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Research Article

Statistical Convergence in Function Spaces

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We study statistical versions of several classical kinds of convergence of sequences of functions between metric spaces (Dini, Arzelà, and Alexandroff) in different function spaces. Also, we discuss a statistical approach to recently introduced notions of strong uniform convergence and exhaustiveness.

1. Introduction

One of central questions in analysis is what precisely must be added to pointwise convergence of a sequence of continuous functions to preserve continuity of the limit function? In 1841, Weierstrass discovered that uniform convergence yields continuity of the limit function. Dini had given in 1878 a sufficient condition, weaker than uniform convergence, for continuity of the limit function. In 1883/1884, Arzelà [1] found out a necessary and sufficient condition under which the pointwise limit of a sequence of real-valued continuous functions on a compact interval is continuous. He called this condition "uniform convergence by segments" ("convergenza uniforme a tratti") [2], and his work initiated a study that led to several outstanding papers. In 1905, Borel in [3] introduced the term "quasiuniform convergence" for the Arzelà condition, and Bartle in [4] extended Arzelà's result to nets of real-valued continuous functions on a topological space. In 1948, Alexandroff studied the question for a sequence of continuous functions from a topological space X, not necessarily compact, to a metric space [5]. The reader may consult [6, 7] for the literature concerning the preservation of continuity of the limit function.

In 2009, Beer and Levi [8] proposed a new approach to this investigation, in the realm of metric spaces, through the notion of strong uniform convergence on bornologies, when this bornology reduces to that of all nonempty finite subsets of X. In [6], a direct proof of the equivalence of Arzelà, Alexandroff, and Beer-Levi conditions was offered.

In [9], Caserta and Kočinac proposed a new model to investigate convergence in function spaces: the statistical one. Actually they obtained results parallel to the classical ones in spite of the fact that statistical convergence has a mild control of the whole set of functions. One of the main goals of this paper is to continue their analysis. In Section 3, we prove that continuity of the limit of a sequence of functions is equivalent to several modes of statistical convergence which are similar to but weaker than the classical ones, namely, Arzelà, Alexandroff, and Beer-Levi. Moreover, we state the novel notion of statistically strong Arzelà convergence, the appropriate tool to investigate strong uniform continuity of the limit of a sequence of strongly uniformly continuous functions, a concept introduced in [8].

In 2008, the definition of exhaustiveness, closely related to equicontinuity [10], for a family of functions (not necessarily continuous), was introduced by Gregoriades and Papanastassiou in [11]. Exhaustiveness describes convergence of a net of functions in terms of properties of the whole net and not of properties of the functions as single members. Thus, statistical versions of exhaustiveness and its variations are natural and the investigation in this direction was initiated by Caserta and Kočinac in [9]. In Section 4, we continue this study and provide additional information about exhaustiveness and its variations. First, we analyze the exact location of exhaustiveness. In fact, in [11] it was shown that equicontinuity implies exhaustiveness. We prove that exhaustiveness lies between equicontinuity and even continuity [10], a classical property weaker than equicontinuity. Furthermore, we propose a notion of statistical uniform exhaustiveness of a sequence of functions which is the appropriate device to study uniform convergence.

2. Notation and Preliminaries

Throughout the paper, (X,d) and (Y,ρ) will be metric spaces, Y^X and $\mathsf{C}(X,Y)$ the sets of all in all continuous mappings from X to Y. The pointwise (resp., uniform) topology on Y^X and $\mathsf{C}(X,Y)$ will be denoted by τ_p (resp. τ_u). We denote by $\mathcal{P}_0(X)$ the family of all nonempty subsets of X, and by $\mathcal{F}(X)$, or simply $\mathcal{F}$, the family of all nonempty finite subsets of X. If $x_0 \in (X,d)$, $A \subset X \setminus \{\emptyset\}$, and $\varepsilon > 0$, we write $S(x_0,\varepsilon)$ for the open ε-ball with center x_0, and $A^\varepsilon = \bigcup_{x\in A} S_\varepsilon(x)$ for the ε-enlargement of A.

Recall that a bornology $\mathfrak{B}$ on a space (X,d) is a hereditary family of subsets of X which covers X and is closed under taking finite unions (see [12, 13]). By a base for a bornology $\mathfrak{B}$, we mean a subfamily $\mathfrak{B}_0$ of $\mathfrak{B}$ that is cofinal with respect to inclusion. The smallest bornology on X is the family $\mathcal{F}(X)$, and the largest is the family $\mathcal{P}_0(X)$.

In [8], as mentioned above, the notions of strong uniform continuity of a function on a bornology $\mathfrak{B}$ and the topology of strong uniform convergence on $\mathfrak{B}$ for function spaces were introduced.

Definition 2.1 (see [8]). Let (X,d) and (Y,ρ) be metric spaces, and let B be a subset of X. A function $f : X \to Y$ is *strongly uniformly continuous on* B if for each $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x,z) < \delta$ and $\{x,z\} \cap B \neq \emptyset$, then $\rho(f(x),f(z)) < \varepsilon$.

