Research Article

Approximation of Analytic Functions by Chebyshev Functions

Soon-Mo Jung¹ and Themistocles M. Rassias²

¹ Mathematics Section, College of Science and Technology, Hongik University,
 Jochiwon 339-701, Republic of Korea

² Department of Mathematics, National Technical University of Athens, Zografou Campus,
 15780 Athens, Greece

Correspondence should be addressed to Soon-Mo Jung, smjung@hongik.ac.kr

Received 7 May 2011; Accepted 11 July 2011

Academic Editor: Yong Zhou

Copyright © 2011 S.-M. Jung and T. M. Rassias. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We solve the inhomogeneous Chebyshev's differential equation and apply this result for approximating analytic functions by the Chebyshev functions.

1. Introduction

Let X be a normed space over a scalar field $\mathbb{K}$, and let $I \subset \mathbb{R}$ be an open interval, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Assume that $a_0, a_1, \dots, a_n : I \rightarrow \mathbb{K}$, and $g : I \rightarrow X$ are given continuous functions and that $y : I \rightarrow X$ is an n times continuously differentiable function satisfying the inequality

$$\left\| a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) + g(t) \right\| \leq \varepsilon \quad (1.1)$$

for all $t \in I$ and for a given $\varepsilon > 0$. If there exists an n times continuously differentiable function $y_0 : I \rightarrow X$ satisfying

$$a_n(t)y_0^{(n)}(t) + a_{n-1}(t)y_0^{(n-1)}(t) + \dots + a_1(t)y_0'(t) + a_0(t)y_0(t) + g(t) = 0 \quad (1.2)$$

and $\|y(t) - y_0(t)\| \leq K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε with $\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0$, then we say that the above differential equation has the Hyers-Ulam stability. For more detailed definitions of the Hyers-Ulam stability, we refer the reader to [1–7].

Obłozna seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [8, 9]). Here, we will introduce a result of Alsina and Ger [10]. They proved that if a differentiable function $f : I \rightarrow \mathbb{R}$ satisfies the inequality $|y'(t) - y(t)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a constant c such that $|f(t) - ce^t| \leq 3\varepsilon$ for any $t \in I$. Their result was generalized by Takahasi et al. Indeed, it was proved in [11] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$ (see also [12, 13]).

Moreover, Miura et al. [14] investigated the Hyers-Ulam stability of n th order linear differential equation with complex coefficients. They [15] also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function.

Jung also proved the Hyers-Ulam stability of various linear differential equations of first order [16–19]. Moreover, he applied the power series method to the study of the Hyers-Ulam stability of Legendre's differential equation (see [20, 21]). Recently, Jung and Kim tried to prove the Hyers-Ulam stability of the Chebyshev's differential equation

$$(1 - x^2)y''(x) - xy'(x) + n^2y(x) = 0 \quad (1.3)$$

for all $x \in (-1, 1)$. However, the obtained theorem unfortunately does not describe the Hyers-Ulam stability of the Chebyshev's differential equation in a strict sense (see [22]).

In Section 2 of this paper, by using the ideas from [20–26], we investigate the general solution of the inhomogeneous Chebyshev's differential equation of the form

$$(1 - x^2)y''(x) - xy'(x) + n^2y(x) = \sum_{m=0}^{\infty} a_m x^m, \quad (1.4)$$

where n is a given positive integer. Section 3 will be devoted to the investigation of the Hyers-Ulam stability and an approximation property of the Chebyshev functions.

2. Inhomogeneous Chebyshev's Equation

Every solution of the Chebyshev's differential equation (1.3) is called a Chebyshev function. The Chebyshev's differential equation has regular singular points at -1 , 1 , and ∞ , and it plays a great role in physics and engineering. In particular, this equation is most useful for treating the boundary value problems exhibiting certain symmetries.

