Research Article

Intuitionistic Fuzzy Stability of Functional Equations Associated with Inner Product Spaces

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In intuitionistic fuzzy normed spaces, we investigate some stability results for the functional equation $\sum_{i=1}^n f(x_i - (1/n)\sum_{j=1}^n x_j) = \sum_{i=1}^n f(x_i) - nf((1/n)\sum_{i=1}^n x_i)$ which is said to be a functional equation associated with inner products space.

1. Introduction and Preliminaries

The aim of this article is to prove an intuitionistic fuzzy version of the Hyers-Ulam-Rassias stability for the functional equation:

$$\sum_{i=1}^n f\left(x_i - \frac{1}{n}\sum_{j=1}^n x_j\right) = \sum_{i=1}^n f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^n x_i\right), \quad (1.1)$$

which is said to be a functional equation associated with inner product spaces. It was shown by Rassias [1] that the norm defined over a real vector space X is induced by an inner product if and only if for a fixed integer $n \geq 2$ it follows

$$\sum_{i=1}^n \left\| x_i - \frac{1}{n}\sum_{j=1}^n x_j \right\|^2 = \sum_{i=1}^n \|x_i\|^2 - n \left\| \frac{1}{n}\sum_{i=1}^n x_i \right\|^2, \quad (1.2)$$

for all $x_i, \dots, x_n \in X$. Interesting new results concerning functional equations associated with inner product spaces have recently been obtained by Park et al. [2, 3] and Najati and Rassias [4] as well as for the fuzzy stability of a functional equation associated with inner product spaces [5].

Stability problem of a functional equation was first posed by Ulam [6] which was answered by Hyers [7] on approximately additive mappings and then generalized by Aoki [8] and Rassias [9] for additive mappings and linear mappings, respectively. Later there have been proved several new results on stability of various classes of functional equations in the Hyers-Ulam sense (cf. the following books and papers [10–18] and the references cited therein), as well as various fuzzy stability results concerning Cauchy, Jensen, quadratic and cubic functional equations (cf. [19–22]). Furthermore some stability results concerning Jensen, cubic, mixed-type additive and cubic functional equations were investigated (cf. [23–26]) in the spirit of intuitionistic fuzzy normed spaces, while the idea of intuitionistic fuzzy normed space was introduced in [27] and further studied in [28–35].

In this section, we recall some notations and basic definitions used in this paper as follows.

Definition 1.1 (cf. [36]). A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t -norm if it satisfies the following conditions:

- (a) $*$ is commutative and associative, (b) $*$ is continuous,
- (c) $a * 1 = a$ for all $a \in [0, 1]$,
- (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 1.2 (cf. [36]). A binary operation $\diamond$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t -conorm if it satisfies the following conditions:

- (a') $\diamond$ is commutative and associative, (b') $\diamond$ is continuous,
- (c') $a \diamond 0 = a$ for all $a \in [0, 1]$,
- (d') $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Using the notions of continuous t -norm and continuous t -conorm, Saadati and Park [27] have recently introduced the concept of intuitionistic fuzzy normed spaces as follows.

Definition 1.3. The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t -norm, $\diamond$ is a continuous t -conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$,

- (i) $\mu(x, t) + \nu(x, t) \leq 1$,
- (ii) $\mu(x, t) > 0$,
- (iii) $\mu(x, t) = 1$ if and only if $x = 0$,
- (iv) $\mu(\alpha x, t) = \mu(x, t/|\alpha|)$ for each $\alpha \neq 0$,
- (v) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,
- (vi) $\mu(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- (vii) $\lim_{t \rightarrow \infty} \mu(x, t) = 1$ and $\lim_{t \rightarrow 0} \mu(x, t) = 0$,
- (viii) $\nu(x, t) < 1$, (ix) $\nu(x, t) = 0$ if and only if $x = 0$,

- (x) $\nu(\alpha x, t) = \nu(x, t/|\alpha|)$ for each $\alpha \neq 0$,
- (xi) $\nu(x, t) \diamond \nu(y, s) \geq \nu(x + y, t + s)$,
- (xii) $\nu(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- (xiii) $\lim_{t \rightarrow \infty} \nu(x, t) = 0$ and $\lim_{t \rightarrow 0} \nu(x, t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm.

Example 1.4 (cf. [37]). Let $(X, \|\cdot\|)$ be a normed space, $a * b = ab$, and $a \diamond b = \min(a + b, 1)$ for all $a, b \in [0, 1]$. For all $x \in X$ and every $t > 0$ and $k = 1, 2$, consider

$$\mu_k(x, t) = \begin{cases} \frac{t}{t + k\|x\|}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases} \quad \nu_k(x, t) = \begin{cases} \frac{k\|x\|}{t + k\|x\|}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases} \quad (1.3)$$

Then $(X, \mu, \nu, *, \diamond)$ is an IFNS.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [27].

Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $\{x_k\}$ is said to be intuitionistic fuzzy convergent to $x \in X$ if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x, t) > 1 - \varepsilon$ and $\nu(x_k - x, t) < \varepsilon$ for all $k \geq k_0$. In this case we write $(\mu, \nu) - \lim x_k = x$. The sequence $\{x_k\}$ is said to be intuitionistic fuzzy Cauchy sequence if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_\ell, t) > 1 - \varepsilon$ and $\nu(x_k - x_\ell, t) < \varepsilon$ for all $k, \ell \geq k_0$. $(X, \mu, \nu, *, \diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamond)$.

2. Intuitionistic Fuzzy Stability

Throughout this section, assume that X , (Z, μ', ν') , and (Y, μ, ν) are linear space, IFNS, and intuitionistic fuzzy Banach space, respectively. For convenience, we use the following abbreviation for a given function $f : X \rightarrow Y$:

$$\Delta f(x_1, \dots, x_n) = \sum_{i=1}^n f\left(x_i - \frac{1}{n} \sum_{j=1}^n x_j\right) - \sum_{i=1}^n f(x_i) + n f\left(\frac{1}{n} \sum_{i=1}^n x_i\right). \quad (2.1)$$

We begin with the Hyers-Ulam-Rassias type theorem in IFNS for the functional (1.1) which is said to be a functional equation associated with inner product spaces.