If $\mathfrak{B}$ is a family of nonempty subsets of X and (Y,ρ) a metric space, a function $f \in Y^X$ is called uniformly continuous (resp., strongly uniformly continuous) on $\mathfrak{B}$ if for each $B \in \mathfrak{B}$, $f \restriction B$ is uniformly continuous (resp., strongly uniformly continuous) on B. We denote by $\mathsf{C}(X,Y)^s_{\mathfrak{B}}$ the set of all strongly uniformly continuous functions on $\mathfrak{B}$.

Given a bornology $\mathfrak{B}$ with closed base on X, Beer and Levi presented a new uniformizable topology on the set Y^X.

Definition 2.2 (see [8]). Let (X, d) and (Y, ρ) be metric spaces, and let $\mathfrak{B}$ be a bornology with a closed base on X. The *topology* $\tau^s_{\mathfrak{B}}$ *of strong uniform convergence* is determined by the uniformity on Y^X having as a base all sets of the form

$$[B;\varepsilon]^s := \left\{(f,g) : \exists\delta > 0 \text{ for each } x \in B^\delta \rho(f(x), g(x)) < \varepsilon\right\}, \quad (B \in \mathfrak{B}, \varepsilon > 0). \tag{2.1}$$

On $\mathsf{C}(X, Y)$, this topology is in general finer than the classical topology of uniform convergence on $\mathfrak{B}$. This new function space has been intensively studied in [6, 8, 14–16].

Let us recall some classical definitions and results.

Definition 2.3 (Arzelà (see [1], [7, page 268])). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued continuous functions defined on an arbitrary set X, and let $f : X \to \mathbb{R}$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *converge to f quasiuniformly* on X if it pointwise converges to f, and for each $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, there exists a finite number of indices $n_1, n_2, \ldots, n_k \geq n_0$ such that for each $x \in X$ at least one of the following inequalities holds:

$$\left|f_{n_i}(x) - f(x)\right| < \varepsilon, \quad i = 1, \ldots, k. \tag{2.2}$$

Definition 2.4 (Alexandroff [5]). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{C}(X, Y)$ and $f \in Y^X$. Then $(f_n)_{n\in\mathbb{N}}$ is *Alexandroff convergent to f* on X, provided it pointwise converges to f, and for each $\varepsilon > 0$ and each $n_0 \in \mathbb{N}$, there exist a countable open cover $\{U_1, U_2, \ldots\}$ of X and a sequence $(n_k)_{k\in\mathbb{N}}$, of positive integers greater than n_0 such that for each $x \in U_k$ we have $\rho(f_{n_k}(x), f(x)) < \varepsilon$.

Theorem 2.5 (see [6]). *If a net $(f_\alpha)_{\alpha\in D}$ in $\mathsf{C}(X, Y)$ pointwise converges to $f \in Y^X$, then the following are equivalent:*

(i) *f is continuous;*

(ii) *$(f_\alpha)_{\alpha\in D}$ Alexandroff converges to f;*

(iii) *$(f_\alpha)_{\alpha\in D}$ converges to f quasiuniformly on compacta;*

(iv) *$(f_\alpha)_{\alpha\in D}$ $\tau^s_{\mathfrak{F}}$-converges to f.*

In the next section, we will show that similar results about continuity of the limit function are true for statistical pointwise convergence of sequences of functions between two metric spaces.

The idea of statistical convergence appeared, under the name almost convergence, in the first edition (Warsaw, 1935) of the celebrated monograph [17] of Zygmund. Explicitly, the notion of statistical convergence of sequences of real numbers was introduced by Fast in [18] and Steinhaus in [19] and is based on the notion of *asymptotic density* $\partial(A)$ of a set $A \subset \mathbb{N}$:

$$\partial(A) = \lim_{n\to\infty} \frac{|\{k \in A : k \leq n\}|}{n}. \tag{2.3}$$

We recall that $\partial(\mathbb{N} \setminus A) = 1 - \partial(A)$ for $A \subset \mathbb{N}$. A set $A \subset X$ is said to be *statistically dense* if $\partial(A) = 1$.

Fact 1. The union and intersection of two statistically dense sets in $\mathbb{N}$ are also statistically dense.

Statistical convergence has many applications in different fields of mathematics: number theory, summability theory, trigonometric series, probability theory, measure theory, optimization, approximation theory, and so on. For more information, see [20] (where statistical convergence was generalized to sequences in topological and uniform spaces) and references therein, and about some applications see [21, 22].

A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X *statistically converges* (or shortly, st-*converges*) to $x \in X$ if for each neighborhood U of x, $\partial(\{n \in \mathbb{N} : x_n \notin U\}) = 0$ [20]. This will be denoted by $(x_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau}{\to} x$, where τ is a topology on X.

It was shown in [20, Theorem 2.2] (see [23, 24] for $X = \mathbb{R}$) that for first countable spaces this definition is equivalent to the following statement.

Fact 2. There exists a subset A of $\mathbb{N}$ with $\partial(A) = 1$ such that the sequence $(x_n)_{n\in A}$ converges to x.

Facts 1 and 2 will be used in the sequel without special mention.