In this section, we set $c_0 = c_1 = 0$ and define, for all $m \in \mathbb{N}$,

$$\begin{aligned} c_{2m} &= \frac{1}{2m} \sum_{i=0}^{m-1} \frac{a_{2i}}{2i+1} \prod_{j=i+1}^{m-1} \frac{(2j)^2 - n^2}{2j(2j+1)}, \\ c_{2m+1} &= \frac{1}{2m+1} \sum_{i=0}^{m-1} \frac{a_{2i+1}}{2i+2} \prod_{j=i+1}^{m-1} \frac{(2j+1)^2 - n^2}{(2j+1)(2j+2)}, \end{aligned} \quad (2.1)$$

where we refer to (1.4) for the a_m 's and we follow the convention $\prod_{j=m}^{m-1} [\dots] = 1$. We can easily check that c_m 's satisfy the following relation:

$$(m+2)(m+1)c_{m+2} - (m^2 - n^2)c_m = a_m \quad (2.2)$$

for any $m \in \{0, 1, 2, \dots\}$.

Theorem 2.1. *Assume that n is a positive integer and the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is $\rho > 0$. Let $\rho_0 = \min\{1, \rho\}$. Then, every solution $y : (-\rho_0, \rho_0) \rightarrow \mathbb{C}$ of the Chebyshev's differential equation (1.4) can be expressed by*

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m, \quad (2.3)$$

where $y_h(x)$ is a Chebyshev function and the c_m 's are given in (2.1).

Proof. It is not difficult to see that, if $j \in \mathbb{N}$ and $|(2j)^2 - n^2| > 2j(2j+1)$, then

$$j < \frac{-1 + \sqrt{1 + 8n^2}}{8} < \frac{\sqrt{8n^2}}{8} \quad (\text{for } 2j < n). \quad (2.4)$$

Hence, we have $1 \leq j \leq n_e$ with $n_e = \lfloor n/\sqrt{8} \rfloor$. If $m > n_e$, then it follows from (2.1) that

$$\begin{aligned} |c_{2m}| &\leq \frac{1}{2m} \sum_{i=0}^{n_e-1} \frac{|a_{2i}|}{2i+1} \left(\prod_{j=i+1}^{n_e} \frac{|(2j)^2 - n^2|}{2j(2j+1)} \right) \left(\prod_{j=n_e+1}^{m-1} \frac{|(2j)^2 - n^2|}{2j(2j+1)} \right) \\ &\quad + \frac{1}{2m} \sum_{i=n_e}^{m-1} \frac{|a_{2i}|}{2i+1} \prod_{j=i+1}^{m-1} \frac{|(2j)^2 - n^2|}{2j(2j+1)} \\ &\leq \frac{1}{2m} \sum_{i=0}^{n_e-1} \frac{|a_{2i}|}{2i+1} \prod_{j=i+1}^{n_e} \frac{n^2 - 4}{2j(2j+1)} + \frac{1}{2m} \sum_{i=n_e}^{m-1} \frac{|a_{2i}|}{2i+1} \\ &\leq \frac{1}{2m} \sum_{i=0}^{n_e-1} \frac{(2i)!(n^2 - 4)^{n_e-i}}{(2n_e + 1)!} |a_{2i}| + \frac{1}{2m} \sum_{i=n_e}^{m-1} \frac{|a_{2i}|}{2n_e + 1} \\ &\leq \frac{\max_{0 \leq i \leq n_e} ((2i)! / (2n_e + 1)!) (n^2 - 4)^{n_e-i}}{2m} \sum_{i=0}^{m-1} |a_{2i}|. \end{aligned} \quad (2.5)$$

We now suppose $1 \leq m \leq n_e$. Then it holds true that $n \geq 3$, and we have

$$\begin{aligned}
 |c_{2m}| &\leq \frac{1}{2m} \sum_{i=0}^{m-1} \frac{|a_{2i}|}{2i+1} \prod_{j=i+1}^{m-1} \frac{|(2j)^2 - n^2|}{2j(2j+1)} \\
 &\leq \frac{1}{2m} \sum_{i=0}^{m-1} \frac{|a_{2i}|}{2i+1} \prod_{j=i+1}^{m-1} \frac{n^2 - 4}{2j(2j+1)} \\
 &= \frac{1}{2m} \sum_{i=0}^{m-1} \frac{(2i)!(n^2 - 4)^{m-1-i}}{(2m-1)!} |a_{2i}| \\
 &\leq \frac{\max_{0 \leq i \leq m-1} ((2i)!/(2m-1)!) (n^2 - 4)^{m-1-i}}{2m} \sum_{i=0}^{m-1} |a_{2i}|.
 \end{aligned} \tag{2.6}$$