Theorem 2.1. *Let $\varphi : X \rightarrow Z$ be a function such that $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $0 < |\alpha| < 4$. Suppose that an even function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned} \mu(\Delta f(x_1, \dots, x_n), t_1 + \dots + t_n) &\geq \mu'(\varphi(x_1), t_1) * \dots * \mu'(\varphi(x_n), t_n), \\ \nu(\Delta f(x_1, \dots, x_n), t_1 + \dots + t_n) &\leq \nu'(\varphi(x_1), t_1) \diamond \dots \diamond \nu'(\varphi(x_n), t_n), \end{aligned} \quad (2.2)$$

for all $x_1, \dots, x_n \in X$ and all $t_1, \dots, t_n > 0$. Then there exists a unique quadratic function $Q : X \rightarrow Y$ such that

$$\mu(Q(x) - f(x), t) \geq \mu_1''\left(x, \frac{(4 - |\alpha|)t}{8}\right), \quad \nu(Q(x) - f(x), t) \leq \nu_1''\left(x, \frac{(4 - |\alpha|)t}{8}\right), \quad (2.3)$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned} \mu_1''(x, t) &:= \mu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\ &\quad * \mu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right), \\ \nu_1''(x, t) &:= \nu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\ &\quad \diamond \nu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right). \end{aligned} \quad (2.4)$$

Proof. Put $x_1 = nx_1$, $x_i = nx_2$ ($i = 2, \dots, n$), $t_i = t$ ($i = 1, \dots, n$) in (2.2), and, using the evenness of f , we obtain

$$\begin{aligned} &\mu(nf(x_1 + (n-1)x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - f(nx_1) - (n-1)f(nx_2), nt) \\ &\quad \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_2), t), \\ &\nu(nf(x_1 + (n-1)x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - f(nx_1) - (n-1)f(nx_2), nt) \\ &\quad \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_2), t), \end{aligned} \quad (2.5)$$

for all $x_1, x_2 \in X$ and $t > 0$. Interchanging x_1 with x_2 in (2.5) and using the evenness of f , we get

$$\begin{aligned} &\mu(nf((n-1)x_1 + x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - (n-1)f(nx_1) - f(nx_2), nt) \\ &\quad \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_2), t), \\ &\nu(nf((n-1)x_1 + x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2) - (n-1)f(nx_1) - f(nx_2), nt) \\ &\quad \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_2), t), \end{aligned} \quad (2.6)$$

for all $x_1, x_2 \in X$ and $t > 0$. It follows from (2.5) and (2.6) that

$$\begin{aligned}
& \mu(nf((n-1)x_1 + x_2) + nf(x_1 + (n-1)x_2) + 2f((n-1)(x_1 - x_2)) \\
& \quad + 2(n-1)f(x_1 - x_2) - nf(nx_1) - nf(nx_2), 2nt) \\
& \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_2), t), \\
& \nu(nf((n-1)x_1 + x_2) + nf(x_1 + (n-1)x_2) + 2f((n-1)(x_1 - x_2)) \\
& \quad + 2(n-1)f(x_1 - x_2) - nf(nx_1) - nf(nx_2), 2nt) \\
& \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_2), t),
\end{aligned} \tag{2.7}$$

for all $x_1, x_2 \in X$ and $t > 0$. Putting $x_1 = nx_1$, $x_2 = -nx_2$, $x_i = 0$ ($i = 3, \dots, n$), $t_i = t$ ($i = 1, \dots, n$) in (2.2) and using the evenness of f , we obtain

$$\begin{aligned}
& \mu(f((n-1)x_1 + x_2) + f(x_1 + (n-1)x_2) + 2(n-1)f(x_1 - x_2) - f(nx_1) - f(nx_2), nt) \\
& \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(-nx_2), t) * \mu'(\varphi(0), t), \\
& \nu(f((n-1)x_1 + x_2) + f(x_1 + (n-1)x_2) + 2(n-1)f(x_1 - x_2) - f(nx_1) - f(nx_2), nt) \\
& \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(-nx_2), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.8}$$

for all $x_1, x_2 \in X$ and $t > 0$. Hence, we obtain from (2.7) and (2.8) that

$$\begin{aligned}
& \mu\left(f((n-1)(x_1 - x_2)) - (n-1)^2 f(x_1 - x_2), \frac{n^2 + 2n}{2}t\right) \\
& \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_2), t) * \mu'(\varphi(-nx_2), t) * \mu'(\varphi(0), t), \\
& \nu\left(f((n-1)(x_1 - x_2)) - (n-1)^2 f(x_1 - x_2), \frac{n^2 + 2n}{2}t\right) \\
& \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_2), t) \diamond \nu'(\varphi(-nx_2), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.9}$$

for all $x_1, x_2 \in X$ and $t > 0$. So

$$\begin{aligned}
& \mu\left(f((n-1)x) - (n-1)^2 f(x), \frac{n^2 + 2n}{2}t\right) \geq \mu'(\varphi(nx), t) * \mu'(\varphi(0), t), \\
& \nu\left(f((n-1)x) - (n-1)^2 f(x), \frac{n^2 + 2n}{2}t\right) \leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.10}$$

for all $x \in X$ and $t > 0$. Putting $x_1 = nx$, $x_i = 0$ ($i = 2, \dots, n$), $t_i = t$ ($i = 1, \dots, n$) in (2.2), we obtain

$$\begin{aligned}\mu(f(nx) - f((n-1)x) - (2n-1)f(x), nt) &\geq \mu'(\varphi(nx), t) * \mu'(\varphi(0), t), \\ \nu(f(nx) - f((n-1)x) - (2n-1)f(x), nt) &\leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi(0), t),\end{aligned}\quad (2.11)$$

for all $x \in X$ and $t > 0$. It follows from (2.10) and (2.11) that

$$\begin{aligned}\mu\left(f(nx) - n^2 f(x), \frac{n^2 + 4n}{2} t\right) &\geq \mu'(\varphi(nx), t) * \mu'(\varphi(0), t), \\ \nu\left(f(nx) - n^2 f(x), \frac{n^2 + 4n}{2} t\right) &\leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi(0), t),\end{aligned}\quad (2.12)$$

for all $x \in X$ and $t > 0$. Letting $x_2 = -(n-1)x_1$ in (2.8) and replacing x_1 by x/n in the obtained inequality, we get

$$\begin{aligned}\mu(f((n-1)x) - f((n-2)x) - (2n-3)f(x), nt) &\geq \mu'(\varphi(x), t) * \mu'(\varphi((n-1)x), t) * \mu'(\varphi(0), t), \\ \nu(f((n-1)x) - f((n-2)x) - (2n-3)f(x), nt) &\leq \nu'(\varphi(x), t) \diamond \nu'(\varphi((n-1)x), t) \diamond \nu'(\varphi(0), t),\end{aligned}\quad (2.13)$$

for all $x \in X$ and $t > 0$. It follows from (2.10), (2.11), (2.12), and (2.13) that

