Research Article

On the q -Bernoulli Numbers and Polynomials with Weight α

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We present a systemic study of some families of higher-order q -Bernoulli numbers and polynomials with weight α . From these studies, we derive some interesting identities on the q -Bernoulli numbers and polynomials with weight α .

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p -adic rational integers, the field of p -adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p -adic norm of $\mathbb{C}_p$ is defined as $|x|_p = p^{-r}$, where $x = p^r m/n$ with $(p, m) = (p, n) = 1$, $r \in \mathbb{Q}$ and $m, n \in \mathbb{Z}$. Let $\mathbb{N}$ and $\mathbb{Z}$ be the set of natural numbers and integers, respectively, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-1/(p-1)}$. The notation of q -number is defined by $[x]_w = (1 - w^x)/(1 - w)$ and $[x]_q = (1 - q^x)/(1 - q)$, (see [1–13]).

As the well known definition, the Bernoulli polynomials are defined by

$$\frac{t}{e^t - 1} e^x = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \quad (1.1)$$

In the special case, $x = 0$, $B_n(0) = B_n$ are called the n th Bernoulli numbers. That is, the recurrence formula for the Bernoulli numbers is given by

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (1.2)$$

with the usual convention about replacing B^i with B_i .

In [1, 2], q -extension of Bernoulli numbers are defined by Carlitz as follows:

$$\beta_{0,q} = 1, \quad q(q\beta + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (1.3)$$

with the usual convention about replacing β^i with $\beta_{i,q}$.

By (1.2) and (1.3), we get $\lim_{q \rightarrow 1} \beta_{i,q} = B_i$. In this paper, we assume that $\alpha \in \mathbb{N}$.

In [7], the q -Bernoulli numbers with weight α are defined by Kim as follows:

$$\tilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q(q^\alpha \tilde{\beta}^\alpha + 1)^n - \tilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (1.4)$$

with the usual convention about replacing $(\tilde{\beta}^{(\alpha)})^i$ with $\tilde{\beta}_{i,q}^{(\alpha)}$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p -adic q -integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (1.5)$$

(see [4, 5]). From (1.5), we note that

$$q^n I_q(f_n) = I_q(f) + (q-1) \sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} q^l f'(l), \quad (1.6)$$

where $f_n(x) = f(x+n)$ and $f'(l) = (df(x)/dx)|_{x=l}$.

By (1.4), (1.5), and (1.6), we set

$$\tilde{\beta}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x), \quad \text{where } n \in \mathbb{Z}_+, \quad (1.7)$$

(see [7]). The q -Bernoulli polynomials are also given by

$$\tilde{\beta}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_q(x) = \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(\alpha)}. \quad (1.8)$$

The purpose of this paper is to derive a new concept of higher-order q -Bernoulli numbers and polynomials with weight α from the fermionic p -adic q -integral on $\mathbb{Z}_p$. Finally, we present a systemic study of some families of higher-order q -Bernoulli numbers and polynomials with weight α .

2. Higher Order q -Bernoulli Numbers with Weight α

Let $\beta \in \mathbb{Z}$ and $\alpha \in \mathbb{N}$ in this paper. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we consider the expansion of higher-order q -Bernoulli polynomials with weight α as follows:

$$\tilde{\beta}_{n,q}^{(\beta,k|\alpha)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{x_1(\beta-1) + \cdots + x_k(\beta-k)} d\mu_q(x_1) \cdots d\mu_q(x_k). \quad (2.1)$$

From (2.1), we note that

$$\begin{aligned} \tilde{\beta}_{n,q}^{(\beta,k|\alpha)}(x) &= \frac{(1-q)^{k-n}}{[\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\prod_{i=0}^{k-1} (\alpha l + \beta - i)}{\prod_{i=0}^{k-1} (1 - q^{\alpha l + \beta - i})} \\ &= \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{\alpha l + \beta}{k}_q k!}{\binom{\alpha l + \beta}{k}_q [k]_q!}, \end{aligned} \quad (2.2)$$

where $\binom{\alpha}{l}_q = ((1 - q^\alpha)(1 - q^{\alpha-1}) \cdots (1 - q^{\alpha-l+1})) / ((1 - q)(1 - q^2) \cdots (1 - q^l))$ and $[k]_q! = [k]_q \cdots [2]_q [1]_q$.