The reader is referred to [7, 10, 25–27] for standard notation and terminology.

3. Statistical Arzelà and Alexandroff Convergence

In [9], a statistical version of the Alexandroff convergence was defined.

Definition 3.1. A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathsf{C}(X,Y)$ is said to be *statistically Alexandroff convergent to* $f \in Y^X$, denoted by $(f_n)_{n\in\mathbb{N}} \overset{\text{st-Al}}{\to} f$, provided $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_p}{\to} f$, and for each $\varepsilon > 0$ and each statistically dense set $A \subset \mathbb{N}$, there exist an open cover $\mathcal{U} = \{U_n : n \in A\}$ and an infinite set $M_A = \{n_1 < n_2 < \cdots n_k < \cdots\} \subset A$ such that for each $x \in U_k$ we have $\rho(f_{n_k}(x), f(x)) < \varepsilon$.

Below, a statistical version of the celebrated Arzelà's quasiuniform convergence is given.

Definition 3.2. A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathsf{C}(X,Y)$ is said to be *statistically Arzelà convergent to* $f \in Y^X$, denoted by $(f_n)_{n\in\mathbb{N}} \overset{\text{st-Arz}}{\to} f$, if $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_p}{\to} f$, and for each $\varepsilon > 0$ and each statistically dense set $A \subset \mathbb{N}$ there exists a finite set $\{n_1, n_2, \ldots, n_k\} \subset A$ such that for each $x \in X$ it holds that $\rho(f_{n_i}(x), f(x)) < \varepsilon$ for at least one $i \leq k$.

Theorem 3.3. *For a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathsf{C}(X,Y)$ such that $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_p}{\to} f \in Y^X$, the following is equivalent:*

(1) *f is continuous;*

(2) *$(f_n)_{n\in\mathbb{N}} \overset{\text{st-Arz}}{\to} f$ on compacta;*

(3) *$(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau^s_{\mathcal{F}}}{\to} f$;*

(4) *$(f_n)_{n\in\mathbb{N}} \overset{\text{st-Al}}{\to} f$.*

Proof. (1) $\Rightarrow$ (2): Let a compact set $K \subset X$, a statistically dense set $A \subset \mathbb{N}$, and $\varepsilon > 0$ be fixed. Since $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_p}{\to} f$, for each $y \in X$ there is a statistically dense set $A_y \subset \mathbb{N}$ such that $\rho(f_n(y), f(y)) < \varepsilon$ for each $n \in A_y$. Choose $n_y \in A_y \cap A$ and set

$$U_y = \left\{x \in X : \rho\left(f_{n_y}(x), f(x)\right) < \varepsilon\right\}. \tag{3.1}$$

Since all functions f_n and f are continuous, the sets U_y are open, and thus $\{U_y : y \in K\}$ is an open cover of K. By compactness of K there are $y_1, y_2, \ldots, y_k \in K$ such that $K = \bigcup_{i=1}^{k} U_{y_i}$. The set $\{n_{y_i} : i = 1, 2, \ldots, k\}$ is a finite subset of A such that for each $x \in K$ it holds $\rho(f_{n_{y_i}}(x), f(x)) < \varepsilon$ for at least one $i \le k$, that is, (2) is true.

(2) $\Rightarrow$ (3): It suffices to show that for each $x \in X$ and each $\varepsilon > 0$ we have $\partial(\{n \in \mathbb{N} : f_n \notin [\{x\}, \varepsilon]^s(f)\}) = 0$. Since $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_p}{\to} f$, there is a set $A \subset \mathbb{N}$ with $\partial(A) = 1$ so that $\rho(f_n(x), f(x)) < \varepsilon/4$ for each $n \in A$. We are going to prove that for each $n \in A$ there is $\delta_n > 0$ such that for each $y \in S(x, \delta_n)$, $\rho(f_n(y), f(y)) < \varepsilon$. Suppose, by contradiction, that this assumption fails. Then there is $n_0 \in A$ and a sequence $(x_j)_{j\in\mathbb{N}}$ converging to x such that $\rho(f_{n_0}(x_j), f(x_j)) \ge \varepsilon$ for each $j \in A$. The set $K = \{x_j : j \in \mathbb{N}\} \cup \{x\}$ is compact so that, by (2), there are $m_1, \cdots, m_k \in A$ such that for each $z \in K$, $\rho(f_{m_i}(z), f(z)) < \varepsilon/4$ holds for at least one $i \le k$. Therefore, we found $i \le k$ such that there is an infinite set $\mathsf{C} \subset K$ with the property that for each $z \in \mathsf{C}$, $\rho(f_{m_i}(z), f(z)) < \varepsilon/4$. For this m_i, we have

$$\rho\left(f_{m_i}(x), f_{n_0}(x)\right) \le \rho\left(f_{m_i}(x), f(x)\right) + \rho\left(f(x), f_{n_0}(x)\right) < \frac{\varepsilon}{2}. \tag{3.2}$$

