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Research Article

Certain Subordination Properties for Subclasses of Analytic Functions Involving Complex Order

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We derive several subordination results for a certain class of analytic functions defined by the Sălăgean operator In the present investigation.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : |z| < 1\}. \tag{1.2}$$

Further, by $\mathcal{S}$ we will denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Also let $\mathcal{S}^*(b)$, $\mathcal{K}(b)$ denote, respectively, the subclasses of $\mathcal{A}$ consisting of functions that are starlike of complex order b $(b \in \mathbb{C} \setminus \{0\})$, convex of complex order b $(b \in \mathbb{C} \setminus \{0\})$ in $\mathbb{U}$. In particular, the classes $\mathcal{S}^* := \mathcal{S}^*(1)$ and $\mathcal{K} := \mathcal{K}(1)$ are the familiar classes of starlike and convex functions in $\mathbb{U}$.

Sălăgean [1] introduced the following operator which is popularly known as the Sălăgean derivative operator:

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = zf'(z), \tag{1.3}$$

and, in general,

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}). \tag{1.4}$$

It is easy to see that from (1.1),

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k. \tag{1.5}$$

Let $G_n(\lambda, b, A, B)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$1 + \frac{1}{b}\left[(1-\lambda)\frac{D^n f(z)}{z} + \lambda\left(D^n f(z)\right)' - 1\right] \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1). \tag{1.6}$$

Equivalently,

$$\left|\frac{(1-\lambda)\left(D^n f(z)/z\right) + \lambda\left(D^n f(z)\right)' - 1}{b(A-B) - B\left((1-\lambda)\left(D^n f(z)/z\right) + \lambda\left(D^n f(z)\right)' - 1\right)}\right| < 1. \tag{1.7}$$

We note that, for $z \in \mathbb{U}$,

(1) $G_n(\lambda, b, 1, -1) = G_n(\lambda, b) = \{f \in \mathcal{A} : \mathrm{Re}\,(1+1/b[(1-\lambda)(D^n f(z)/z) + \lambda(D^n f(z))' - 1])\} > 0$,

(2) $G_n(0, b, 1, -1) = G_n(b) = \{f \in \mathcal{A} : \mathrm{Re}\,(1 + 1/b[D^n f(z)/z - 1])\} > 0$,

(3) $G_n(1, b, 1, -1) = R_n(b) = \{f \in \mathcal{A} : \mathrm{Re}\,(1 + 1/b[(D^n f(z))' - 1])\} > 0$,

(4) $G_0(0, b, 1, -1) = G(b) = \{f \in \mathcal{A} : \mathrm{Re}\,(1 + 1/b[f(z)/z - 1])\} > 0$,

(5) $G_0(1, b, 1, -1) = R(b) = \{f \in \mathcal{A} : \mathrm{Re}\,(1 + 1/b[f'(z) - 1])\} > 0$,

(6) $G_0(0, 1-\alpha, 1, -1) = G(\alpha) = \{f \in \mathcal{A} : \mathrm{Re}\,(f(z)/z) > \alpha,\ 0 \le \alpha < 1\}$,

(7) $G_0(1, 1-\alpha, 1, -1) = R(\alpha) = \{f \in \mathcal{A} : \mathrm{Re}\,(f'(z)) > \alpha,\ 0 \le \alpha < 1\}$.

The class $R(b)$ was studied by Abdul Halim [2], while the class $G(\alpha)$ was studied by Chen [3, 4] and the class $R(\alpha)$ was studied by Èzrohi [5] (see also the works of Altintas and Özkan [6], Aouf et al. [7], Attiya [8], Kamali and Akbulut [9], Özkan [10], and Shanmugam et al. [11]). A systematic investigation of the starlike and convex functions involving Sălăgean derivative was done by Aouf et al. very recently [7].

In our proposed investigation of functions in these subclasses of the normalized analytic function class $\mathcal{A}$, we need the following definitions and results.

Definition 1.1 (Hadamard product or convolution). For functions f and g in the class $\mathcal{A}$, where $f(z)$ of the form (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} c_k z^k, \tag{1.8}$$

the Hadamard product (or convolution) $(f * g)(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \quad z \in \mathbb{U}. \tag{1.9}$$

Definition 1.2 (subordination principle). For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function w such that

$$\begin{gathered} w(0) = 0, \qquad |w(z)| < 1, \\ g(z) = h(w(z)), \end{gathered} \tag{1.10}$$

for all $z \in \mathbb{U}$.