$$\begin{aligned}\mu\left(f((n-2)x) - (n-1)^2 f(x), \frac{n^2 + 4n}{2} t\right) &\geq \mu'(\varphi(nx), t) * \mu'(\varphi((n-1)x), t) * \mu'(\varphi(x), t) * \mu'(\varphi(0), t), \\ \nu\left(f((n-2)x) - (n-1)^2 f(x), \frac{n^2 + 4n}{2} t\right) &\leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi((n-1)x), t) \diamond \nu'(\varphi(x), t) \diamond \nu'(\varphi(0), t),\end{aligned}\quad (2.14)$$

for all $x \in X$ and $t > 0$. Applying (2.12) and (2.14), we obtain

$$\begin{aligned}\mu\left(f(nx) - f((n-2)x) - 4(n-1)f(x), (n^2 + 4n)t\right) &\geq \mu'(\varphi(nx), t) * \mu'(\varphi((n-1)x), t) * \mu'(\varphi(x), t) * \mu'(\varphi(0), t), \\ \nu\left(f(nx) - f((n-2)x) - 4(n-1)f(x), (n^2 + 4n)t\right) &\leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi((n-1)x), t) \diamond \nu'(\varphi(x), t) \diamond \nu'(\varphi(0), t),\end{aligned}\quad (2.15)$$

for all $x \in X$ and $t > 0$. Setting $x_1 = x_2 = nx$, $x_i = 0$ ($i = 3, \dots, n$), $t_i = t$ ($i = 1, \dots, n$) in (2.2), we obtain

$$\begin{aligned}\mu\left(f((n-2)x) + (n-1)f(2x) - f(nx), \frac{n}{2}t\right) &\geq \mu'(\varphi(nx), t) * \mu'(\varphi(0), t), \\ \nu\left(f((n-2)x) + (n-1)f(2x) - f(nx), \frac{n}{2}t\right) &\leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi(0), t),\end{aligned}\quad (2.16)$$

for all $x \in X$ and $t > 0$. It follows from (2.15) and (2.16) that

$$\begin{aligned}\mu\left(f(2x) - 4f(x), \frac{2n^2 + 9n}{2n-2}t\right) \\ \geq \mu'(\varphi(nx), t) * \mu'(\varphi((n-1)x), t) * \mu'(\varphi(x), t) * \mu'(\varphi(0), t), \\ \nu\left(f(2x) - 4f(x), \frac{2n^2 + 9n}{2n-2}t\right) \\ \leq \nu'(\varphi(nx), t) \diamond \nu'(\varphi((n-1)x), t) \diamond \nu'(\varphi(x), t) \diamond \nu'(\varphi(0), t).\end{aligned}\quad (2.17)$$

It follows that

$$\begin{aligned}\mu\left(f(x) - 4^{-1}f(2x), t\right) &\geq \mu'\left(\varphi(nx), \frac{8(n-1)}{2n^2 + 9n}t\right) * \mu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2 + 9n}t\right) \\ &\quad * \mu'\left(\varphi(x), \frac{8(n-1)}{2n^2 + 9n}t\right) * \mu'\left(\varphi(0), \frac{8(n-1)}{2n^2 + 9n}t\right), \\ \nu\left(f(x) - 4^{-1}f(2x), t\right) &\leq \nu'\left(\varphi(nx), \frac{8(n-1)}{2n^2 + 9n}t\right) \diamond \nu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2 + 9n}t\right) \\ &\quad \diamond \nu'\left(\varphi(x), \frac{8(n-1)}{2n^2 + 9n}t\right) \diamond \nu'\left(\varphi(0), \frac{8(n-1)}{2n^2 + 9n}t\right).\end{aligned}\quad (2.18)$$

Define

$$\begin{aligned}\mu_1''(x, t) &:= \mu'\left(\varphi(nx), \frac{8(n-1)}{2n^2 + 9n}t\right) * \mu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2 + 9n}t\right) \\ &\quad * \mu'\left(\varphi(x), \frac{8(n-1)}{2n^2 + 9n}t\right) * \mu'\left(\varphi(0), \frac{8(n-1)}{2n^2 + 9n}t\right), \\ \nu_1''(x, t) &:= \nu'\left(\varphi(nx), \frac{8(n-1)}{2n^2 + 9n}t\right) \diamond \nu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2 + 9n}t\right) \\ &\quad \diamond \nu'\left(\varphi(x), \frac{8(n-1)}{2n^2 + 9n}t\right) \diamond \nu'\left(\varphi(0), \frac{8(n-1)}{2n^2 + 9n}t\right).\end{aligned}\quad (2.19)$$

Then, by our assumption,

$$\mu_1''(2x, t) = \mu_1''\left(x, \frac{t}{\alpha}\right), \quad \nu_1''(2x, t) = \nu_1''\left(x, \frac{t}{\alpha}\right). \quad (2.20)$$

Replacing x by $2^n x$ in (2.18) and applying (2.20), we get

$$\begin{aligned} \mu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1} x)}{4^{n+1}}, \frac{\alpha^n t}{4^n}\right) &= \mu\left(f(2^n x) - \frac{f(2^{n+1} x)}{4}, \alpha^n t\right) \geq \mu_1''(2^n x, \alpha^n t) = \mu_1''(x, t), \\ \nu\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1} x)}{4^{n+1}}, \frac{\alpha^n t}{4^n}\right) &= \nu\left(f(2^n x) - \frac{f(2^{n+1} x)}{4}, \alpha^n t\right) \leq \nu_1''(2^n x, \alpha^n t) = \nu_1''(x, t). \end{aligned} \quad (2.21)$$

Thus for each $n > m$, we have

$$\begin{aligned} \mu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right) &= \mu\left(\sum_{k=m}^{n-1} \frac{f(2^k x)}{4^k} - \frac{f(2^{k+1} x)}{4^{k+1}}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right) \\ &\geq \prod_{k=m}^{n-1} \mu\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1} x)}{4^{k+1}}, \frac{\alpha^k t}{4^k}\right) \geq \mu_1''(x, t), \\ \nu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right) &= \nu\left(\sum_{k=m}^{n-1} \frac{f(2^k x)}{4^k} - \frac{f(2^{k+1} x)}{4^{k+1}}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right) \\ &\leq \prod_{k=m}^{n-1} \nu\left(\frac{f(2^k x)}{4^k} - \frac{f(2^{k+1} x)}{4^{k+1}}, \frac{\alpha^k t}{4^k}\right) \leq \nu_1''(x, t), \end{aligned} \quad (2.22)$$