Since f_{m_i} and f are continuous at x, there are $\delta_{m_i} > 0$ and $\delta_0 > 0$ such that $\rho(f_{m_i}(u), f_{m_i}(x)) < \varepsilon/8$ for each $u \in S(x, \delta_{m_i})$, and $\rho(f_{n_0}(u), f_{n_0}(x)) < \varepsilon/8$ for each $u \in S(x, \delta_0)$. If $\delta = \min\{\delta_0, \delta_{m_i}\}$, then for each $z \in \mathsf{C} \cap S(x, \delta)$ we have

$$\rho\left(f_{m_i}(z), f_{n_0}(z)\right) \le \rho\left(f_{m_i}(z), f_{m_i}(x)\right) + \rho\left(f_{m_i}(x), f_{n_0}(x)\right) + \rho\left(f_{n_0}(x), f_{n_0}(z)\right) < \frac{3\varepsilon}{4}. \tag{3.3}$$

Since $(x_j)_{j\in\mathbb{N}}$ converges to x, there is $j^* \in A$ such that $x_{j^*} \in C \cap S(x, \delta)$. For this j^*, we have

$$\rho\left(f_{n_0}(x_{j^*}), f(x_{j^*})\right) \le \rho\left(f_{n_0}(x_{j^*}), f_{m_i}(x_{j^*})\right) + \rho\left(f_{m_i}(x_{j^*}), f(x_{j^*})\right) < \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \tag{3.4}$$

which is a contradiction.

(3) $\Rightarrow$ (4): Let $\varepsilon > 0$ and a statistically dense set $A \subset \mathbb{N}$ be given. Since $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_{\mathcal{F}}^s}{\to} f$, given $[\{x\}, \varepsilon]^s(f)$, there is $B_x \subset \mathbb{N}$ statistically dense, such that for each $n \in B_x$ we have $f_n \in [\{x\}, \varepsilon]^s(f)$. Hence there is a $\delta_{n,x}$ such that for each $y \in S(x, \delta_{n,x})$ and each $n \in B_x$ we have $\rho(f_n(y), f(y)) < \varepsilon$. Let $B = \bigcup_{x\in X} B_x$. For each $n \in B \cap A$, define

$$E_n := \left\{x \in X : \forall m \in B_x \cap A, m \ge n, \rho\left(f_m(y), f(y)\right) < \varepsilon \ \forall y \in S(x, \delta_{n,x})\right\}. \tag{3.5}$$

Note that $X = \bigcup_{n\in A} E_n$. For each $n \in A$, let U_n be the following open set:

$$U_n = \begin{cases} \emptyset, & \text{if } n \in A \setminus B, \\ \bigcup_{x\in E_n} S(x, \delta_{n,x}), & \text{if } n \in A \cap B. \end{cases} \tag{3.6}$$

Then $\{U_n : n \in A\}$ is an open cover of X. Thus for each $k \in A$ and each $x \in U_k$, there is some $m \in B$ such that it holds that $\rho(f_m(x), f(x)) < \varepsilon$, that is, the set $A \cap B = \{n_1 < n_2 < \cdots\} \subset A$ and the cover $\{U_n : n \in A\}$ witness that (4) is true.

(4) $\Rightarrow$ (1): It is proved in [9, Theorem 4.7]. □

The following two theorems use other kinds of statistical convergence, related to Dini convergence [28][29, pages 105-106] and Arzelà convergence, which imply continuity and strong uniform continuity of the limit function.

Definition 3.4. A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathsf{C}(X, Y)$ is said to be *statistically Dini convergent to* $f \in Y^X$, denoted by $(f_n)_{n\in\mathbb{N}} \overset{\text{st-Di}}{\to} f$ if $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_p}{\to} f$ and for each $\varepsilon > 0$ and each statistically dense set $A \subset \mathbb{N}$ there exists an increasing sequence $m_1 < m_2 < \cdots$ in A such that $\rho(f_{m_i}(x), f(x)) < \varepsilon$ for each $x \in X$ and each $i \in \mathbb{N}$.

Theorem 3.5. *If a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathsf{C}(X, Y)$ statistically Dini converges to $f \in Y^X$, then f is continuous.*

Proof. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Since $(f_n(x_0))_{n\in\mathbb{N}} \overset{\text{st}}{\to} f(x_0)$, there is a statistically dense set $A \subset \mathbb{N}$ such that $\rho(f_n(x_0), f(x_0)) < \varepsilon/3$ for each $n \in A$. Because $(f_n)_{n\in\mathbb{N}} \overset{\text{st-Di}}{\to} f$, there exists an increasing sequence $m_1 < m_2 < \cdots$ in A such that $\rho(f_{m_i}(x), f(x)) < \varepsilon$ for each $x \in X$ and each $i \in \mathbb{N}$. Take some n_k. Since f_{n_k} is continuous at x_0, there is $\delta > 0$ such that $\rho(f_{n_k}(x), f_{n_k}(x_0)) < \varepsilon/3$ whenever $d(x, x_0) < \delta$. So, for each $x \in S(x_0, \delta)$, we have

$$\rho(f(x), f(x_0)) \le \rho(f(x), f_{m_k}(x)) + \rho(f_{m_k}(x), f_{m_k}(x_0)) + \rho(f_{m_k}(x_0), f(x_0)) < \varepsilon, \tag{3.7}$$