Therefore, we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, we have

$$\tilde{\beta}_{n,q}^{(\beta,k|\alpha)}(x) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{\alpha l + \beta}{k}_q k!}{\binom{\alpha l + \beta}{k}_q [k]_q!}. \quad (2.3)$$

In the special case, $x = 0$, $\tilde{\beta}_{n,q}^{(\beta,k|\alpha)}(0) = \tilde{\beta}_{n,q}^{(\beta,k|\alpha)}$ are called the n th higher order q -Bernoulli numbers with weight α .

From (2.1) and (2.2), we can derive

$$\tilde{\beta}_{n,q}^{(\beta,k|\alpha)} = (q^\alpha - 1)\tilde{\beta}_{n+1,q}^{(\beta-\alpha,k|\alpha)} + \tilde{\beta}_{n,q}^{(\beta-\alpha,k|\alpha)}. \quad (2.4)$$

By Theorem 2.1 and (2.4), we get

$$\begin{aligned} \tilde{\beta}_{0,q}^{(m\alpha,k|\alpha)} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^k (\alpha m - j)x_j} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \sum_{l=0}^m \binom{m}{l} (q^\alpha - 1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^l q^{-\sum_{j=1}^k jx_j} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \sum_{l=0}^m \binom{m}{l} (q^\alpha - 1)^l \tilde{\beta}_{l,q}^{(0,k|\alpha)} \\ &= \frac{(1-q)^k}{\prod_{i=0}^{k-1} (1 - q^{\alpha m - k + 1 + i})} = (1-q)^k \sum_{l=0}^k \binom{k+l-1}{l}_q q^{(\alpha m - k + 1)l}. \end{aligned} \quad (2.5)$$

From (2.1), we have

$$\begin{aligned} \sum_{j=0}^i \binom{i}{j} (q^\alpha - 1)^j \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^{n-i+j} q^{(\beta-\alpha-1)x_1 + \cdots + (\beta-\alpha-k)x_k} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^{n-i} q^{(\beta-1)x_1 + \cdots + (\beta-k)x_k} q^{\alpha(x_1 + \cdots + x_k)(i-1)} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ = \sum_{j=0}^{i-1} \binom{i-1}{j} (q^\alpha - 1)^j \tilde{\beta}_{n-i+j,q}^{(\beta,k|\alpha)}. \end{aligned} \quad (2.6)$$

Thus, we obtain the following theorem.

Theorem 2.2. For $i \in \mathbb{N}$, we have

$$\sum_{j=0}^i \binom{i}{j} (q^\alpha - 1)^j \tilde{\beta}_{n-i+j,q}^{(\beta-\alpha,k|\alpha)} = \sum_{j=0}^{i-1} \binom{i-1}{j} (q^\alpha - 1)^j \tilde{\beta}_{n-i+j,q}^{(\beta,k|\alpha)}. \quad (2.7)$$

It is easy to show that

$$\begin{aligned} \sum_{j=0}^m \binom{m}{j} (q^\alpha - 1)^j \tilde{\beta}_{j,q}^{(0,k|\alpha)} &= (1-q)^k \sum_{l=0}^k \binom{k+l-1}{l}_q q^{(\alpha m - k + 1)l} \\ &= \frac{(1-q)^k}{\prod_{i=0}^{k-1} (1 - q^{\alpha m - k + 1 + i})}. \end{aligned}$$

3. Polynomials $\tilde{\beta}_{n,q}^{(0,k|\alpha)}(x)$

In this section, we consider the polynomials $\tilde{\beta}_{n,q}^{(0,k|\alpha)}(x)$ as follows:

$$\tilde{\beta}_{n,q}^{(0,k|\alpha)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{-\sum_{j=1}^k jx_j} d\mu_q(x_1) \cdots d\mu_q(x_k). \quad (3.1)$$

From (3.1), we can easily derive the following equation:

$$\begin{aligned} \tilde{\beta}_{n,q}^{(0,k|\alpha)}(x) &= \frac{(1-q)^k}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\prod_{i=0}^k (\alpha l - i)}{\prod_{i=0}^{k-1} (1 - q^{\alpha l - i})} \\ &= \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{\alpha l}{k} k!}{\binom{\alpha l}{k}_q [k]_q!}. \end{aligned} \quad (3.2)$$