where $\prod_{j=1}^n a_j = a_1 * a_2 * \cdots * a_n$, $\prod_{j=1}^n a_j = a_1 \diamond a_2 \diamond \cdots \diamond a_n$. Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t \rightarrow \infty} \mu_1''(x, t) = 1$ and $\lim_{t \rightarrow \infty} \nu_1''(x, t) = 0$, there exists some $t_0 > 0$ such that $\mu_1''(x, t_0) > 1 - \varepsilon$ and $\nu_1''(x, t_0) < \varepsilon$. Since $\sum_{k=0}^{\infty} (\alpha^k t_0 / 4^k) < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} (\alpha^k t_0 / 4^k) < \delta$ for each $n > m \geq n_0$. It follows that

$$\begin{aligned} \mu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \delta\right) &\geq \mu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t_0}{4^k}\right) \geq \mu_1''(x, t_0) > 1 - \varepsilon, \\ \nu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \delta\right) &\leq \nu\left(\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t_0}{4^k}\right) \leq \nu_1''(x, t_0) < \varepsilon, \end{aligned} \quad (2.23)$$

for all $t > t_0$. This shows that the sequence $\{f(2^n x)/4^n\}$ is Cauchy in (Y, μ, ν) . Since (Y, μ, ν) is intuitionistic fuzzy Banach space, $\{f(2^n x)/4^n\}$ converges to some point $Q(x) \in Y$. Thus,

we can define a mapping $Q(x) : X \rightarrow Y$ such that $Q(x) := (\mu, \nu) - \lim_{n \rightarrow \infty} (f(2^n x)/4^n)$. Moreover, if we put $m = 0$ in (2.22), we get

$$\mu\left(\frac{f(2^n x)}{4^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{4^k}\right) \geq \mu_1''(x, t), \quad \nu\left(\frac{f(2^n x)}{4^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{4^k}\right) \leq \nu_1''(x, t). \quad (2.24)$$

Thus,

$$\mu\left(\frac{f(2^n x)}{4^n} - f(x), t\right) \geq \mu_1''\left(x, \frac{t}{\sum_{k=0}^{n-1} (\alpha/4)^k}\right), \quad \nu\left(\frac{f(2^n x)}{4^n} - f(x), t\right) \leq \nu_1''\left(x, \frac{t}{\sum_{k=0}^{n-1} (\alpha/4)^k}\right). \quad (2.25)$$

Now, we will show that Q is quadratic. Setting $x_i = 2^m x_i$ ($i = 1, \dots, n$) and $t_i = (t/n)$ ($i = 1, \dots, n$) in (2.2), we obtain

$$\begin{aligned} \mu\left(\frac{\Delta f(2^m x_1, \dots, 2^m x_n)}{4^m}, t\right) &\geq \mu'\left(\varphi(2^m x_1), 4^m \frac{t}{n}\right) * \dots * \mu'\left(\varphi(2^m x_n), 4^m \frac{t}{n}\right) \\ &= \mu'\left(\varphi(x_1), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right) * \dots * \mu'\left(\varphi(x_n), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right), \\ \nu\left(\frac{\Delta f(2^m x_1, \dots, 2^m x_n)}{4^m}, t\right) &\leq \nu'\left(\varphi(2^m x_1), 4^m \frac{t}{n}\right) \diamond \dots \diamond \nu'\left(\varphi(2^m x_n), 4^m \frac{t}{n}\right) \\ &= \nu'\left(\varphi(x_1), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right) \diamond \dots \diamond \nu'\left(\varphi(x_n), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right), \end{aligned} \quad (2.26)$$

for all $x_1, \dots, x_n \in X$ and $t > 0$. Letting $n \rightarrow \infty$ in (2.26), we obtain

$$\mu(\Delta Q(x_1, \dots, x_n), t) = 1, \quad \nu(\Delta Q(x_1, \dots, x_n), t) = 0, \quad (2.27)$$

for all $x_1, \dots, x_n \in X$ and all $t > 0$. This means that Q satisfies the functional (1.1) and so it is quadratic (see Lemma 2.2 of [4]).

Next, we approximate the difference between f and Q in intuitionistic fuzzy sense. By (2.25), we have

$$\begin{aligned} \mu(Q(x) - f(x), t) &\geq \mu\left(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}\right) * \mu\left(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2}\right) \\ &\geq \mu_1''\left(x, \frac{t}{2 \sum_{k=0}^{\infty} (\alpha/4)^k}\right) = \mu_1''\left(x, \frac{(4-\alpha)t}{8}\right), \\ \nu(Q(x) - f(x), t) &\leq \nu\left(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}\right) \diamond \nu\left(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2}\right) \\ &\leq \nu_1''\left(x, \frac{t}{2 \sum_{k=0}^{\infty} (\alpha/4)^k}\right) = \nu_1''\left(x, \frac{(4-\alpha)t}{8}\right), \end{aligned} \quad (2.28)$$

for every $x \in X$, $t > 0$ and large enough n . To prove the uniqueness of Q , assume that Q' is another quadratic mapping from X to Y , which satisfies the required inequality. Then, for each $x \in X$ and $t > 0$,

$$\begin{aligned}\mu(Q(x) - Q'(x), t) &\geq \mu\left(Q(x) - f(x), \frac{t}{2}\right) * \mu\left(Q'(x) - f(x), \frac{t}{2}\right) \geq \mu_1''\left(x, \frac{(4-\alpha)t}{16}\right), \\ \nu(Q(x) - Q'(x), t) &\leq \nu\left(Q(x) - f(x), \frac{t}{2}\right) \diamond \nu\left(Q'(x) - f(x), \frac{t}{2}\right) \leq \nu_1''\left(x, \frac{(4-\alpha)t}{16}\right).\end{aligned}\tag{2.29}$$

Since Q and Q' are quadratic, we have

$$\begin{aligned}\mu(Q(x) - Q'(x), t) &= \mu(Q(2^n x) - Q'(2^n x), 4^n t) \geq \mu_1''\left(x, \frac{(4/\alpha)^n(4-\alpha)t}{16}\right), \\ \nu(Q(x) - Q'(x), t) &= \nu(Q(2^n x) - Q'(2^n x), 4^n t) \leq \nu_1''\left(x, \frac{(4/\alpha)^n(4-\alpha)t}{16}\right),\end{aligned}\tag{2.30}$$

for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$. Since $0 < \alpha < 4$ and $\lim_{n \rightarrow \infty} (4/\alpha)^n = \infty$, we get

$$\lim_{n \rightarrow \infty} \mu_1''\left(x, \frac{(4/\alpha)^n(4-\alpha)t}{16}\right) = 1, \quad \lim_{n \rightarrow \infty} \nu_1''\left(x, \frac{(4/\alpha)^n(4-\alpha)t}{16}\right) = 0.\tag{2.31}$$