that is, f is continuous at x_0, hence on X. □

Definition 3.6. A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathsf{C}(X, Y)$ *statistically strongly Arzelà converges* to a function $f \in Y^X$ on a bornology $\mathfrak{B}$ on X, denoted by $(f_n)_{n\in\mathbb{N}} \overset{\text{st-s.Arz}}{\longrightarrow} f$, if $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_{\mathfrak{B}}}{\to} f$ and for each $B \in \mathfrak{B}$, $\varepsilon > 0$ and each statistically dense set A in $\mathbb{N}$, there are finitely many $n_1, \ldots, n_k \in A$ such that $f_{n_i} \in [B, \varepsilon]^s(f)$ for at least one $i \le k$.

Theorem 3.7. *If a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathsf{C}^s_{\mathfrak{B}}(X, Y)$ statistically strongly Arzelà converges to $f \in Y^X$ on a bornology $\mathfrak{B}$ with closed base on X, then f is a strongly uniformly continuous on $\mathfrak{B}$.*

Proof. Let $B \in \mathfrak{B}$ and $\varepsilon > 0$. As $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_{\mathfrak{B}}}{\to} f$, there is a statistically dense set $A \subset \mathbb{N}$ such that for each $n \in$ A we have $f_n \in [B, \varepsilon/3](f)$, that is, for each $x \in B$, $\rho(f_n(x), f(x)) < \varepsilon/3$. By assumption, there are $n_1, \ldots, n_k \in A$ with $f_{n_m} \in [B, \varepsilon/3]^s(f)$ for some $m \le k$, that is, there exists $\delta_m > 0$ such that $\rho(f_{n_m}(x), f(x)) < \varepsilon/3$ for each $x \in B^{\delta_m}$. Since f_{n_m} is strongly uniformly continuous on B, there is $\delta_0 > 0$ so that for each $x \in B$ and each $y \in B^{\delta_0}$ with $d(x, y) < \delta_0$,

we have $\rho(f_{n_m}(x), f_{n_m}(y)) < \varepsilon/3$. Set $\delta = \min\{\delta_m, \delta_0\}$. Then for each $x \in B$ and $y \in X$ with $d(x,y) < \delta$ by the above relations, it follows

$$\rho(f(x), f(y)) \leq \rho(f(x), f_{n_m}(x)) + \rho(f_{n_m}(x), f_{n_m}(y)) + \rho(f_{n_m}(y), f(y)) < \varepsilon, \tag{3.8}$$

that is, f is strongly uniformly continuous on B, hence on $\mathfrak{B}$. □

Theorem 3.8. *Let X be a compact space, $\mathfrak{B}$ a bornology on X with closed base, and $(f_n)_{n\in\mathbb{N}}$ a sequence in $\mathsf{C}^s_{\mathfrak{B}}(X,Y)$. If $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_{\mathfrak{B}}}{\to} f$ and f is strongly uniformly continuous on $\mathfrak{B}$, then $(f_n)_{n\in\mathbb{N}}$ statistically strongly Arzelà converges to f on $\mathfrak{B}$.*

Proof. Let $B \in \mathfrak{B}$, $\varepsilon > 0$, and a statistically dense set $A \subset \mathbb{N}$ be given. Since f is strongly uniformly continuous on B, there is $\delta_0 > 0$ such that for each $x \in B$ and each $y \in B^{\delta_0}$ with $d(x,y) < \delta_0$ we have $\rho(f(x), f(y)) < \varepsilon/3$. From $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_{\mathfrak{B}}}{\to} f$, it follows that there is a statistically dense set $B \subset \mathbb{N}$ such that for each $n \in C$, $C = A \cap B$, we have $f_n \in [B, \varepsilon/3](f)$, that is, for each $n \in C$, and each $x \in B$ it holds that $\rho(f_n(x), f(x)) < \varepsilon/3$. For each $n \in C$ set $U_n = \{x \in X : \rho(f_n(x), f(x)) < \varepsilon/3\}$. Since f and f_n's are continuous, each U_n is open in X, so that $\{U_n : n \in \mathbb{N}\}$ is an open cover of X. By compactness of X, there are finitely many $n_1, \ldots, n_k \in \mathsf{C}$ such that $X = \bigcup_{i=1}^k U_{n_i}$. But each f_{n_i} is strongly uniformly continuous on B, so that for each $i \leq k$ there is $\delta_i > 0$ such that $\rho(f_{n_i}(x), f_{n_i}(y)) < \varepsilon/3$ whenever $x \in B$ and $y \in B^{\delta_i}$, $\rho(x,y) < \delta_i$. Let $\delta = \min\{\delta_0, \delta_1, \ldots, \delta_k\}$. Then for $x \in B$ and $y \in B^{\delta}$ with $d(x,y) < \delta$, since $x \in U_{n_m}$ for some $m \leq k$, we have

$$\rho(f_{n_m}(y), f(y)) \leq \rho(f_{n_m}(y), f_{n_m}(x)) + \rho(f_{n_m}(x), f(x)) + \rho(f(x), f(y)) < \varepsilon. \tag{3.9}$$

