z}[-a,a]} |\Delta u_k(n - 1)|^{p-1}|\Delta u(n - 1)|
\]
\[
- \sum_{n \in \mathbb{Z}[-a,a]} |\Delta u(n - 1)|^{p-1}|\Delta u_k(n - 1)|
\]
\[
+ \sum_{n \in \mathbb{Z}[-a,a]} (\nabla K(n, u_k(n)) - \nabla K(n, u(n)), u_k(n) - u(n))
\]
\[
- \sum_{n \in \mathbb{Z}[-a,a]} (\nabla W(n, u_k(n)) - \nabla W(n, u(n)), u_k(n) - u(n))
\]
\[
\geq \|\Delta u_k\|_{p\mathbb{Z}[-a,a]}^p + \|\Delta u\|_{p\mathbb{Z}[-a,a]}^p - \|\Delta u_k\|_{p\mathbb{Z}[-a,a]}^{p-1}|\Delta u_k\|_{p\mathbb{Z}[-a,a]}^{p-1} - \|\Delta u\|_{p\mathbb{Z}[-a,a]}^{p-1}|\Delta u\|_{p\mathbb{Z}[-a,a]}^{p-1}
\]
\[
+ \sum_{n \in \mathbb{Z}[-a,a]} (\nabla K(n, u_k(n)) - \nabla K(n, u(n)), u_k(n) - u(n))
\]
\[
- \sum_{n \in \mathbb{Z}[-a,a]} (\nabla W(n, u_k(n)) - \nabla W(n, u(n)), u_k(n) - u(n))
\]
\[
= \left(\|\Delta u_k\|_{p\mathbb{Z}[-a,a]}^{p-1} - \|\Delta u\|_{p\mathbb{Z}[-a,a]}^{p-1}\right) \left(\|\Delta u_k\|_{p\mathbb{Z}[-a,a]}^{p-1} - \|\Delta u\|_{p\mathbb{Z}[-a,a]}^{p-1}\right)
\]
\[
+ \sum_{n \in \mathbb{Z}[-a,a]} (\nabla K(n, u_k(n)) - \nabla K(n, u(n)), u_k(n) - u(n))
\]
\[
- \sum_{n \in \mathbb{Z}[-a,a]} (\nabla W(n, u_k(n)) - \nabla W(n, u(n)), u_k(n) - u(n))
\]
(3.32)

Since \(\varphi'(u_k) \to 0\) as \(k \to +\infty\) and \(u_k \rightharpoonup u\) in \(E\), it follows from (3.14) that

\[
\langle \varphi'(u_k) - \varphi'(u), \chi_a(u_k - u) \rangle \to 0 \quad \text{as} \quad k \to \infty,
\]
\[
\sum_{n \in \mathbb{Z}[-a,a]} (\nabla K(n, u_k(n)) - \nabla K(n, u(n)), u_k(n) - u(n)) \to 0 \quad \text{as} \quad k \to \infty,
\]
(3.33)
\[
\sum_{n \in \mathbb{Z}[-a,a]} (\nabla W(n, u_k(n)) - \nabla W(n, u(n)), u_k(n) - u(n)) \to 0 \quad \text{as} \quad k \to \infty.
\]

It follows from (3.32) and (3.33) that \(\|\Delta u_k\|_{p\mathbb{Z}[-a,a]} \to \|\Delta u\|_{p\mathbb{Z}[-a,a]}\) as \(k \to +\infty\).

For any \(u \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)\), and assume that for some \(A \in \mathbb{Z}\) with \(A > 0\), \(\text{supp}(w) \subset \mathbb{Z}[-A,A]\). Since

\[
\lim_{k \to \infty} \Delta u_k(n - 1) = \Delta u(n - 1), \quad \forall a.e. \quad n \in \mathbb{Z},
\]
\[
\left| \left( |\Delta u_k(n - 1)|^{p-2}\Delta u_k(n - 1), \Delta w(n - 1) \right) \right|
\]
The proof of Theorem 1.1 is complete.

In problem 4.1.

