Research Article

Some Identities on the $q$-Bernoulli Numbers and Polynomials with Weight 0

T. Kim, J. Choi, and Y. H. Kim

Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

Correspondence should be addressed to J. Choi, jeschoi@kw.ac.kr

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Recently, Kim (2011) has introduced the $q$-Bernoulli numbers with weight $\alpha$. In this paper, we consider the $q$-Bernoulli numbers and polynomials with weight $\alpha = 0$ and give $p$-adic $q$-integral representation of Bernstein polynomials associated with $q$-Bernoulli numbers and polynomials with weight 0. From these integral representation on $\mathbb{Z}_p$, we derive some interesting identities on the $q$-Bernoulli numbers and polynomials with weight 0.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$.

Let $| \cdot |_p$ be a $p$-adic norm with $|x|_p = p^{-r}$, where $x = p^r s/t$ and $(p, s) = (p, t) = 1$, $r \in \mathbb{Q}$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$, and $[x]_q = (1 - q^x)/(1 - q)$.

Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_q(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \mu_q(x + p^N \mathbb{Z}_p)$$

(1.1)

(see [1–5]). For $n \in \mathbb{N}$, let $f_n(x) = f(x + n)$. From (1.1), we note that

$$q^n I_q(f_n) - I_q(f) = (q - 1) \sum_{l=0}^{n-1} q^{l} f(l) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} q^l f'(l),$$

(1.2)
where \( f'(l) = df(x)/dx \rfloor_{x=l} \) (see [3, 6, 7]). In the special case, \( n = 1 \), we get

\[
q \int_{z_q} f(x+1)d\mu_q(x) - \int_{z_q} f(x)d\mu_q(x) = (q-1)f(0) + \frac{q-1}{\log q} f'(0). \tag{1.3}
\]

Throughout this paper, we assume that \( \alpha \in \mathbb{Q} \).
The \( q \)-Bernoulli numbers with weight \( \alpha \) are defined by Kim [8] as follows:

\[
\tilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q \left( q^{\alpha} \tilde{\beta}_{q}^{(\alpha)} + 1 \right)^n - \tilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{|\alpha|_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.4}
\]

with the usual convention about replacing \((\tilde{\beta}_{q}^{(\alpha)})^n\) with \(\tilde{\beta}_{n,q}^{(\alpha)}\). From (1.4), we can derive the following equation:

\[
\tilde{\beta}_{n,q}^{(\alpha)} = \frac{1}{(1-q)^n|\alpha|_q} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{\alpha l + 1}{|\alpha l + 1|_q}.
\]

\[
= -\frac{n\alpha}{|\alpha|_q} \sum_{m=0}^{\infty} q^{m\alpha + m}[m] q^{-1} + (1-q) \sum_{m=0}^{\infty} q^m[m] q^{-n}. \tag{1.5}
\]

By (1.1), (1.4), and (1.5), we get

\[
\tilde{\beta}_{n,q}^{(\alpha)} = \int_{z_q} [x]^n d\mu_q(x) = \frac{1}{(1-q)^n|\alpha|_q} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{\alpha l + 1}{|\alpha l + 1|_q}. \tag{1.6}
\]

The \( q \)-Bernoulli polynomials with weight \( \alpha \) are defined by

\[
\tilde{\beta}_{n,q}^{(\alpha)}(x) = \int_{z_q} [x+y]^n d\mu_q(y) = \sum_{l=0}^{n} \binom{n}{l} q^\alpha [x] q^{-l} \tilde{\beta}_{l,q}^{(\alpha)}. \tag{1.7}
\]

By (1.6) and (1.7), we easily see that

\[
\tilde{\beta}_{n,q}^{(\alpha)}(x) = \int_{z_q} [x+y]^n d\mu_q(y) = \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l} \frac{\alpha l + 1}{|\alpha l + 1|_q}. \tag{1.8}
\]

Let \( C(\mathbb{Z}_p) \) be the set of continuous functions on \( \mathbb{Z}_p \). For \( f \in C(\mathbb{Z}_p) \), the \( p \)-adic analogue of Bernstein operator of order \( n \) for \( f \) is given by

\[
\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B_{k,n}(x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k}, \tag{1.9}
\]
where \( n, k \in \mathbb{Z}^+ \) (see [1, 9, 10]). For \( n, k \in \mathbb{Z}^+ \), the \( p \)-adic Bernstein polynomials of degree \( n \) are defined by \( B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \) for \( x \in \mathbb{Z}_p \), (see [1, 10, 11]).