Hence, we conclude from the above two inequalities that

$$|c_{2m}| \leq \frac{M_e}{2m} \sum_{i=0}^{m-1} |a_{2i}| \tag{2.7}$$

for all $m \in \mathbb{N}$, where we set

$$M_e = \max_{0 \leq i \leq \ell \leq n_e} \frac{(2i)!}{(2\ell+1)!} (n^2 - 4)^{\ell-i}. \tag{2.8}$$

On the other hand, if $j \in \mathbb{N}$ and $|(2j+1)^2 - n^2| > (2j+1)(2j+2)$, then

$$j < \frac{\sqrt{8n^2+1}-5}{8} < \frac{\sqrt{8n^2}-4}{8} < \frac{n}{2} - \frac{1}{2} \quad (\text{for } 2j+1 < n). \tag{2.9}$$

Hence, we get $1 \leq j \leq n_o$ with $n_o = \lfloor n/\sqrt{8} - 1/2 \rfloor$. If $m > n_o$, then it follows from (2.1) that

$$\begin{aligned}
 |c_{2m+1}| &\leq \frac{1}{2m+1} \sum_{i=0}^{n_o-1} \frac{|a_{2i+1}|}{2i+2} \left(\prod_{j=i+1}^{n_o} \frac{|(2j+1)^2 - n^2|}{(2j+1)(2j+2)} \right) \left(\prod_{j=n_o+1}^{m-1} \frac{|(2j+1)^2 - n^2|}{(2j+1)(2j+2)} \right) \\
 &\quad + \frac{1}{2m+1} \sum_{i=n_o}^{m-1} \frac{|a_{2i+1}|}{2i+2} \prod_{j=i+1}^{m-1} \frac{|(2j+1)^2 - n^2|}{(2j+1)(2j+2)} \\
 &\leq \frac{1}{2m+1} \sum_{i=0}^{n_o-1} \frac{|a_{2i+1}|}{2i+2} \prod_{j=i+1}^{n_o} \frac{n^2 - 9}{(2j+1)(2j+2)} + \frac{1}{2m+1} \sum_{i=n_o}^{m-1} \frac{|a_{2i+1}|}{2i+2} \\
 &\leq \frac{1}{2m+1} \sum_{i=0}^{n_o-1} \frac{(2i+1)!(n^2 - 9)^{n_o-i}}{(2n_o+2)!} |a_{2i+1}| + \frac{1}{2m+1} \sum_{i=n_o}^{m-1} \frac{|a_{2i+1}|}{2n_o+2} \\
 &\leq \frac{\max_{0 \leq i \leq n_o} ((2i+1)!/(2n_o+2)!) (n^2 - 9)^{n_o-i}}{2m+1} \sum_{i=0}^{m-1} |a_{2i+1}|.
 \end{aligned} \tag{2.10}$$

If $1 \leq m \leq n_o$, then we have $n \geq 5$, and it follows from (2.1) that

$$\begin{aligned} |c_{2m+1}| &\leq \frac{1}{2m+1} \sum_{i=0}^{m-1} \frac{|a_{2i+1}|}{2i+2} \prod_{j=i+1}^{m-1} \frac{|(2j+1)^2 - n^2|}{(2j+1)(2j+2)} \\ &\leq \frac{1}{2m+1} \sum_{i=0}^{m-1} \frac{|a_{2i+1}|}{2i+2} \prod_{j=i+1}^{m-1} \frac{n^2 - 9}{(2j+1)(2j+2)} \end{aligned} \quad (2.11)$$

since $j < n_o$ and hence $2j+1 < 2n/\sqrt{8} < n$. Furthermore, we have

$$\begin{aligned} |c_{2m+1}| &\leq \frac{1}{2m+1} \sum_{i=0}^{m-1} \frac{(2i+1)!(n^2-9)^{m-1-i}}{(2m)!} |a_{2i+1}| \\ &\leq \frac{\max_{0 \leq i \leq m-1} ((2i+1)!/(2m)!) (n^2-9)^{m-1-i}}{2m+1} \sum_{i=0}^{m-1} |a_{2i+1}|. \end{aligned} \quad (2.12)$$