By (3.1) and (3.2), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^k (\alpha n - j)x_j + \alpha n x} d\mu_q(x_1) \cdots d\mu_q(x_k) = \sum_{l=0}^n \binom{n}{l} [\alpha]_q^l (q-1)^l \tilde{\beta}_{l,q}^{(0,k|\alpha)}(x), \quad (3.3)$$

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^k (\alpha n - l)x_l + \alpha n x} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \frac{q^{\alpha n x} (1-q)^k \left(\prod_{j=0}^{k-1} (\alpha n - j) \right)}{\prod_{j=0}^{k-1} (1 - q^{\alpha n - j})} = \frac{q^{\alpha n x} \binom{\alpha n}{k} k!}{\binom{\alpha n}{k}_q [k]_q!}. \end{aligned} \quad (3.4)$$

Therefore, by (3.3) and (3.4), we obtain the following theorem.

Theorem 3.1. For $n \in \mathbb{Z}_+$, we have

$$(1-q)^n \tilde{\beta}_{n,q}^{(0,k|\alpha)}(x) = \frac{1}{[\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\binom{\alpha l}{k} k!}{\binom{\alpha l}{k}_q [k]_q!}. \quad (3.5)$$

Moreover,

$$\sum_{l=0}^n \binom{n}{l} [\alpha]_q^l (q-1)^l \tilde{\beta}_{l,q}^{(0,k|\alpha)}(x) = \frac{q^{\alpha n x} \binom{\alpha n}{k} k!}{\binom{\alpha n}{k}_q [k]_q!}. \quad (3.6)$$

Let $d \in \mathbb{N}$. Then, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{-\sum_{j=1}^k j x_j} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{a_1, \dots, a_k=0}^{d-1} q^{-\sum_{j=2}^k (j-1)a_j} \\ & \quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\frac{x + \sum_{j=1}^k a_j}{d} + \sum_{i=1}^k x_i \right]_{q^{\alpha d}}^n q^{-d \sum_{j=1}^k j x_j} d\mu_{q^d}(x_1) \cdots d\mu_{q^d}(x_k). \end{aligned} \quad (3.7)$$

Thus, by (3.1) and (3.7), we obtain the following theorem.

Theorem 3.2. For $d, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, we have

$$\tilde{\beta}_{n,q}^{(0,k|\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{a_1, \dots, a_k=0}^{d-1} q^{-\sum_{j=2}^k (j-1)a_j} \tilde{\beta}_{n,q^\alpha}^{(0,k|\alpha)}\left(\frac{x + a_1 + \cdots + a_k}{d}\right). \quad (3.8)$$

From (3.1), we note that

$$\begin{aligned} \tilde{\beta}_{n,q}^{(0,k|\alpha)}(x) &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(0,k|\alpha)}, \\ \tilde{\beta}_{n,q}^{(0,k|\alpha)}(x+y) &= \sum_{l=0}^n \binom{n}{l} [y]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(0,k|\alpha)}(x). \end{aligned} \quad (3.9)$$

4. Polynomials $\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x)$

For $h \in \mathbb{Z}$, let us define weighted (h, q) -Bernoulli polynomials $\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x)$ as follows:

$$\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_q(x_1). \quad (4.1)$$

By (4.1), we easily see that

$$\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \frac{1}{[\alpha]_q^n (1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\alpha l + h}{[\alpha l + h]_q}. \quad (4.2)$$

Therefore, by (4.2), we obtain the following theorem.

Theorem 4.1. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \frac{1}{[\alpha]_q^n (1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\alpha l + h}{[\alpha l + h]_q}. \quad (4.3)$$

From (4.1), we can derive the following equation:

$$\begin{aligned} q^{\alpha x} \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_q(x_1) \\ = (q^\alpha - 1) \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^{n+1} q^{x_1(h-\alpha-1)} d\mu_q(x_1) + \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-\alpha-1)} d\mu_q(x_1). \end{aligned} \quad (4.4)$$

By (4.4), we easily get

$$q^{\alpha x} \tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = (q^\alpha - 1) \tilde{\beta}_{n+1,q}^{(h-\alpha-1,1|\alpha)}(x) + \tilde{\beta}_{n,q}^{(h-\alpha-1,1|\alpha)}(x). \quad (4.5)$$

From (4.1), we have

$$\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_q(x_1) = \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(h,1|\alpha)}, \quad (4.6)$$

where $\tilde{\beta}_{l,q}^{(h,1|\alpha)}(0) = \tilde{\beta}_{l,q}^{(h,1|\alpha)}$.