In this paper, we consider Bernstein polynomials to express the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) and investigate some interesting identities of Bernstein polynomials associated with the \( q \)-Bernoulli numbers and polynomials with weight 0 by using the expression of \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) of these polynomials.

### 2. \( q \)-Bernoulli Numbers with Weight 0 and Bernstein Polynomials

In the special case, \( \alpha = 0 \), the \( q \)-Bernoulli numbers with weight 0 will be denoted by \( \tilde{\beta}^{(0)}_{n,q} = \tilde{\beta}_{n,q} \). From (1.4), (1.5), and (1.6), we note that

\[
\sum_{n=0}^{\infty} \tilde{\beta}_{n,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_q(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{xt} d\mu_q(x) = \left( \frac{q-1}{\log q} \right) \left( \frac{t + \log q}{qe^t - 1} \right).
\]

It is easy to show that

\[
\frac{t + \log q}{qe^t - 1} = \frac{t}{q-1} \left( \frac{1 - q^{-1}}{e^t - q^{-1}} \right) + \frac{\log q}{q-1} \left( \frac{1 - q^{-1}}{e^t - q^{-1}} \right)
\]

\[
= \frac{t}{q-1} \sum_{n=0}^{\infty} H_n(q^{-1}) \frac{t^n}{n!} + \frac{\log q}{q-1} \sum_{n=0}^{\infty} H_n(q^{-1}) \frac{t^n}{n!}
\]

\[
= \frac{1}{q-1} \sum_{n=1}^{\infty} nH_{n-1}(q^{-1}) \frac{t^n}{n!} + \frac{\log q}{q-1} \sum_{n=0}^{\infty} H_n(q^{-1}) \frac{t^n}{n!},
\]

where \( H_n(q^{-1}) \) are the \( n \)th Frobenius-Euler numbers.

By (2.1) and (2.2), we get

\[
\tilde{\beta}_{n,q} = \begin{cases} 
1 & \text{if } n = 0, \\
\frac{n}{\log q} H_{n-1}(q^{-1}) + H_n(q^{-1}) & \text{if } n > 0.
\end{cases}
\]

Therefore, we obtain the following theorem.

**Theorem 2.1.** For \( n \in \mathbb{Z}_+ \), we have

\[
\tilde{\beta}_{n,q} = \begin{cases} 
1 & \text{if } n = 0, \\
\frac{n}{\log q} H_{n-1}(q^{-1}) + H_n(q^{-1}) & \text{if } n > 0.
\end{cases}
\]

where \( H_n(q^{-1}) \) are the \( n \)th Frobenius-Euler numbers.
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From (1.5), (1.6), and (1.7), we have

$$
\tilde{\beta}_{0,q} = 1, \quad q (\tilde{\beta}_q + 1)^n - \tilde{\beta}_{n,q} = \begin{cases} 
\frac{q - 1}{\log q} & \text{if } n = 1, \\
0 & \text{if } n > 1,
\end{cases}
$$

(2.5)

with the usual convention about replacing \((\tilde{\beta}_q)^n\) with \(\tilde{\beta}_{n,q}\). By (1.7), the \(n\)th \(q\)-Bernoulli polynomials with weight 0 are given by

$$
\tilde{\beta}_{n,q}(x) = \int_{\mathbb{Z}_p} (x + y)^n \, d\mu_q(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \tilde{\beta}_{l,q}.
$$

(2.6)

From (2.6), we can derive the following function equation:

$$
\left( \frac{q - 1}{\log q} \right) \left( \frac{t + \log q}{q^e t - 1} \right) e^{xt} = \sum_{n=0}^{\infty} \tilde{\beta}_{n,q}(x) \frac{t^n}{n!}.
$$

(2.7)

Thus, by (2.7), we get that

$$
\tilde{\beta}_{n,q}(-1) = (-1)^n \tilde{\beta}_{n,q}(x), \quad \text{for } n \in \mathbb{Z}_+.
$$

(2.8)

By the definition of \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\), we see that

$$
\int_{\mathbb{Z}_p} (1 - x)^n d\mu_q(x) = (-1)^n \int_{\mathbb{Z}_p} (x - 1)^n d\mu_q(x) = (-1)^n \tilde{\beta}_{n,q}(-1).
$$

(2.9)

Therefore, by (2.8) and (2.9), we obtain the following theorem.