Thus, we may conclude from the last two inequalities that

$$|c_{2m+1}| \leq \frac{M_o}{2m+1} \sum_{i=0}^{m-1} |a_{2i+1}| \quad (2.13)$$

for any $m \in \mathbb{N}$, where

$$M_o = \max_{0 \leq i \leq \ell \leq n_o} \frac{(2i+1)!}{(2\ell+2)!} (n^2-9)^{\ell-i}. \quad (2.14)$$

Let ρ_1 be an arbitrary positive number less than ρ_0 . Then it follows from (2.7) and (2.13) that

$$\begin{aligned} \left| \sum_{m=2}^{\infty} c_m x^m \right| &\leq \sum_{m=1}^{\infty} |c_{2m}| |x|^{2m} + \sum_{m=1}^{\infty} |c_{2m+1}| |x|^{2m+1} \\ &\leq M_e \sum_{m=1}^{\infty} \frac{|x|^{2m}}{2m} \sum_{i=0}^{m-1} |a_{2i}| + M_o \sum_{m=1}^{\infty} \frac{|x|^{2m+1}}{2m+1} \sum_{i=0}^{m-1} |a_{2i+1}| \\ &= M_e |a_0| \left(\frac{|x|^2}{2} + \frac{|x|^4}{4} + \frac{|x|^6}{6} + \frac{|x|^8}{8} + \frac{|x|^{10}}{10} + \dots \right) \\ &\quad + M_e |a_2| |x|^2 \left(\frac{|x|^2}{4} + \frac{|x|^4}{6} + \frac{|x|^6}{8} + \frac{|x|^8}{10} + \frac{|x|^{10}}{12} + \dots \right) \end{aligned}$$

$$\begin{aligned}
& + M_e |a_4| |x|^4 \left(\frac{|x|^2}{6} + \frac{|x|^4}{8} + \frac{|x|^6}{10} + \frac{|x|^8}{12} + \frac{|x|^{10}}{14} + \dots \right) \\
& + \dots \\
& + M_o |a_1| |x| \left(\frac{|x|^2}{3} + \frac{|x|^4}{5} + \frac{|x|^6}{7} + \frac{|x|^8}{9} + \frac{|x|^{10}}{11} + \dots \right) \\
& + M_o |a_3| |x|^3 \left(\frac{|x|^2}{5} + \frac{|x|^4}{7} + \frac{|x|^6}{9} + \frac{|x|^8}{11} + \frac{|x|^{10}}{13} + \dots \right) \\
& + M_o |a_5| |x|^5 \left(\frac{|x|^2}{7} + \frac{|x|^4}{9} + \frac{|x|^6}{11} + \frac{|x|^8}{13} + \frac{|x|^{10}}{15} + \dots \right) \\
& + \dots \\
& = M_e \sum_{m=0}^{\infty} |a_{2m}| |x|^{2m} \sum_{i=1}^{\infty} \frac{|x|^{2i}}{2(m+i)} + M_o \sum_{m=0}^{\infty} |a_{2m+1}| |x|^{2m+1} \sum_{i=1}^{\infty} \frac{|x|^{2i}}{2(m+i)+1}
\end{aligned} \tag{2.15}$$

for any $x \in [-\rho_1, \rho_1]$.