4. An Example

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\[ \lim_{n \to \infty} \sum_{n \in \mathbb{Z}[-A, A]} \left[ \frac{p-1}{p} |\Delta u_k(n-1)|^p + \frac{1}{p} |\Delta w(n-1)|^p \right] \]

\[ = \frac{p-1}{p} \lim_{k \to \infty} \|\Delta u_k\|_{p\mathbb{Z}[-A, A]}^p + \frac{1}{p} \|\Delta w\|_{p\mathbb{Z}[-A, A]}^p \]

\[ = \frac{p-1}{p} \|\Delta u\|_{p\mathbb{Z}[-A, A]}^p + \frac{1}{p} \|\Delta w\|_{p\mathbb{Z}[-A, A]}^p \]

\[ = \sum_{n \in \mathbb{Z}[-A, A]} \left[ \frac{p-1}{p} |\Delta u(n-1)|^p + \frac{1}{p} |\Delta w(n-1)|^p \right] < +\infty, \]

\[ (3.34) \]

then, we have

\[ \sum_{n \in \mathbb{Z}[-A, A]} (|\Delta u_k(n-1)|^{p-2} \Delta u_k(n-1), \Delta w(n-1)) \quad \to \quad \sum_{n \in \mathbb{Z}[-A, A]} (|\Delta u(n-1)|^{p-2} \Delta u(n-1), \Delta w(n-1)) \]

as \( k \to \infty \). Noting that

\[ \sum_{n \in \mathbb{Z}[-A, A]} (\nabla K(n, u_k(n)), w(n)) \to \sum_{n \in \mathbb{Z}[-A, A]} (\nabla K(n, u(n)), w(n)) \quad \text{as} \quad k \to \infty, \quad (3.36) \]

\[ \sum_{n \in \mathbb{Z}[-A, A]} (\nabla W(n, u_k(n)), w(n)) \to \sum_{n \in \mathbb{Z}[-A, A]} (\nabla W(n, u(n)), w(n)) \quad \text{as} \quad k \to \infty. \]

Hence, we have

\[ \langle \varphi'(u), w \rangle = \lim_{k \to \infty} \langle \varphi'(u_k), w \rangle = 0, \]

\[ (3.37) \]

which implies that \( \varphi'(u) = 0 \); that is, \( u \) is a critical point of \( \varphi \). From (K1) and (W1), we know that \( u \neq 0 \). In fact, if \( u = 0 \), we have from (2.5), (K1), and (W1) that \( \varphi(u) = 0 \). On the other hand, from Step 1, Step 2, and Lemma 2.5, we know that \( \varphi(u) = c > 0 \). This is a contradiction. The proof of Theorem 1.1 is complete. \( \square \)

4. An Example

Example 4.1. In problem (1.1), let \( p = 3/2 \), and

\[ K(n, x) = \left( 1 + \frac{1}{|x|^{3/2} + 1} \right) |x|^{3/2}, \quad W(n, x) = a(n)|x|^{3/2} \left( 1 - \frac{1}{(\ln(e + |x|))^{1/2}} \right), \]

\[ (4.1) \]
where \( a \in L^{\infty}(\mathbb{Z}, \mathbb{R}^+) \) with \( \inf_{n} a(n) > 3 \). One can easily check that \( K \) satisfies conditions (K1) and (K2) with \( b_1 = 1, b_2 = 2, \) and \( b_3 = 3/2 \). An easy computation shows that

\[
\nabla W(n, x) = \frac{3}{2} a(n)|x|^{-1/2} x \left( 1 - \frac{1}{(\ln(e + |x|))^{1/2}} \right) + \frac{a(n)|x|^{1/2}}{2(e + |x|)(\ln(e + |x|))^{3/2}},
\]

(4.2)

\[
(\nabla W(n, x), x) - \frac{3}{2} W(n, x) = \frac{a(n)|x|^{5/2}}{2(e + |x|)(\ln(e + |x|))^{3/2}}.
\]

Then it is easy to check that \( W \) satisfies (W1)–(W4). Hence, \( K(n, x) \) and \( W(n, x) \) satisfy all the conditions of Theorem 1.1 and then problem (1.1) has at least one nontrivial homoclinic solution.

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**References**


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