By (4.6), we get the following recurrence formula:

$$\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \left(q^{\alpha x} \tilde{\beta}_q^{(h,1|\alpha)} + [x]_{q^\alpha} \right)^n, \quad \text{for } n \geq 1, \quad (4.7)$$

with the usual convention about replacing $(\tilde{\beta}_q^{(h,1|\alpha)})^n$ with $\tilde{\beta}_{n,q}^{(h,1|\alpha)}$.

From (1.6), we note that

$$qI_q(f_1) = I_q(f) + (q-1)f(0) + \frac{q-1}{\log q} f'(0). \quad (4.8)$$

For $h \in \mathbb{Z}_+$, by (4.8), we have

$$q^h \int_{\mathbb{Z}_p} f(x+1) q^{(h-1)x} d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + (q-1)hf(0) + \frac{q-1}{\log q} f'(0). \quad (4.9)$$

If $h \in \{-1, -2, -3, \dots\}$, then we get

$$q^h \int_{\mathbb{Z}_p} f(x+1) q^{(h-1)x} d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + (1-q)hf(0) + \frac{q-1}{\log q} f'(0). \quad (4.10)$$

Let $h \in \mathbb{Z}_+$. By (4.9), we get

$$\begin{aligned} & q^h \int_{\mathbb{Z}_p} [x + x_1 + 1]_{q^\alpha}^n q^{(h-1)x_1} d\mu_q(x_1) - \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{(h-1)x_1} d\mu_q(x_1) \\ &= (q-1)h[x]_{q^\alpha}^n + \frac{\alpha}{[\alpha]_q} [x]_{q^\alpha}^{n-1} q^{\alpha x}. \end{aligned} \quad (4.11)$$

From (4.6) and (4.11), we note that

$$q^h \tilde{\beta}_{n,q}^{(h,1|\alpha)}(x+1) - \tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = (q-1)h[x]_{q^\alpha}^n + n \frac{\alpha}{[\alpha]_q} [x]_{q^\alpha}^{n-1} q^{\alpha x}. \quad (4.12)$$

If we take $x = 0$ in (4.12), then we have

$$\tilde{\beta}_{0,q}^{(h,1|\alpha)} = \frac{h}{[h]_q}, \quad q^h \tilde{\beta}_{n,q}^{(h,1|\alpha)}(1) - \tilde{\beta}_{n,q}^{(h,1|\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \quad (4.13)$$

Therefore, by (4.12) and (4.13), we obtain the following theorem.

Theorem 4.2. For $h \in \mathbb{Z}_+$, we have

$$\tilde{\beta}_{0,q}^{(h,1|\alpha)} = \frac{h}{[h]_q}, \quad q^h \tilde{\beta}_{n,q}^{(h,1|\alpha)}(1) - \tilde{\beta}_{n,q}^{(h,1|\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \quad (4.14)$$

By (4.7) and Theorem 4.2, we obtain the following corollary.

Corollary 4.3. For $h \in \mathbb{Z}_+$, we have

$$\tilde{\beta}_{0,q}^{(h,1|\alpha)} = \frac{h}{[h]_q}, \quad q^h \left(q^\alpha \tilde{\beta}_q^{(h,1|\alpha)} + 1 \right)^n - \tilde{\beta}_{n,q}^{(h,1|\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (4.15)$$

with the usual convention about replacing $(\tilde{\beta}_q^{(h,1|\alpha)})^n$ with $\tilde{\beta}_{n,q}^{(h,1|\alpha)}$.