Because of $0 < \rho_1 < \rho_0 \leq 1$, we obtain

$$\sum_{i=1}^{\infty} \frac{|x|^{2i}}{2(m+i)} \leq \frac{1}{2m+2} \frac{|x|^2}{1-|x|^2}, \quad \sum_{i=1}^{\infty} \frac{|x|^{2i}}{2(m+i)+1} \leq \frac{1}{2m+3} \frac{|x|^2}{1-|x|^2} \tag{2.16}$$

for all $x \in [-\rho_1, \rho_1]$. Thus, we have

$$\begin{aligned}
\left| \sum_{m=2}^{\infty} c_m x^m \right| & \leq M_e \sum_{m=0}^{\infty} \frac{|a_{2m} x^{2m}|}{2m+2} \frac{|x|^2}{1-|x|^2} + M_o \sum_{m=0}^{\infty} \frac{|a_{2m+1} x^{2m+1}|}{2m+3} \frac{|x|^2}{1-|x|^2} \\
& \leq M_e \frac{|x|^2}{1-|x|^2} \sum_{m=0}^{\infty} \frac{|a_m x^m|}{m+2}
\end{aligned} \tag{2.17}$$

for all $x \in [-\rho_1, \rho_1]$. Since ρ_1 is arbitrarily given with $0 < \rho_1 < \rho_0$, inequality (2.17) holds true for all $x \in (-\rho_0, \rho_0)$. Moreover, the power series $\sum_{m=0}^{\infty} a_m x^m$ is absolutely convergent on $(-\rho, \rho)$. Hence, we conclude that

$$\left| \sum_{m=2}^{\infty} c_m x^m \right| < \infty \tag{2.18}$$

for all $x \in (-\rho_0, \rho_0)$. That is, the power series $\sum_{m=2}^{\infty} c_m x^m$ is convergent for each $x \in (-\rho_0, \rho_0)$.

We will now prove that $\sum_{m=2}^{\infty} c_m x^m$ satisfies the inhomogeneous Chebyshev's differential equation (1.4) for all $x \in (-\rho_0, \rho_0)$. If we substitute $\sum_{m=2}^{\infty} c_m x^m = \sum_{m=1}^{\infty} c_{2m} x^{2m} + \sum_{m=1}^{\infty} c_{2m+1} x^{2m+1}$ for $y(x)$ in (1.4), then it follows from (2.2) that

$$\begin{aligned}
 & (1-x^2)y''(x) - xy'(x) + n^2 y(x) \\
 &= \sum_{m=0}^{\infty} (2m+2)(2m+1)c_{2m+2}x^{2m} + \sum_{m=0}^{\infty} (2m+3)(2m+2)c_{2m+3}x^{2m+1} \\
 &\quad - \sum_{m=1}^{\infty} 2m(2m-1)c_{2m}x^{2m} - \sum_{m=1}^{\infty} (2m+1)(2m)c_{2m+1}x^{2m+1} \\
 &\quad - \sum_{m=1}^{\infty} 2mc_{2m}x^{2m} - \sum_{m=1}^{\infty} (2m+1)c_{2m+1}x^{2m+1} \\
 &\quad + \sum_{m=1}^{\infty} n^2 c_{2m}x^{2m} + \sum_{m=1}^{\infty} n^2 c_{2m+1}x^{2m+1} \\
 &= 2c_2 + 6c_3x + \sum_{m=1}^{\infty} \left[(2m+2)(2m+1)c_{2m+2} + (n^2 - (2m)^2)c_{2m} \right] x^{2m} \\
 &\quad + \sum_{m=1}^{\infty} \left[(2m+3)(2m+2)c_{2m+3} + (n^2 - (2m+1)^2)c_{2m+1} \right] x^{2m+1} \\
 &= 2c_2 + 6c_3x + \sum_{m=1}^{\infty} a_{2m}x^{2m} + \sum_{m=1}^{\infty} a_{2m+1}x^{2m+1} \\
 &= \sum_{m=0}^{\infty} a_m x^m
 \end{aligned} \tag{2.19}$$

for all $x \in (-\rho_0, \rho_0)$. That is, $\sum_{m=2}^{\infty} c_m x^m$ is a particular solution of the inhomogeneous Chebyshev's differential equation (1.4), and hence every solution $y : (-\rho_0, \rho_0) \rightarrow \mathbb{C}$ of (1.4) can be expressed by