So, $f_{n_m} \in [B, \varepsilon]^s(f)$ which completes the proof. □

Theorem 3.9. *Let $\mathfrak{B}$ be a bornology on X with closed base, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{C}^s_{\mathfrak{B}}(X,Y)$ such that $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_{\mathfrak{B}}}{\to} f$. Then f is strongly uniformly continuous on $\mathfrak{B}$ if and only if $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau^s_{\mathfrak{B}}}{\to} f$.*

Proof. By [8, Proposition 6.5], we have $\tau^s_{\mathfrak{B}} = \tau_{\mathfrak{B}}$. So, it suffices to prove that $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau^s_{\mathfrak{B}}}{\to} f$ implies strong uniform continuity of f on $\mathfrak{B}$.

Assume that f is not strongly uniformly continuous on $\mathfrak{B}$. There are a $B \in \mathfrak{B}$ and $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there are points $x_n, z_n \in B^{1/n}$ with $d(x_n, z_n) < 1/n$ such that $\rho(f(x_n), f(z_n)) \geq \varepsilon$. Since $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau^s_{\mathfrak{B}}}{\to} f$, the density of the set $A = \{n \in \mathbb{N} : f_n \notin [B, \varepsilon/3]^s(f)\}$ is 0. Let $M = \mathbb{N} \setminus A$. Then M is statistically dense in $\mathbb{N}$, and there exist $m \in M$, $x_m, z_m \in B^{1/m}$, $d(x_m, z_m) < 1/m$, such that $\rho(f(x_m), f_m(x_m)) < \varepsilon/3$, and $\rho(f(z_m), f_m(z_m)) < \varepsilon/3$. Thus

$$\varepsilon \leq \rho(f(x_m), f(z_m)) \leq \rho(f(x_m), f_m(x_m)) + \rho(f_m(x_m), f_m(z_m)) + \rho(f_m(z_m), f(z_m)), \tag{3.10}$$

and so

$$\rho(f_m(x_m), f_m(z_m)) \geq \varepsilon - \rho(f(x_m), f_m(x_m)) - \rho(f_m(z_m), f(z_m)) > \frac{\varepsilon}{3}, \tag{3.11}$$

that is, f_m is not strongly uniformly continuous on B. A contradiction. □

4. More on (Statistical) Exhaustiveness

As we mentioned in Introduction, in 2008 the notion of exhaustiveness was introduced in [11]. We recall the definition for both families and nets of functions [11].

Definition 4.1. Let $\mathcal{M}$ be a family and $(f_n)_{n\in\mathbb{N}}$ a sequence in Y^X. If case $\mathcal{M}$ is finite, we say that $\mathcal{M}$ is exhaustive at $x \in X$ if all functions in $\mathcal{M}$ are continuous at x. If $\mathcal{M}$ is infinite, then $\mathcal{M}$ is exhaustive at $x \in X$ if for each $\varepsilon > 0$ there exist $\delta > 0$ and a finite set $\mathcal{A} \subset \mathcal{M}$ such that for each $y \in S(x,\delta)$ and for each $f \in \mathcal{M} \setminus \mathcal{A}$, we have $\rho(f(x), f(y)) < \varepsilon$. The sequence $(f_n)_{n\in\mathbb{N}}$ is exhaustive at x if the family $\{f_n : n \in \mathbb{N}\}$ is exhaustive at x. The family $\mathcal{M}$ (sequence $(f_n)_{n\in\mathbb{N}}$) is *exhaustive on* X if it is exhaustive at each $x \in X$.

In [15], it was shown that exhaustiveness for a net of functions at each point of the domain is the property that must be added to pointwise convergence to have uniform convergence on compacta.

The notion of weak exhaustiveness was also introduced in [11], and it was proved that it gives a necessary and sufficient condition under which the pointwise limit of a sequence of (not necessarily continuous) functions is continuous.

In [9], two of the authors investigated the continuity of the statistical pointwise limit of a sequence of functions via the notion of statistical exhaustiveness.

Definition 4.2 (see [9]). A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is said to be *statistically exhaustive* (shortly, st-*exhaustive*) at a point $x \in X$ if for each $\varepsilon > 0$ there are $\delta > 0$ and a statistically dense set $\mathrm{M} \subset \mathbb{N}$ such that for each $y \in S(x,\delta)$ we have $\rho(f_n(y), f_n(x)) < \varepsilon$ for each $n \in \mathrm{M}$. The sequence $(f_n)_{n\in\mathbb{N}}$ is st-*exhaustive* if it is *st*-exhaustive at each $x \in X$.

In this section, we continue this study and provide some additional information about exhaustiveness and its variations.

First, we show that exhaustiveness is a property between equicontinuity and even continuity. It is well known that equicontinuity implies even continuity, and in [11] it was shown that equicontinuity implies exhaustiveness.