Definition 1.3 (see [12], subordinating factor sequence). A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z)$ is of the form (1.1) is analytic, univalent, and convex in $\mathbb{U}$, one has the subordination given by

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) = \sum_{k=1}^{\infty} a_k z^k, \quad z \in \mathbb{U},\ a_1 = 1. \tag{1.11}$$

Lemma 1.4 (see [12]). *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence for the class $\mathcal{K}$ of convex univalent functions if and only if*

$$\mathfrak{Re}\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0, \quad z \in \mathbb{U}. \tag{1.12}$$

2. Main Results

Theorem 2.1. *Let the function f of the form* (1.1) *satisfy the following condition:*

$$\sum_{n=2}^{\infty} (1 + |B|)[(1 + \lambda(k-1))]k^n |a_k| \le (A - B)|b|. \tag{2.1}$$

Then $f \in G_n(\lambda, b, A, B)$.

Proof. Suppose the inequality (2.1) holds. Then we have for $z \in \mathbb{U}$

$$\begin{aligned}
&\left|(1-\lambda)\frac{D^n f(z)}{z} + \lambda\left(D^n f(z)\right)' - 1\right| - \left|(A-B)b - B\left((1-\lambda)\frac{D^n f(z)}{z} + \lambda\left(D^n f(z)\right)' - 1\right)\right| \\
&\quad = \left|\sum_{k=2}^{\infty}[(1+\lambda(k-1))]k^n a_k z^{k-1}\right| - \left|(A-B)b - B\sum_{k=2}^{\infty}[(1+\lambda(k-1))]k^n a_k z^{k-1}\right| \\
&\quad \le \sum_{k=2}^{\infty}[(1+\lambda(k-1))]k^n |a_k||z|^{k-1} - \left\{(A-B)|b| - |B|\sum_{k=2}^{\infty}[(1+\lambda(k-1))]k^n |a_k||z|^{k-1}\right\} \\
&\quad \le \sum_{k=2}^{\infty}(1+|B|)[(1+\lambda(k-1))]k^n |a_k| - (A-B)|b| \le 0
\end{aligned} \tag{2.2}$$

which shows that f belongs to the class $G_n(\lambda, b, A, B)$. □

In view of Theorem 2.1, we now introduce the subclass $G_n^*(\lambda, b, A, B)$ which consists of functions $f \in \mathcal{A}$ whose Taylor-Maclaurin coefficients satisfy the inequality (2.1). We note that $G_n^*(\lambda, b, A, B) \subset G_n(\lambda, b, A, B)$.

In this work, we prove several subordination relationships involving the function class $G_n^*(\lambda, b, A, B)$ employing the technique used earlier by Attiya [13] and Srivastava and Attiya [14].

Theorem 2.2. *Let $f \in G_n^*(\lambda, b, A, B)$, and let $g(z)$ be any function in the usual class of convex functions $\mathcal{K}$, then*

$$\frac{(1+\lambda)(1+|B|)2^n}{2[(1+\lambda)(1+|B|)2^n + (A-B)|b|]}(f * g)(z) \prec g(z) \tag{2.3}$$

for every function $g \in \mathcal{K}$. Further,

$$\operatorname{Re}\{f(z)\} > -\frac{[(1+\lambda)(1+|B|)2^n + (A-B)|b|]}{(1+\lambda)(1+|B|)2^n}, \quad z \in \mathbb{U}. \tag{2.4}$$

The constant factor $(1+\lambda)(1+|B|)2^n/2[(1+\lambda)(1+|B|)2^n + (A-B)|b|]$ in (2.3) cannot be replaced by a larger number.