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m, \tag{2.20}$$

where $y_h(x)$ is a Chebyshev function. □

3. Approximate Chebyshev Differential Equation

In this section, let $K \geq 0$ and $\rho > 0$ be constants. We denote by $\mathcal{C}_K$ the set of all functions $y : (-\rho, \rho) \rightarrow \mathbb{C}$ with the following properties:

- (a) $y(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least ρ ;

(b) $\sum_{m=0}^{\infty} |a_m x^m| \leq K |\sum_{m=0}^{\infty} a_m x^m|$ for any $x \in (-\rho, \rho)$, where

$$a_m = (m+2)(m+1)b_{m+2} - (m^2 - n^2)b_m \quad (3.1)$$

for all $m \in \mathbb{N}_0$ and set $b_0 = b_1 = 0$.

We now investigate the (local) Hyers-Ulam stability problem of the Chebyshev differential equation. More precisely, we try to answer the question, whether there exists a Chebyshev function near any approximate Chebyshev function.

Theorem 3.1. *Let n be a positive integer, and assume that a function $y \in \mathcal{C}_K$ satisfies the differential inequality*

$$\left| (1-x^2)y''(x) - xy'(x) + n^2y(x) \right| \leq \varepsilon \quad (3.2)$$

for all $x \in (-\rho, \rho)$ and for some $\varepsilon > 0$. Let $\rho_0 = \min\{1, \rho\}$. Then there exists a Chebyshev function $y_h : (-\rho_0, \rho_0) \rightarrow \mathbb{C}$ such that

$$|y(x) - y_h(x)| \leq \frac{KM_e \varepsilon}{2} \frac{x^2}{1-x^2} \quad (3.3)$$

for all $x \in (-\rho_0, \rho_0)$, where the constant M_e is defined in (2.8).

Proof. It follows from (a) and (b) that

$$(1-x^2)y''(x) - xy'(x) + n^2y(x) = \sum_{m=0}^{\infty} a_m x^m \quad (3.4)$$

for all $x \in (-\rho, \rho)$ (cf. (2.19)). Moreover, by using (b) and (3.2), we get

$$\sum_{m=0}^{\infty} |a_m x^m| \leq K \left| \sum_{m=0}^{\infty} a_m x^m \right| \leq K\varepsilon \quad (3.5)$$

for any $x \in (-\rho, \rho)$.

According to Theorem 2.1 and (3.4), $y(x)$ can be written as $y_h(x) + \sum_{m=2}^{\infty} c_m x^m$ for all $x \in (-\rho_0, \rho_0)$, where y_h is some Chebyshev function and c_m 's are given in (2.1). It moreover follows from (2.17) and (3.5) that

$$|y(x) - y_h(x)| = \left| \sum_{m=2}^{\infty} c_m x^m \right| \leq M_e \frac{x^2}{1-x^2} \frac{K}{2} \varepsilon \quad (3.6)$$

for all $x \in (-\rho_0, \rho_0)$. □

If ρ is assumed to be less than 1, then $\rho_0 = \rho < 1$ and Theorem 3.1 implies the Hyers-Ulam stability of the Chebyshev's differential equation (1.3).

Table 1

n	n_e	n_o	M_e	M_o
1	0	-1	1	$-\infty$
2	0	0	1	1/2
3	1	0	1	1/2
4	1	0	2	1/2
5	1	1	7/2	2/3
6	2	1	128/15	9/8

Remark 3.2. We give some values for n_e , n_o , M_e , and M_o in Table 1.

Corollary 3.3. Let n be a positive integer, and assume that a function $y \in C_K$ satisfies the differential inequality (3.2) for all $x \in (-\rho, \rho)$ and for some $\varepsilon > 0$. Let $\rho_0 = \min\{1, \rho\}$. Then there exists a Chebyshev function $y_h : (-\rho_0, \rho_0) \rightarrow \mathbb{C}$ such that

$$|y(x) - y_h(x)| = O(x^2) \quad (3.7)$$

as $x \rightarrow 0$.

Acknowledgments

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (no. 2011-0004919).