Therefore $\mu(Q(x) - Q'(x), t) = 1$ and $\nu(Q(x) - Q'(x), t) = 0$ for all $x \in X$ and $t > 0$. Hence, $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of the theorem. $\square$

Theorem 2.2. Let $\varphi : X \rightarrow Z$ be a function such that $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $0 < |\alpha| < 2$. Suppose that an odd function $f : X \rightarrow Y$ satisfies the inequality

$$\begin{aligned}\mu(\Delta f(x_1, \dots, x_n), t_1 + \dots + t_n) &\geq \mu'(\varphi(x_1), t_1) * \dots * \mu'(\varphi(x_n), t_n), \\ \nu(\Delta f(x_1, \dots, x_n), t_1 + \dots + t_n) &\leq \nu'(\varphi(x_1), t_1) \diamond \dots \diamond \nu'(\varphi(x_n), t_n),\end{aligned}\tag{2.32}$$

for all $x_1, \dots, x_n \in X$ and all $t_1, \dots, t_n > 0$. Then there exists a unique additive function $A : X \rightarrow Y$ such that

$$\mu(A(x) - f(x), t) \geq \mu_2''\left(x, \frac{(2-|\alpha|)t}{4}\right), \quad \nu(A(x) - f(x), t) \leq \nu_2''\left(x, \frac{(2-|\alpha|)t}{4}\right),\tag{2.33}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned}
 \mu_2''(x, t) &:= \mu' \left(\varphi(2x), \frac{4}{n^2 + 4n} t \right) * \mu' \left(\varphi(x), \frac{4}{n^2 + 4n} t \right) \\
 &\quad * \mu' \left(\varphi(-x), \frac{4}{n^2 + 4n} t \right) * \mu' \left(\varphi(0), \frac{4}{n^2 + 4n} t \right), \\
 \nu_2''(x, t) &:= \nu' \left(\varphi(2x), \frac{4}{n^2 + 4n} t \right) \diamond \nu' \left(\varphi(x), \frac{4}{n^2 + 4n} t \right) \\
 &\quad \diamond \nu' \left(\varphi(-x), \frac{4}{n^2 + 4n} t \right) \diamond \nu' \left(\varphi(0), \frac{4}{n^2 + 4n} t \right).
 \end{aligned} \tag{2.34}$$

Proof. Put $x_1 = nx_1$, $x_i = nx'_1$ ($i = 2, \dots, n$), $t_i = t$ ($i = 1, \dots, n$) in (2.32) and using the oddness of f , we obtain

$$\begin{aligned}
 &\mu(nf(x_1 + (n-1)x'_1) + f((n-1)(x_1 - x'_1)) - (n-1)f(x_1 - x'_1) - f(nx_1) - (n-1)f(nx'_1), nt) \\
 &\quad \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx'_1), t), \\
 &\nu(nf(x_1 + (n-1)x'_1) + f((n-1)(x_1 - x'_1)) - (n-1)f(x_1 - x'_1) - f(nx_1) - (n-1)f(nx'_1), nt) \\
 &\quad \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx'_1), t),
 \end{aligned} \tag{2.35}$$

for all $x_1, x'_1 \in X$ and $t > 0$. Interchanging x_1 with x'_1 in (2.35) and using the oddness of f , we get

$$\begin{aligned}
 &\mu(nf((n-1)x_1 + x'_1) - f((n-1)(x_1 - x'_1)) + (n-1)f(x_1 - x'_1) - (n-1)f(nx_1) - f(nx'_1), nt) \\
 &\quad \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx'_1), t), \\
 &\nu(nf((n-1)x_1 + x'_1) - f((n-1)(x_1 - x'_1)) + (n-1)f(x_1 - x'_1) - (n-1)f(nx_1) - f(nx'_1), nt) \\
 &\quad \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx'_1), t),
 \end{aligned} \tag{2.36}$$

for all $x_1, x'_1 \in X$ and $t > 0$. It follows from (2.35) and (2.36) that

$$\begin{aligned}
 &\mu(nf(x_1 + (n-1)x'_1) - nf((n-1)x_1 + x'_1) + 2f((n-1)(x_1 - x'_1)) \\
 &\quad - 2(n-1)f(x_1 - x'_1) + (n-2)f(nx_1) - (n-2)f(nx'_1), 2nt) \\
 &\quad \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx'_1), t), \\
 &\nu(nf(x_1 + (n-1)x'_1) - nf((n-1)x_1 + x'_1) + 2f((n-1)(x_1 - x'_1)) \\
 &\quad - 2(n-1)f(x_1 - x'_1) + (n-2)f(nx_1) - (n-2)f(nx'_1), 2nt) \\
 &\quad \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx'_1), t),
 \end{aligned} \tag{2.37}$$

for all $x_1, x'_1 \in X$ and $t > 0$. Setting $x_1 = nx_1$, $x_2 = -nx'_1$, $x_i = 0$ ($i = 3, \dots, n$), $t_i = t$ ($i = 1, \dots, n$) in (2.32) and using the oddness of f , we get

$$\begin{aligned}
& \mu(f((n-1)x_1 + x'_1) - f(x_1 + (n-1)x'_1) + 2f(x_1 - x'_1) - f(nx_1) + f(nx'_1), nt) \\
& \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(-nx'_1), t) * \mu'(\varphi(0), t), \\
& \nu(f((n-1)x_1 + x'_1) - f(x_1 + (n-1)x'_1) + 2f(x_1 - x'_1) - f(nx_1) + f(nx'_1), nt) \\
& \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(-nx'_1), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.38}$$

for all $x_1, x'_1 \in X$ and $t > 0$. It follows from (2.37) and (2.38) that

$$\begin{aligned}
& \mu\left(f((n-1)(x_1 - x'_1)) + f(x_1 - x'_1) - f(nx_1) + f(nx'_1), \frac{n^2 + 2n}{2}t\right) \\
& \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx'_1), t) * \mu'(\varphi(-nx'_1), t) * \mu'(\varphi(0), t), \\
& \nu\left(f((n-1)(x_1 - x'_1)) + f(x_1 - x'_1) - f(nx_1) + f(nx'_1), \frac{n^2 + 2n}{2}t\right) \\
& \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx'_1), t) \diamond \nu'(\varphi(-nx'_1), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.39}$$

for all $x_1, x'_1 \in X$ and $t > 0$. Putting $x_1 = n(x_1 - x'_1)$, $x_i = 0$ ($i = 2, \dots, n$), $t_i = t$ ($i = 1, \dots, n$) in (2.32), we obtain