From (4.1), we have

$$\tilde{\beta}_{0,q}^{(h,1|\alpha)} = \int_{\mathbb{Z}_p} q^{x_1(h-1)} d\mu_q(x_1) = \frac{h}{[h]_q}, \quad \text{if } h \in \mathbb{Z}_+. \quad (4.16)$$

It is not difficult to show that

$$\begin{aligned} \tilde{\beta}_{n,q^{-1}}^{(h,1|\alpha)}(1-x) &= \int_{\mathbb{Z}_p} [1-x+x_1]_{q^{-\alpha}}^n q^{-x_1(h-1)} d\mu_{q^{-1}}(x_1) \\ &= (-1)^n q^{\alpha n+h-1} \int_{\mathbb{Z}_p} [x+x_1]_{q^\alpha} q^{x_1(h-1)} d\mu_q(x_1) \\ &= (-1)^n q^{\alpha n+h-1} \tilde{\beta}_{n,q}^{(h,1|\alpha)}(x). \end{aligned} \quad (4.17)$$

Therefore, by (4.17), we obtain the following theorem.

Theorem 4.4. *For $h, n \in \mathbb{Z}_+$, we have*

$$\tilde{\beta}_{n,q^{-1}}^{(h,1|\alpha)}(1-x) = (-1)^n q^{\alpha n+h-1} \tilde{\beta}_{n,q}^{(h,1|\alpha)}(x). \quad (4.18)$$

For $x = 1$ in Theorem 4.4, we get

$$\begin{aligned} \tilde{\beta}_{n,q^{-1}}^{(h,1|\alpha)} &= (-1)^n q^{\alpha n+h-1} \tilde{\beta}_{n,q}^{(h,1|\alpha)}(1) \\ &= (-1)^n q^{\alpha n-1} \tilde{\beta}_{n,q}^{(h,1|\alpha)} \quad \text{if } n > 1. \end{aligned} \quad (4.19)$$

Therefore, by (4.19), we obtain the following corollary.

Corollary 4.5. *For $h \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ with $n > 1$, we have*

$$\tilde{\beta}_{n,q^{-1}}^{(h,1|\alpha)} = (-1)^n q^{\alpha n-1} \tilde{\beta}_{n,q}^{(h,1|\alpha)}. \quad (4.20)$$

Let $d \in \mathbb{N}$. By (4.1), we see that

$$\int_{\mathbb{Z}_p} q^{(h-1)x_1} [x+x_1]_{q^\alpha}^n d\mu_q(x_1) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} q^{ha} \int_{\mathbb{Z}_p} \left[\frac{x+a}{d} + x_1 \right]_{q^{ad}}^n q^{x_1(h-1)d} d\mu_{q^\alpha}(x_1). \quad (4.21)$$

By (4.1) and (4.21), we obtain the following equation:

$$\tilde{\beta}_{n,q}^{(h,1|\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} q^{ha} \tilde{\beta}_{n,q}^{(h,1|\alpha)}\left(\frac{x+a}{d}\right), \quad (4.22)$$

where $d \in \mathbb{N}$ and $h \in \mathbb{Z}_+$.

5. Polynomials $\tilde{\beta}_{n,q}^{(h,k|\alpha)}(x)$ and $h = k$

From (2.1), we note that

$$\begin{aligned} \tilde{\beta}_{n,q}^{(h,k|\alpha)}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{(\alpha l + h) \cdots (\alpha l + h - k + 1)}{[\alpha l + h]_q \cdots [\alpha l + h - k + 1]_q}, \end{aligned} \quad (5.1)$$

$$\begin{aligned} &q^h \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + 1 + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &\quad + (q-1)h \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_2 + \cdots + x_k]_{q^\alpha}^n q^{(h-1)x_2 + \cdots + (h-k)x_k} d\mu_q(x_2) \cdots d\mu_q(x_k) \\ &\quad + n \frac{\alpha}{[\alpha]_q} q^{\alpha x} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_2 + \cdots + x_k]_{q^\alpha}^{n-1} \\ &\quad \times q^{\alpha(h-1)x_2 + \cdots + \alpha(h-k+1)x_k} d\mu_q(x_2) \cdots d\mu_q(x_k). \end{aligned} \quad (5.2)$$

From (5.1), we have

$$q^h \tilde{\beta}_{n,q}^{(h,k|\alpha)}(x+1) = \tilde{\beta}_{n,q}^{(h,k|\alpha)}(x) + (q-1)h \tilde{\beta}_{n,q}^{(h-1,k-1|\alpha)}(x) + q^{\alpha x} \frac{n\alpha}{[\alpha]_q} \tilde{\beta}_{n,q}^{(h,k-1|\alpha)}(x). \quad (5.3)$$

Therefore, by (5.3), we obtain the following theorem.