Proof. Let $f \in G_n^*(\lambda, b, A, B)$, and suppose that $g(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{K}$. Then

$$\begin{aligned}
&\frac{(1+\lambda)(1+|B|)2^n}{2[(1+\lambda)(1+|B|)2^n + (A-B)|b|]}(f * g)(z) \\
&\quad = \frac{(1+\lambda)(1+|B|)2^n}{2[(1+\lambda)(1+|B|)2^n + (A-B)|b|]}\left(z + \sum_{k=2}^{\infty} c_k a_k z^k\right).
\end{aligned} \tag{2.5}$$

Thus, by Definition 1.3, the subordination result holds true if

$$\left\{\frac{(1+\lambda)(1+|B|)2^n}{2[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}a_k\right\}_{k=1}^{\infty} \tag{2.6}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.4, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{1+2\sum_{k=1}^{\infty}\frac{(1+\lambda)(1+|B|)2^n}{2[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}a_k z^k\right\} > 0, \quad z \in \mathbb{U}. \tag{2.7}$$

Since $\Psi(k) = (1+|B|)[(1+\lambda(k-1))]k^n$ is an increasing function of k $(k \geq 2)$, we have, for $|z| = r < 1$,

$$\begin{aligned}
&\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}2\frac{(1+\lambda)(1+|B|)2^n}{2[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}a_k z^k\right\} \\
&\quad = \operatorname{Re}\left\{1+\frac{(1+\lambda)(1+|B|)2^n}{[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}z+\sum_{k=2}^{\infty}\frac{(1+\lambda)(1+|B|)2^n}{[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}a_k z^k\right\} \\
&\quad \geq 1-\frac{(1+\lambda)(1+|B|)2^n}{[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}r-\frac{1}{[(1+\lambda)(1+|B|)2^n+(A-B)|b|]} \\
&\qquad \times\sum_{k=2}^{\infty}(1+|B|)[1+\lambda(k-1)]k^n|a_k|r^k \geq 1-\frac{(1+\lambda)(1+|B|)2^n}{[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}r \\
&\qquad -\frac{|b|(A-B)}{[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}r = 1-r > 0, \quad |z| = r < 1,
\end{aligned} \tag{2.8}$$

where we have also made use of the assertion (2.1) of Theorem 2.1. This evidently proves the inequality (2.3) and hence also the subordination result (2.3) asserted by Theorem 2.2. The inequality (2.4) follows from (2.3) by taking

$$g(z) = \frac{z}{1-z} = z+\sum_{k=2}^{\infty}z^k \in \mathcal{K}. \tag{2.9}$$

To prove the sharpness of the constant $(1+\lambda)(1+|B|)2^n/[(1+\lambda)(1+|B|)2^n+(A-B)|b|]$, we consider the function $F \in G_n^*(\lambda, b, A, B)$ defined by

$$F(z) = z-\frac{(A-B)|b|}{(1+\lambda)(1+|B|)2^n}. \tag{2.10}$$

Thus, from (2.3), we have

$$\frac{(1+\lambda)(1+|B|)2^n}{2[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}F(z) \prec \frac{z}{1-z}. \tag{2.11}$$

It is easily verified that

$$\min\left\{\operatorname{Re}\left(\frac{(1+\lambda)(1+|B|)2^n}{2[(1+\lambda)(1+|B|)2^n+(A-B)|b|]}F(z)\right)\right\} = -\frac{1}{2}, \quad z \in \mathbb{U}. \tag{2.12}$$

This shows that the constant $(1+\lambda)(1+|B|)2^n/2[(1+\lambda)(1+|B|)2^n+(A-B)|b|]$ cannot be replaced by any larger one. □

For the choices of $A-1=0$ and $B+1=0$, we get the following corollary.

Corollary 2.3. *Let $f \in G_n^*(\lambda, b)$ let and $g(z)$ be any function in the usual class of convex functions $\mathcal{K}$, then*

$$\frac{(1+\lambda)2^n}{[(1+\lambda)(1+|B|)2^n+|b|]}(f*g)(z) \prec g(z), \tag{2.13}$$

where $b \in \mathbb{C} \setminus \{0\}$ and $0 \le \lambda < 1$,

$$\operatorname{Re}\{f(z)\} > -\frac{2[(1+\lambda)2^n+|b|]}{(1+\lambda)2^n}, \quad z \in \mathbb{U}. \tag{2.14}$$

The constant factor $(1+\lambda)2^n/2[(1+\lambda)2^n+|b|]$ in (2.13) cannot be replaced by a larger number.

For the choices of $A-1=B+1=0$ and $n=0$, one gets the following.