$$\begin{aligned}
& \mu(f(n(x_1 - x'_1)) - f((n-1)(x_1 - x'_1)) - f(x_1 - x'_1), nt) \geq \mu'(\varphi(n(x_1 - x'_1)), t) * \mu'(\varphi(0), t), \\
& \nu(f(n(x_1 - x'_1)) - f((n-1)(x_1 - x'_1)) - f(x_1 - x'_1), nt) \leq \nu'(\varphi(n(x_1 - x'_1)), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.40}$$

for all $x_1, x'_1 \in X$ and $t > 0$. It follows from (2.39) and (2.40) that

$$\begin{aligned}
& \mu\left(f(n(x_1 - x'_1)) - f(nx_1) + f(nx'_1), \frac{n^2 + 4n}{2}t\right) \\
& \geq \mu'(\varphi(n(x_1 - x'_1)), t) * \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx'_1), t) * \mu'(\varphi(-nx'_1), t) * \mu'(\varphi(0), t), \\
& \nu\left(f(n(x_1 - x'_1)) - f(nx_1) + f(nx'_1), \frac{n^2 + 4n}{2}t\right) \\
& \leq \nu'(\varphi(n(x_1 - x'_1)), t) \diamond \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx'_1), t) \diamond \nu'(\varphi(-nx'_1), t) \diamond \nu'(\varphi(0), t),
\end{aligned} \tag{2.41}$$

for all $x_1, x'_1 \in X$ and $t > 0$. Replacing x_1 and x'_1 by x/n and $-x/n$ in (2.41); respectively, we have

$$\begin{aligned}
 & \mu\left(f(2x) - 2f(x), \frac{n^2 + 4n}{2}t\right) \\
 & \geq \mu'(\varphi(2x), t) * \mu'(\varphi(x), t) * \mu'(\varphi(-x), t) * \mu'(\varphi(0), t), \\
 & \nu\left(f(2x) - 2f(x), \frac{n^2 + 4n}{2}t\right) \\
 & \leq \nu'(\varphi(2x), t) \diamond \nu'(\varphi(x), t) \diamond \nu'(\varphi(-x), t) \diamond \nu'(\varphi(0), t).
 \end{aligned} \tag{2.42}$$

It follows that

$$\begin{aligned}
 \mu\left(f(x) - 2^{-1}f(2x), t\right) & \geq \mu'\left(\varphi(2x), \frac{4}{n^2 + 4n}t\right) * \mu'\left(\varphi(x), \frac{4}{n^2 + 4n}t\right) \\
 & \quad * \mu'\left(\varphi(-x), \frac{4}{n^2 + 4n}t\right) * \mu'\left(\varphi(0), \frac{4}{n^2 + 4n}t\right), \\
 \nu\left(f(x) - 2^{-1}f(2x), t\right) & \leq \nu'\left(\varphi(2x), \frac{4}{n^2 + 4n}t\right) \diamond \nu'\left(\varphi(x), \frac{4}{n^2 + 4n}t\right) \\
 & \quad \diamond \nu'\left(\varphi(-x), \frac{4}{n^2 + 4n}t\right) \diamond \nu'\left(\varphi(0), \frac{4}{n^2 + 4n}t\right).
 \end{aligned} \tag{2.43}$$

Define

$$\begin{aligned}
 \mu''_2(x, t) & := \mu'\left(\varphi(2x), \frac{4}{n^2 + 4n}t\right) * \mu'\left(\varphi(x), \frac{4}{n^2 + 4n}t\right) \\
 & \quad * \mu'\left(\varphi(-x), \frac{4}{n^2 + 4n}t\right) * \mu'\left(\varphi(0), \frac{4}{n^2 + 4n}t\right), \\
 \nu''_2(x, t) & := \nu'\left(\varphi(2x), \frac{4}{n^2 + 4n}t\right) \diamond \nu'\left(\varphi(x), \frac{4}{n^2 + 4n}t\right) \\
 & \quad \diamond \nu'\left(\varphi(-x), \frac{4}{n^2 + 4n}t\right) \diamond \nu'\left(\varphi(0), \frac{4}{n^2 + 4n}t\right).
 \end{aligned} \tag{2.44}$$

Then by the assumption

$$\mu''_2(2x, t) = \mu''_2\left(x, \frac{t}{\alpha}\right), \quad \nu''_2(2x, t) = \nu''_2\left(x, \frac{t}{\alpha}\right). \tag{2.45}$$

Replacing x by $2^n x$ in (2.43) and using (2.45), we obtain

$$\begin{aligned}
 \mu\left(\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1} x)}{2^{n+1}}, \frac{\alpha^n t}{2^n}\right) &= \mu\left(f(2^n x) - \frac{f(2^{n+1} x)}{2}, \alpha^n t\right) \\
 &\geq \mu_2''(2^n x, \alpha^n t) = \mu_2''(x, t), \\
 \nu\left(\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1} x)}{2^{n+1}}, \frac{\alpha^n t}{2^n}\right) &= \nu\left(f(2^n x) - \frac{f(2^{n+1} x)}{2}, \alpha^n t\right) \\
 &\leq \nu_2''(2^n x, \alpha^n t) = \nu_2''(x, t).
 \end{aligned} \tag{2.46}$$

Thus, for each $n > m$, we have

$$\begin{aligned}
 \mu\left(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{2^k}\right) &= \mu\left(\sum_{k=m}^{n-1} \frac{f(2^k x)}{2^k} - \frac{f(2^{k+1} x)}{2^{k+1}}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{2^k}\right) \\
 &\geq \prod_{k=m}^{n-1} \mu\left(\frac{f(2^k x)}{2^k} - \frac{f(2^{k+1} x)}{2^{k+1}}, \frac{\alpha^k t}{2^k}\right) \geq \mu_2''(x, t), \\
 \nu\left(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{2^k}\right) &= \nu\left(\sum_{k=m}^{n-1} \frac{f(2^k x)}{2^k} - \frac{f(2^{k+1} x)}{2^{k+1}}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{2^k}\right) \\
 &\leq \prod_{k=m}^{n-1} \nu\left(\frac{f(2^k x)}{2^k} - \frac{f(2^{k+1} x)}{2^{k+1}}, \frac{\alpha^k t}{2^k}\right) \leq \nu_2''(x, t),
 \end{aligned} \tag{2.47}$$

where $\prod_{j=1}^n a_j = a_1 * a_2 * \cdots * a_n$, $\prod_{j=1}^n a_j = a_1 \diamond a_2 \diamond \cdots \diamond a_n$. Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t \rightarrow \infty} \mu_2''(x, t) = 1$ and $\lim_{t \rightarrow \infty} \nu_2''(x, t) = 0$, there exists some $t_0 > 0$ such that $\mu_2''(x, t_0) > 1 - \varepsilon$ and $\nu_2''(x, t_0) < \varepsilon$. Since $\sum_{k=0}^{\infty} \alpha^k t_0 / 2^k < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} \alpha^k t_0 / 2^k < \delta$ for each $n > m \geq n_0$. It follows that