**Theorem 2.2.** For \(n \in \mathbb{Z}_+\), we have

$$
(-1)^n \tilde{\beta}_{n,q}(x) = \tilde{\beta}_{n,q}(-1).
$$

(2.10)

In particular, \(x = -1\), we get

$$
\int_{\mathbb{Z}_p} (1 - y)^n d\mu_q(y) = (-1)^n \tilde{\beta}_{n,q}(-1) = \tilde{\beta}_{n,q}(-1).
$$

(2.11)

From (2.5), we can derive the following equation:

$$
q^2 \tilde{\beta}_{n,q}(2) = q^2 + nq \frac{q - 1}{\log q} - q + \tilde{\beta}_{n,q}, \quad \text{if } n > 1.
$$

(2.12)

Therefore, by (2.12), we obtain the following theorem.
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**Theorem 2.3.** For \( n \in \mathbb{N} \) with \( n > 1 \), we have

\[
\tilde{\beta}_{n,q}(2) = 1 + \frac{n}{q} \left( \frac{q-1}{\log q} \right) - \frac{1}{q} + \frac{1}{q^2} \tilde{\beta}_{n,q}. \tag{2.13}
\]

Taking the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) for one Bernstein polynomials in (1.9), we get

\[
\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_q(x) = \binom{n}{k} \int_{\mathbb{Z}_p} x^k(1-x)^{n-k}d\mu_q(x)
\]

\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} x^{k+l}d\mu_q(x) \tag{2.14}
\]

From the symmetry of Bernstein polynomials, we note that

\[
\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_q(x) = \int_{\mathbb{Z}_p} B_{n-k,n}(1-x)d\mu_q(x)
\]

\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} (1-x)^{n-l}d\mu_q(x). \tag{2.15}
\]

Let \( n > k + 1 \). Then, by Theorem 2.3 and (2.15), we get

\[
\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_q(x) = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( 1 - \frac{n-l}{q} \left( \frac{q-1}{\log q} \right) - q + q^2 \tilde{\beta}_{n-l,q} \right)
\]

\[
= \begin{cases} 
1 + n \left( \frac{q-1}{\log q} \right) - q + q^2 \tilde{\beta}_{n,q} & \text{if } k = 0, \\
n \left( \frac{1-q}{\log q} \right) + nq^2 \tilde{\beta}_{n,q} + nq^2 \tilde{\beta}_{n-1,q} & \text{if } k = 1, \\
n \frac{1}{q} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \tilde{\beta}_{n-l,q} & \text{if } k > 1.
\end{cases} \tag{2.16}
\]

By comparing the coefficients on the both sides of (2.14) and (2.16), we obtain the following theorem.
Theorem 2.4. For $n, k \in \mathbb{Z}_+$ with $n > k + 1$, we have
\[
\sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \hat{\beta}_{1+l,q} = \frac{1 - q}{\log q} + q^2 \beta_{n,q^{-1}} + q^2 \beta_{n-1,q^{-1}},
\]
(2.17)
\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \hat{\beta}_{k+l,q} = q^2 \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \beta_{n-l,q^{-1}}, \quad \text{if } k > 1.
\]

In particular, when $k = 0$, we have
\[
\sum_{l=0}^{n} \binom{n}{l} (-1)^l \beta_{l,q} = 1 + n \frac{q - 1}{\log q} - q + q^2 \beta_{n,q^{-1}}.
\]
(2.18)