Theorem 5.1. For $h, n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, we have

$$q^h \tilde{\beta}_{n,q}^{(h,k|\alpha)}(x+1) - \tilde{\beta}_{n,q}^{(h,k|\alpha)}(x) = (q-1)h \tilde{\beta}_{n,q}^{(h-1,k-1|\alpha)}(x) + nq^{\alpha x} \frac{\alpha}{[\alpha]_q} \tilde{\beta}_{n,q}^{(h,k-1|\alpha)}(x). \quad (5.4)$$

It is easy to show that

$$\begin{aligned}
& q^{\alpha x} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{hx_1 + (h-1)x_2 + \cdots + (h+1-k)x_k} d\mu_q(x_1) \cdots d\mu_q(x_k) \\
&= (q^\alpha - 1) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^{n+1} q^{(h-\alpha)x_1 + \cdots + (h-\alpha+1-k)x_k} d\mu_q(x_1) \cdots d\mu_q(x_k) \\
&\quad + \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(h-\alpha)x_1 + \cdots + (h-\alpha+1-k)x_k} d\mu_q(x_1) \cdots d\mu_q(x_k) \\
&= (q^\alpha - 1) \tilde{\beta}_{n+1,q}^{(h+1-\alpha,k|\alpha)}(x) + \tilde{\beta}_{n,q}^{(h+1-\alpha,k|\alpha)}(x).
\end{aligned} \tag{5.5}$$

Thus, by (5.5), we obtain the following proposition.

Proposition 5.2. *For $h, n \in \mathbb{Z}_+$, we have*

$$q^{\alpha x} \tilde{\beta}_{n,q}^{(h+1,k|\alpha)}(x) = (q^\alpha - 1) \tilde{\beta}_{n+1,q}^{(h+1-\alpha,k|\alpha)}(x) + \tilde{\beta}_{n,q}^{(h+1-\alpha,k|\alpha)}(x). \tag{5.6}$$

For $d \in \mathbb{N}$, we get

$$\begin{aligned}
& \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[x + \sum_{j=1}^k x_j \right]_{q^\alpha}^n q^{\sum_{j=1}^k (h-j)x_j} d\mu_q(x_1) \cdots d\mu_q(x_k) \\
&= \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{a_1, \dots, a_k=0}^{d-1} q^{h \sum_{j=1}^k a_j - \sum_{j=2}^k (j-1)a_j} \\
&\quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\frac{x + \sum_{j=1}^k a_j}{d} + \sum_{j=1}^k x_j \right]_{q^{\alpha d}}^n q^{d \sum_{j=1}^k (h-j)x_j} d\mu_{q^d}(x_1) \cdots d\mu_{q^d}(x_k).
\end{aligned} \tag{5.7}$$

Thus, we obtain

$$\tilde{\beta}_{n,q}^{(h,k|\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q^k} \sum_{a_1, \dots, a_k=0}^{d-1} q^{h \sum_{j=1}^k a_j - \sum_{j=2}^k (j-1)a_j} \tilde{\beta}_{n,q^d}^{(h,k|\alpha)}\left(\frac{x + a_1 + \cdots + a_k}{d}\right). \tag{5.8}$$

Equation (5.8) is multiplication formula for the q -Bernoulli polynomials of order (h, k) with weight α .