$$\begin{aligned}
 \mu\left(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \delta\right) &\geq \mu\left(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t_0}{2^k}\right) \geq \mu_2''(x, t_0) > 1 - \varepsilon, \\
 \nu\left(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \delta\right) &\leq \nu\left(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \sum_{k=m}^{n-1} \frac{\alpha^k t_0}{2^k}\right) \leq \nu_2''(x, t_0) < \varepsilon,
 \end{aligned} \tag{2.48}$$

for all $t > t_0$. This shows that the sequence $\{f(2^n x)/2^n\}$ is Cauchy in (Y, μ, ν) . Since (Y, μ, ν) is intuitionistic fuzzy Banach space, $\{f(2^n x)/2^n\}$ converges to some point $A(x) \in Y$. Thus, we can define a mapping $A(x) : X \rightarrow Y$ such that $A(x) := (\mu, \nu) - \lim_{n \rightarrow \infty} f(2^n x)/2^n$. Moreover, if we put $m = 0$ in (2.47), we get

$$\mu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{2^k}\right) \geq \mu_2''(x, t), \quad \nu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{2^k}\right) \leq \nu_2''(x, t). \tag{2.49}$$

Thus,

$$\mu\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \geq \mu_2''\left(x, \frac{t}{\sum_{k=0}^{n-1} (\alpha/2)^k}\right), \quad \nu\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \leq \nu_2''\left(x, \frac{t}{\sum_{k=0}^{n-1} (\alpha/2)^k}\right). \quad (2.50)$$

Next we will show that A is additive. Putting $x_i = 2^m x_i$ ($i = 1, \dots, n$) and $t_i = (t/n)$ ($i = 1, \dots, n$) in (2.32), we obtain

$$\begin{aligned} \mu\left(\frac{\Delta f(2^m x_1, \dots, 2^m x_n)}{2^m}, t\right) &\geq \mu'\left(\varphi(2^m x_1), 2^m \frac{t}{n}\right) * \dots * \mu'\left(\varphi(2^m x_n), 2^m \frac{t}{n}\right) \\ &= \mu'\left(\varphi(x_2), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right) * \dots * \mu'\left(\varphi(x_n), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right), \\ \nu\left(\frac{\Delta f(2^m x_1, \dots, 2^m x_n)}{2^m}, t\right) &\leq \nu'\left(\varphi(2^m x_1), 2^m \frac{t}{n}\right) \diamond \dots \diamond \nu'\left(\varphi(2^m x_n), 2^m \frac{t}{n}\right) \\ &= \nu'\left(\varphi(x_2), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right) \diamond \dots \diamond \nu'\left(\varphi(x_n), \left(\frac{2}{\alpha}\right)^m \frac{t}{n}\right), \end{aligned} \quad (2.51)$$

for all $x_1, \dots, x_n \in X$ and $t > 0$. Letting $n \rightarrow \infty$ in (2.51), we obtain

$$\mu(\Delta A(x_1, \dots, x_n), t) = 1, \quad \nu(\Delta A(x_1, \dots, x_n), t) = 0, \quad (2.52)$$

for all $x_1, \dots, x_n \in X$ and all $t > 0$. This means that A satisfies the functional (1.1), and so it is additive (see Lemma 2.1 of [4]).

Now, we approximate the difference between f and A in intuitionistic fuzzy sense. For every $x \in X$, $t > 0$, and sufficiently large n , by (2.50), we have

$$\begin{aligned} \mu(A(x) - f(x), t) &\geq \mu\left(A(x) - \frac{f(2^n x)}{2^n}, \frac{t}{2}\right) * \mu\left(\frac{f(2^n x)}{2^n} - f(x), \frac{t}{2}\right) \\ &\geq \mu_2''\left(x, \frac{t}{2 \sum_{k=0}^{\infty} (\alpha/2)^k}\right) = \mu_2''\left(x, \frac{(2-\alpha)t}{4}\right), \\ \nu(A(x) - f(x), t) &\leq \nu\left(A(x) - \frac{f(2^n x)}{2^n}, \frac{t}{2}\right) \diamond \nu\left(\frac{f(2^n x)}{2^n} - f(x), \frac{t}{2}\right) \\ &\leq \nu_2''\left(x, \frac{t}{2 \sum_{k=0}^{\infty} (\alpha/2)^k}\right) = \nu_2''\left(x, \frac{(2-\alpha)t}{4}\right). \end{aligned} \quad (2.53)$$

To prove the uniqueness of A , assume that A' is another additive mapping from X to Y , which satisfies the required inequality. Then, for each $x \in X$ and $t > 0$,

$$\begin{aligned}\mu(A(x) - A'(x), t) &\geq \mu\left(A(x) - f(x), \frac{t}{2}\right) * \mu\left(A'(x) - f(x), \frac{t}{2}\right) \geq \mu_2''\left(x, \frac{(2-\alpha)t}{8}\right), \\ \nu(A(x) - A'(x), t) &\leq \nu\left(A(x) - f(x), \frac{t}{2}\right) \diamond \nu\left(A'(x) - f(x), \frac{t}{2}\right) \leq \nu_2''\left(x, \frac{(2-\alpha)t}{8}\right).\end{aligned}\quad (2.54)$$

Therefore, by the additivity of A and A' , we have

$$\begin{aligned}\mu(A(x) - A'(x), t) &= \mu(A(2^n x) - A'(2^n x), 2^n t) \geq \mu_2''\left(x, \frac{(2/\alpha)^n(2-\alpha)t}{8}\right), \\ \nu(A(x) - A'(x), t) &= \nu(A(2^n x) - A'(2^n x), 2^n t) \leq \nu_2''\left(x, \frac{(2/\alpha)^n(2-\alpha)t}{8}\right),\end{aligned}\quad (2.55)$$

for all $x \in X$, $t > 0$ and $n \in \mathbb{N}$. Since $0 < \alpha < 2$ and $\lim_{n \rightarrow \infty} (2/\alpha)^n = \infty$, we get

$$\lim_{n \rightarrow \infty} \mu_2''\left(x, \frac{(2/\alpha)^n(2-\alpha)t}{8}\right) = 1, \quad \lim_{n \rightarrow \infty} \nu_2''\left(x, \frac{(2/\alpha)^n(2-\alpha)t}{8}\right) = 0. \quad (2.56)$$

Therefore, $\mu(A(x) - A'(x), t) = 1$ and $\nu(A(x) - A'(x), t) = 0$ for all $x \in X$ and $t > 0$. Hence, $A(x) = A'(x)$ for all $x \in X$. This completes the proof of the theorem. $\square$