Definition 4.3 (see [10, L p. 241]). A family $\mathcal{M} \subset Y^X$ is *evenly continuous* if for each net $(f_\alpha, x_\alpha)_{\alpha\in\Lambda}$ in $\mathcal{M} \times X$ such that $(x_\alpha)_{\alpha\in\Lambda}$ converges to $x \in X$ and $(f_\alpha(x))_{\alpha\in\Lambda}$ converges to $y \in Y$, the net $(f_\alpha(x_\alpha))_{\alpha\in\Lambda}$ converges to y.

Definition 4.4 (see [10]). A family $\mathcal{M} \subset Y^X$ is *equicontinuous at a point* x if and only if for each $\varepsilon > 0$ there is a neighborhood U of x such that $f(U) \subset S(f(x),\varepsilon)$ for each member f of $\mathcal{M}$. A family $\mathcal{M}$ is *equicontinuous* if it is equicontinuous at each $x \in X$.

Theorem 4.5. *If a family $\mathcal{M} \subset Y^X$ is exhaustive, then $\mathcal{M}$ is evenly continuous.*

Proof. If $\mathcal{M}$ is finite there is nothing to prove, and thus we assume that $\mathcal{M}$ is infinite. Let $(f_\alpha, x_\alpha)_{\alpha\in\Lambda}$ be a net in $\mathcal{M} \times X$ satisfying $(x_\alpha)_{\alpha\in\Lambda}$ converges to x and $(f_\alpha(x))_{\alpha\in\Lambda}$ converges to $y \in Y$, and let $\varepsilon > 0$. As $\mathcal{M}$ is exhaustive at x, there exist $\delta_x > 0$ and $\{f_{\beta_1}, f_{\beta_2}, \ldots, f_{\beta_k}\} \subset \mathcal{M}$ such that for each $z \in S(x,\delta_x)$ and each $f \in \mathcal{M} \setminus \{f_{\beta_1}, f_{\beta_2}, \ldots, f_{\beta_k}\}$ we have $\rho(f(x), f(z)) < \varepsilon/2$. Because the set $\{f_{\beta_1}, f_{\beta_2}, \ldots, f_{\beta_k}\}$ is finite, there is some $\alpha^* \in \Lambda$ such that $f_\alpha \neq f_{\beta_i}$ for each $\alpha \geq \alpha^*$ and each $i \leq k$.

Let $\delta = \min\{\delta_x, \varepsilon/2\}$. Since $(x_\alpha)_{\alpha\in\Lambda}$ converges to x and $(f_\alpha(x))_{\alpha\in\Lambda}$ converges to y there is $\alpha_0 \geq \alpha^*$ in Λ such that $x_\alpha \in S(x,\delta)$ and $f_\alpha(x) \in S(y,\delta)$ for each $\alpha \geq \alpha_0$. Then $f_\alpha(x_\alpha) \in S(f_\alpha(x), \varepsilon/2)$ and $f_\alpha(x) \in S(y,\delta) \subset S(y,\varepsilon/2)$ for each $\alpha \geq \alpha_0$. So for each $\alpha \geq \alpha_0$, we have

$$\rho(f_\alpha(x_\alpha), y) \leq \rho(f_\alpha(x_\alpha), f_\alpha(x)) + \rho(f_\alpha(x), y) < \varepsilon, \tag{4.1}$$

that is, $(f_\alpha(x_\alpha))_{\alpha\in\Lambda}$ converges to y. □

Recall that the concept of uniform exhaustiveness was defined in [15, Definition 4.1] under the name strong exhaustiveness: a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is strongly exhaustive on X if for each $\varepsilon > 0$ there are $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $x, y \in X$ with $d(x,y) < \delta$, $\rho(f_n(x), f_n(y)) < \varepsilon$ for each $n \geq n_0$.

The novel notion of statistical uniform exhaustiveness for a sequence is related to uniform convergence.

Definition 4.6. A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is st-*uniformly exhaustive* on X if for each $\varepsilon > 0$ there are $\delta > 0$ and a statistically dense set $A \subset \mathbb{N}$ such that for all $x, y \in X$ with $d(x,y) < \delta$, $\rho(f_n(x), f_n(y)) < \varepsilon$ for all $n \in A$.

Theorem 4.7. *Let (X,d) be a compact space, and let $(f_n)_{n\in\mathbb{N}}$ be a st-exhaustive sequence in Y^X such that $(f_n)_{n\in\mathbb{N}} \overset{\text{st-}\tau_p}{\longrightarrow} f$. Then*

(a) *$(f_n)_{n\in\mathbb{N}}$ is statistically uniformly exhaustive;*

(b) *there is a statistically dense set $M \subset \mathbb{N}$ such that $(f_m)_{m\in M}$ is uniformly exhaustive and $(f_m)_{m\in M} \overset{\tau_u}{\longrightarrow} f$.*

Proof. (a) Let $\varepsilon > 0$ and $x \in X$ be fixed. Since, by hypothesis, $(f_n)_{n\in\mathbb{N}}$ is statistically exhaustive at x, there are $\delta_x > 0$ and a statistically dense set $A_x \subset \mathbb{N}$ such that for each $z \in S(x,\delta_x)$ and each $n \in A_x$, it holds $\rho(f_n(x), f_n(z)) < \varepsilon/2$. From $X = \bigcup_{x\in X} S(x, \delta_x/2)$ and compactness of X, it follows the existence of finitely many points $x_1, \ldots, x_k$ in X such that $X = \bigcup_{i=1}^k S(x_i, \delta_{x_i}/2)$. Let $\delta^* = \min\{\delta_{x_i}/2 : i \leq k\}$ and $A = \bigcap_{i=1}^k A_{x_i}$. The set A is statistically dense in $\mathbb{N}$. We claim that δ^* and A witness that (a) is true.