Let us define $\tilde{\beta}_{n,q}^{(k,k|\alpha)}(x) = \tilde{\beta}_{n,q}^{(k|\alpha)}(x)$. Then we see that

$$\tilde{\beta}_{n,q}^{(k|\alpha)}(x) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{(\alpha l + k) \cdots (\alpha l + 1)}{[\alpha l + k]_q \cdots [\alpha l + 1]_q}, \quad (5.9)$$

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [k - x + x_1 + \cdots + x_k]_q^n q^{-(k-1)x_1 - \cdots - (k-k)x_k} d\mu_{q^{-1}}(x_1) \cdots d\mu_{q^{-1}}(x_k) \\ &= \frac{(-1)^n q^{\alpha n - k + \binom{k+1}{2}}}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{(\alpha l + k) \cdots (\alpha l + 1)}{[\alpha l + k]_q \cdots [\alpha l + 1]_q} \\ &= (-1)^n q^{\alpha n - k + \binom{k+1}{2}} \tilde{\beta}_{n,q}^{(k|\alpha)}(x). \end{aligned} \quad (5.10)$$

Therefore, by (5.9) and (5.10), we obtain the following theorem.

Theorem 5.3. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, we have

$$\tilde{\beta}_{n,q^{-1}}^{(k|\alpha)}(k - x) = (-1)^n q^{\alpha n - k + \binom{k+1}{2}} \tilde{\beta}_{n,q}^{(k|\alpha)}(x). \quad (5.11)$$

Let $x = k$ in Theorem 5.3. Then we see that

$$\tilde{\beta}_{n,q^{-1}}^{(k|\alpha)} = (-1)^n q^{\alpha n - k + \binom{k+1}{2}} \tilde{\beta}_{n,q}^{(k|\alpha)}(k). \quad (5.12)$$

From Theorem 5.1, we can derive the following equation:

$$q^k \tilde{\beta}_{n,q}^{(k|\alpha)}(x+1) - \tilde{\beta}_{n,q}^{(k|\alpha)}(x) = k(q-1) \tilde{\beta}_{n,q}^{(k-1|\alpha)}(x) + nq^{\alpha x} \frac{\alpha}{[\alpha]_q} \tilde{\beta}_{n,q}^{(k,k-1|\alpha)}(x). \quad (5.13)$$

By (5.9), we easily get

$$\tilde{\beta}_{n,q}^{(k|\alpha)} = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{(\alpha l + k) \cdots (\alpha l + 1)}{[\alpha l + k]_q \cdots [\alpha l + 1]_q}. \quad (5.14)$$

From the definition of p -adic q -integral on $\mathbb{Z}_p$, we note that

$$\begin{aligned} & \sum_{l=0}^n \binom{n}{l} (q^\alpha - 1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^l q^{\sum_{i=0}^{k-1} (k-l)x_i} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \frac{(\alpha n + k) \cdots (\alpha n + 1)}{[\alpha n + k]_q \cdots [\alpha n + 1]_q} = \frac{\binom{\alpha n}{k} k!}{\binom{\alpha n}{k}_q [k]_q!}. \end{aligned} \quad (5.15)$$

Thus, by (5.15), we get

$$\sum_{l=0}^n \binom{n}{l} (q^\alpha - 1)^l \tilde{\beta}_{n,q}^{(k|\alpha)} = \frac{(\frac{\alpha n}{k})_q k!}{(\frac{\alpha n}{k})_q [k]_q!}. \quad (5.16)$$

By the definition of polynomial $\tilde{\beta}_{n,q}^{(k|\alpha)}(x)$, we see that

$$\begin{aligned} \tilde{\beta}_{n,q}^{(k|\alpha)}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(k-1)x_1 + \cdots + (k-k)x_k} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \sum_{l=0}^n \binom{n}{l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(k|\alpha)} [x]_q^{n-l} = \left(q^{\alpha x} \tilde{\beta}_q^{(k|\alpha)} + [x]_{q^\alpha} \right)^n, \quad \text{where } n \in \mathbb{Z}_+, \end{aligned} \quad (5.17)$$

with the usual convention about replacing $(\tilde{\beta}_q^{(k|\alpha)})^n$ with $\tilde{\beta}_{n,q}^{(k|\alpha)}$.

Let $x = 0$ in (5.13). Then we have

$$q^k \tilde{\beta}_{n,q}^{(k|\alpha)}(1) - \tilde{\beta}_{n,q}^{(k|\alpha)} = k(q-1) \tilde{\beta}_{n,q}^{(k-1|\alpha)} + \frac{n\alpha}{[\alpha]_q} \tilde{\beta}_{n,q}^{(k,k-1|\alpha)}. \quad (5.18)$$

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