Theorem 2.3. Let $\varphi : X \rightarrow Z$ be a function such that $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $0 < |\alpha| < 2$. Suppose that a function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\begin{aligned}\mu(\Delta f(x_1, \dots, x_n), t_1 + \dots + t_n) &\geq \mu'(\varphi(x_1), t_1) * \dots * \mu'(\varphi(x_n), t_n), \\ \nu(\Delta f(x_1, \dots, x_n), t_1 + \dots + t_n) &\leq \nu'(\varphi(x_1), t_1) \diamond \dots \diamond \nu'(\varphi(x_n), t_n),\end{aligned}\quad (2.57)$$

for all $x_1, \dots, x_n \in X$ and all $t_1, \dots, t_n > 0$. Then there exists a unique quadratic function $Q : X \rightarrow Y$ and a unique additive function $A : X \rightarrow Y$ such that

$$\begin{aligned}\mu(Q(x) - A(x) - f(x), t) &\geq M_1''\left(x, \frac{(4-|\alpha|)t}{16}\right) * \widetilde{M}_1''\left(x, \frac{(2-|\alpha|)t}{8}\right), \\ \nu(Q(x) - A(x) - f(x), t) &\leq M_2''\left(x, \frac{(4-|\alpha|)t}{16}\right) \diamond \widetilde{M}_2''\left(x, \frac{(2-|\alpha|)t}{8}\right),\end{aligned}\quad (2.58)$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned}
M_1''(x, t) &:= \mu' \left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t \right) * \mu' \left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t \right) * \mu' \left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t \right) \\
&\quad * \mu' \left(\varphi(-nx), \frac{8(n-1)}{2n^2+9n}t \right) * \mu' \left(\varphi(-(n-1)x), \frac{8(n-1)}{2n^2+9n}t \right) \\
&\quad * \mu' \left(\varphi(-x), \frac{8(n-1)}{2n^2+9n}t \right) * \mu' \left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t \right), \\
\widetilde{M}_1''(x, t) &:= \mu' \left(\varphi(2x), \frac{4}{n^2+4n}t \right) * \mu' \left(\varphi(x), \frac{4}{n^2+4n}t \right) * \mu' \left(\varphi(-x), \frac{4}{n^2+4n}t \right) \\
&\quad * \mu' \left(\varphi(-2x), \frac{4}{n^2+4n}t \right) * \mu' \left(\varphi(0), \frac{4}{n^2+4n}t \right), \\
M_2''(x, t) &:= \nu' \left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t \right) \diamond \nu' \left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t \right) \diamond \nu' \left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t \right) \\
&\quad \diamond \nu' \left(\varphi(-nx), \frac{8(n-1)}{2n^2+9n}t \right) \diamond \nu' \left(\varphi(-(n-1)x), \frac{8(n-1)}{2n^2+9n}t \right) \\
&\quad \diamond \nu' \left(\varphi(-x), \frac{8(n-1)}{2n^2+9n}t \right) \diamond \nu' \left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t \right), \\
\widetilde{M}_2''(x, t) &:= \nu' \left(\varphi(2x), \frac{4}{n^2+4n}t \right) \diamond \nu' \left(\varphi(x), \frac{4}{n^2+4n}t \right) \diamond \nu' \left(\varphi(-x), \frac{4}{n^2+4n}t \right) \\
&\quad \diamond \nu' \left(\varphi(-2x), \frac{4}{n^2+4n}t \right) \diamond \nu' \left(\varphi(0), \frac{4}{n^2+4n}t \right).
\end{aligned} \tag{2.59}$$

Proof. Passing to the even part f_e and odd part f_o of f , we deduce from (2.57) that

$$\begin{aligned}
\mu(\Delta f_e(x_1, \dots, x_n), t_1 + \dots + t_n) &\geq \mu'(\varphi(x_1), t_1) * \mu'(\varphi(-x_1), t_1) \\
&\quad * \dots * \mu'(\varphi(x_n), t_n) * \mu'(\varphi(-x_n), t_n), \\
\nu(\Delta f_e(x_1, \dots, x_n), t_1 + \dots + t_n) &\leq \nu'(\varphi(x_1), t_1) \diamond \nu'(\varphi(-x_1), t_1) \\
&\quad \diamond \dots \diamond \nu'(\varphi(x_n), t_n) \diamond \nu'(\varphi(-x_n), t_n).
\end{aligned} \tag{2.60}$$

On the other hand,

$$\begin{aligned}
\mu(\Delta f_o(x_1, \dots, x_n), t_1 + \dots + t_n) &\geq \mu'(\varphi(x_1), t_1) * \mu'(\varphi(-x_1), t_1) \\
&\quad * \dots * \mu'(\varphi(x_n), t_n) * \mu'(\varphi(-x_n), t_n), \\
\nu(\Delta f_o(x_1, \dots, x_n), t_1 + \dots + t_n) &\leq \nu'(\varphi(x_1), t_1) \diamond \nu'(\varphi(-x_1), t_1) \\
&\quad \diamond \dots \diamond \nu'(\varphi(x_n), t_n) \diamond \nu'(\varphi(-x_n), t_n).
\end{aligned} \tag{2.61}$$

Applying the proofs of Theorems 2.1 and 2.2, we get a unique quadratic function Q and a unique additive function A satisfying

$$\mu(Q(x) - f_e(x), t) \geq M_1''\left(x, \frac{(4 - |\alpha|)t}{8}\right), \quad \nu(Q(x) - f_e(x), t) \leq M_2''\left(x, \frac{(4 - |\alpha|)t}{8}\right). \quad (2.62)$$

Also,

$$\mu(A(x) - f_o(x), t) \geq \widetilde{M}_1''\left(x, \frac{(2 - |\alpha|)t}{4}\right), \quad \nu(A(x) - f_o(x), t) \leq \widetilde{M}_2''\left(x, \frac{(2 - |\alpha|)t}{4}\right). \quad (2.63)$$

Therefore,

$$\begin{aligned} \mu(Q(x) - A(x) - f(x), t) &\geq M_1''\left(x, \frac{(4 - |\alpha|)t}{16}\right) * \widetilde{M}_1''\left(x, \frac{(2 - |\alpha|)t}{8}\right), \\ \nu(Q(x) - A(x) - f(x), t) &\leq M_2''\left(x, \frac{(4 - |\alpha|)t}{16}\right) \diamond \widetilde{M}_2''\left(x, \frac{(2 - |\alpha|)t}{8}\right). \end{aligned} \quad (2.64)$$

This completes the proof of the theorem. $\square$

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