References

- [1] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Co. Inc., River Edge, NJ, 2002.
- [2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [3] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and their Applications, 34, Birkhäuser, Boston, Mass, USA, 1998.
- [4] D. H. Hyers and T. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.
- [5] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, vol. 48 of Springer Optimization and Its Applications, Springer, New York, NY, USA, 2011.
- [6] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
- [7] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, New York, NY, USA, 1964.
- [8] M. Obłozza, "Hyers stability of the linear differential equation," *Rocznik Naukowo-Dydaktyczny. Prace Matematyczne*, vol. 14, no. 13, pp. 259–270, 1993.
- [9] M. Obłozza, "Connections between Hyers and Lyapunov stability of the ordinary differential equations," *Rocznik Naukowo-Dydaktyczny. Prace Matematyczne*, no. 14, pp. 141–146, 1997.
- [10] C. Alsina and R. Ger, "On some inequalities and stability results related to the exponential function," *Journal of Inequalities and Applications*, vol. 2, no. 4, pp. 373–380, 1998.

- [11] S.-E. Takahasi, T. Miura, and S. Miyajima, "On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$," *Bulletin of the Korean Mathematical Society*, vol. 39, no. 2, pp. 309–315, 2002.
- [12] T. Miura, "On the Hyers-Ulam stability of a differentiable map," *Scientiae Mathematicae Japonicae*, vol. 55, no. 1, pp. 17–24, 2002.
- [13] T. Miura, S.-M. Jung, and S.-E. Takahasi, "Hyers-Ulam-Rassias stability of the Banach space valued linear differential equations $y' = \lambda y$," *Journal of the Korean Mathematical Society*, vol. 41, no. 6, pp. 995–1005, 2004.
- [14] T. Miura, S. Miyajima, and S.-E. Takahasi, "Hyers-Ulam stability of linear differential operator with constant coefficients," *Mathematische Nachrichten*, vol. 258, pp. 90–96, 2003.
- [15] T. Miura, S. Miyajima, and S.-E. Takahasi, "A characterization of Hyers-Ulam stability of first order linear differential operators," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 1, pp. 136–146, 2003.
- [16] S.-M. Jung, "Hyers-Ulam stability of linear differential equations of first order," *Applied Mathematics Letters*, vol. 17, no. 10, pp. 1135–1140, 2004.
- [17] S.-M. Jung, "Hyers-Ulam stability of linear differential equations of first order. II," *Applied Mathematics Letters*, vol. 19, no. 9, pp. 854–858, 2006.
- [18] S.-M. Jung, "Hyers-Ulam stability of linear differential equations of first order. III," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 1, pp. 139–146, 2005.
- [19] S.-M. Jung, "Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 549–561, 2006.
- [20] S.-M. Jung, "Legendre's differential equation and its Hyers-Ulam stability," *Abstract and Applied Analysis*, vol. 2007, Article ID 56419, 14 pages, 2007.
- [21] S.-M. Jung, "Approximation of analytic functions by Legendre functions," *Nonlinear Analysis*, vol. 71, no. 12, pp. e103–e108, 2009.
- [22] S.-M. Jung and B. Kim, "Chebyshev's differential equation and its Hyers-Ulam stability," *Differential Equations & Applications*, vol. 1, no. 2, pp. 199–207, 2009.
- [23] S.-M. Jung, "Approximation of analytic functions by Airy functions," *Integral Transforms and Special Functions*, vol. 19, no. 11–12, pp. 885–891, 2008.
- [24] S.-M. Jung, "Approximation of analytic functions by Hermite functions," *Bulletin des Sciences Mathématiques*, vol. 133, no. 7, pp. 756–764, 2009.
- [25] B. Kim and S.-M. Jung, "Bessel's differential equation and its Hyers-Ulam stability," *Journal of Inequalities and Applications*, vol. 2007, Article ID 21640, 8 pages, 2007.
- [26] T. M. Rassias, H. M. Srivastava, and A. Yanushauskas, *Topics in Polynomials of One and Several Variables and Their Applications*, World Scientific Publishing, River Edge, NJ, USA, 1993, Volume Dedicated to the Memory of P. L. Chebyshev.