Let $x, z \in X$ such that $d(x,z) < \delta^*$. There is $j \leq k$ such that $x, z \in S(x_j, \delta_{x_j})$. Therefore, for each $n \in A$ and all $x, z \in X$ with $d(x,z) < \delta^*$, we have $\rho(f_n(x), f_n(z)) < \rho(f_n(x), f_n(x_j)) + \rho(f_n(x_j), f_n(z)) < \varepsilon$, that is, (a) is true.

(b) By (a) for each $j \in \mathbb{N}$, there are a statistically dense set $A_j \subset \mathbb{N}$ and $\delta_j > 0$ such that $x, y \in X$ and $d(x,y) < \delta_j$ imply $\rho(f_n(x), f_n(y)) < 1/j$ for each $n \in A_j$. Then $\{\mathbb{N} \setminus A_j : j \in \mathbb{N}\}$ is a family of density zero sets, that is, this family is contained in the ideal $\mathcal{D}_d$ of all subsets of $\mathbb{N}$ having density zero. It is known that $\mathcal{D}_d$ is a P-ideal (i.e., for each countable collection $\mathcal{J} \subset \mathcal{D}_d$ there is some $L \in \mathcal{D}_d$ such that $J \setminus L$ is finite for each $J \in \mathcal{J}$), so there exists $D \in \mathcal{D}_d$ such that the set $(\mathbb{N} \setminus A_j) \setminus D$ is finite for each $j \in \mathbb{N}$. Let $N_1 = \mathbb{N} \setminus D := \{n_1 < n_2 < \cdots < n_k < \cdots\}$. Then N_1 is a statistically dense subset of $\mathbb{N}$.

Claim 1. The sequence $(f_n)_{n\in N_1}$ is uniformly exhaustive.

Let $\varepsilon > 0$ be fixed. Choose $j \in \mathbb{N}$ such that $1/j < \varepsilon$. Let $t_\varepsilon \in \mathbb{N}$ be such that $(\mathbb{N} \setminus A_j) \setminus D \subset \{n_1, n_2, \ldots, n_{t_\varepsilon}\}$. It follows that for each $t > t_\varepsilon$, we have $n_t \in A_j$. Thus there is $n^* > n_{t_\varepsilon}$ such that for all $x, y \in X$ with $d(x,y) < 1/j$ we have $\rho(f_n(x), f_n(y)) < \varepsilon$ for each $n \in N_1$ greater than n^*.

Claim 2. There is $M \subset \mathbb{N}$ with $\partial(M) = 1$ such that $(f_m)_{m\in M} \xrightarrow{\tau_u} f$.

From assumptions, according to [9, Theorem 3.5], it follows that the function f is continuous on X, and so uniformly continuous since X is compact. Fix $\varepsilon > 0$. There is $\delta > 0$ such that $\rho(f(x), f(y)) < \varepsilon/4$ for all $x, y \in X$ satisfying $d(x, y) < \delta$. By Claim 1, $(f_n)_{n\in N_1}$ is uniformly exhaustive on X, so that there exist $\delta_0 > 0$ and $n_{k_0} \in N_1$ such that $d(x, y) < \delta_0$ implies $\rho(f_{n_k}(x), f_{n_k}(y)) < \varepsilon/4$ for all $n_k \in N_1$ with $n_k \geq n_{k_0}$. Let $\delta^* = \min\{\delta, \delta_0\}$. Using compactness of X choose a finite set $\{x_1, \ldots, x_s\} \subset X$ such that $X = \bigcup_{i=1}^{s} S(x_i, \delta^*)$. Since $(f_n)_{n\in\mathbb{N}} \xrightarrow{\text{st-}\tau_p} f$, for each $i \leq s$ there is a statistically dense set $A_i \subset \mathbb{N}$ such that for each $n \in A_0 = \bigcap_{i=1}^{s} A_i$ we have $\rho(f_n(x_i), f(x_i)) < \varepsilon/4$, $i \leq s$. Set $M = A_0 \cap N_1$. Then M is statistically dense and the sequence $(f_n)_{n\in\mathbb{N}}$ is still uniformly exhaustive. Each $y \in X$ belongs to $S(x_i, \delta^*)$ for some $i \leq s$, and thus for each $n_k \in M$ with $n_k \geq n_{k_0}$ we have

$$\rho(f_{n_k}(y), f(y)) \leq \rho(f_{n_k}(y), f_{n_k}(x_i)) + \rho(f_{n_k}(x_i), f(x_i)) + \rho(f(x_i), f(y)) < \varepsilon, \tag{4.2}$$

which completes the proof of (Claim 2 and) the theorem. □

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