Corollary 2.4. *Let $f \in G^*(\lambda, b)$, and let $g(z)$ be any function in the usual class of convex functions $\mathcal{K}$, then*

$$\frac{(1+\lambda)}{2[1+\lambda+|b|]}(f*g)(z) \prec g(z), \tag{2.15}$$

where $b \in \mathbb{C} \setminus \{0\}$ and $0 \le \lambda < 1$,

$$\operatorname{Re}\{f(z)\} > -\frac{[1+\lambda+|b|]}{(1+\lambda)}, \quad z \in \mathbb{U}. \tag{2.16}$$

The constant factor $(1+\lambda)/2[1+\lambda+|b|]$ in (2.15) cannot be replaced by a larger number.

For the choices of $A-1=B+1=0, n=0$, and $\lambda=0$, one gets the following.

Corollary 2.5. *Let $f \in G^*(b)$, and $g(z)$ let be any function in the usual class of convex functions $\mathcal{K}$, then*

$$\frac{1}{2[1+|b|]}(f*g)(z) \prec g(z), \tag{2.17}$$

where $b \in \mathbb{C} \setminus \{0\}$ and $0 \le \lambda < 1$,

$$\operatorname{Re}\{f(z)\} > -[1+|b|], \quad z \in \mathbb{U}. \tag{2.18}$$

The constant factor $1/2[1+|b|]$ in (2.17) cannot be replaced by a larger number.

For the choices of $A-1 = B+1 = 0,\ n = 0,\ \lambda = 0$, and $b = 1-\alpha$, one gets the following.

Corollary 2.6. *Let $f \in G^*(\alpha)$, and let $g(z)$ be any function in the usual class of convex functions $\mathcal{K}$, then*

$$\frac{1}{2(2-\alpha)}(f*g)(z) \prec g(z), \tag{2.19}$$

where $0 \le \alpha < 1$,

$$\operatorname{Re}(f(z)) > -(2-\alpha), \quad z \in \mathbb{U}. \tag{2.20}$$

The constant factor $1/2(2-\alpha)$ in (2.19) cannot be replaced by a larger number.

For the choices of $A-1 = B+1 = 0,\ n = 0, \lambda = 1$, and $b = 1-\alpha\ (0 \le \alpha < 1)$, one gets the following.

Corollary 2.7. *Let $f \in R^*(\alpha)$, and $g(z)$ let be any function in the usual class of convex functions $\mathcal{K}$, then*

$$\frac{2}{2(3-\alpha)}(f*g)(z) \prec g(z) \tag{2.21}$$

where $0 \le \alpha < 1$,

$$\operatorname{Re}(f(z)) > -\left(\frac{3-\alpha}{2}\right), \quad z \in \mathbb{U}. \tag{2.22}$$

The constant factor $1/(3-\alpha)$ in (2.21) cannot be replaced by a larger number.

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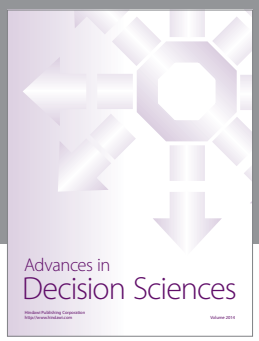

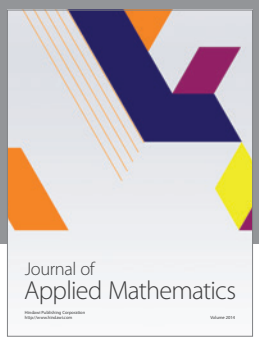

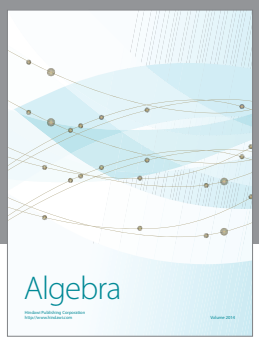

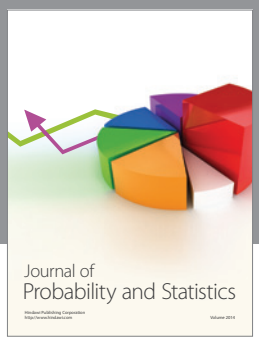

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