Let $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k + 1$. Then we see that
\[
\int_{Z_q} B_{k,n}(x) B_{k,m}(x) d\mu_q(x)
\]
\[
= \binom{n}{k} \binom{m}{k} \int_{Z_q} x^{2k} (1 - x)^{n+m-2k} d\mu_q(x)
\]
\[
= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{Z_q} (1 - x)^{n+m-1} d\mu_q(x)
\]
\[
= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} (1 - (n + m - l)) \left( \frac{1 - q}{\log q} \right) - q + q^2 \beta_{n+m-l,q^{-1}}
\]
(2.19)
\[
= \begin{cases} 
1 + (n + m) \left( \frac{q - 1}{\log q} \right) - q + q^2 \beta_{n+m,q^{-1}} & \text{if } k = 0, \\
\binom{n}{k} \binom{m}{k} q^2 \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \beta_{n+m-l,q^{-1}} & \text{if } k > 0.
\end{cases}
\]

For $m, n, k \in \mathbb{Z}_+$, we have
\[
\int_{Z_q} B_{k,n}(x) B_{k,m}(x) d\mu_q(x) = \binom{n}{k} \binom{m}{k} \int_{Z_q} x^{2k} (1 - x)^{n+m-2k} d\mu_q(x)
\]
\[
= \binom{n}{k} \binom{m}{k} n+m-2k \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \int_{Z_q} x^{2k+l} d\mu_q(x)
\]
(2.20)
\[
= \binom{n}{k} \binom{m}{k} n+m-2k \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \beta_{l+2k,q^{-1}}.
\]
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By comparing the coefficients on the both sides of (2.19) and (2.20), we obtain the following theorem.

**Theorem 2.5.** For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k + 1$, we have

$$\sum_{l=0}^{n+m} \binom{n+m}{l} (-1)^l \tilde{p}_{l,q} = 1 + (n + m) \left( \frac{q - 1}{\log q} \right) - q + q^2 \tilde{p}_{n+m-1,q}. \quad (2.21)$$

In particular, when $k \neq 0$, we have

$$\sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \tilde{p}_{l+2k,q} = q^2 \sum_{l=0}^{2k} \binom{2k}{l} (-1)^l \tilde{p}_{n+m-1,q}. \quad (2.22)$$

For $s \in \mathbb{N}$, let $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$. By the same method above, we get

$$\int_{\mathbb{Z}_+} \left( \prod_{i=1}^{s} B_{k,n_i}(x) \right) d\mu_q(x) = \begin{cases} 1 + \left( \sum_{i=1}^{s} n_i \right) \left( \frac{q - 1}{\log q} \right) - q + q^2 \tilde{p}_{n_1+n_2+\cdots+n_s-1,q} & \text{if } k = 0, \\ \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) q^2 \sum_{l=0}^{2k} \binom{sk}{l} (-1)^l \tilde{p}_{n_1+n_2+\cdots+n_s-1,q} & \text{if } k > 0. \end{cases} \quad (2.23)$$

From the binomial theorem, we note that

$$\int_{\mathbb{Z}_+} \left( \prod_{i=1}^{s} B_{k,n_i}(x) \right) d\mu_q(x) = \left( \prod_{i=1}^{s} \binom{n_i}{k} \right)^{n_1+n_2+\cdots+n_s} \sum_{l=0}^{2k} \binom{sk}{l} (-1)^l \tilde{p}_{l+sk,q}. \quad (2.24)$$

By comparing the coefficients on the both sides of (2.23) and (2.24), we obtain the following theorem.

**Theorem 2.6.** For $s \in \mathbb{N}$, let $k, n_1, \ldots, n_s \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk + 1$. Then, we have

$$\sum_{l=0}^{n_1+\cdots+n_s} \binom{n_1+\cdots+n_s}{l} (-1)^l \tilde{p}_{l,q} = 1 + \left( \sum_{i=1}^{s} n_i \right) \left( \frac{q - 1}{\log q} \right) - q + q^2 \tilde{p}_{n_1+n_2+\cdots+n_s-1,q}. \quad (2.25)$$

In particular, when $k \neq 0$, we have

$$\sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-2k}{l} (-1)^l \tilde{p}_{l+sk,q} = q^2 \sum_{l=0}^{2k} \binom{sk}{l} (-1)^l \tilde{p}_{n_1+n_2+\cdots+n_s-1,q}. \quad (2.26)$$
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References

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