A Class of Analytic Functions with Missing Coefficients

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1. Introduction

Let $T_n(A, B, \gamma, \alpha)$ denote the class of functions of the form

$$ f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N}) $$

which are analytic in the open unit disk $U = \{ z : |z| < 1 \}$. Let $S_n$ and $S^*_n$ denote the subclasses of $A_n$ whose members are univalent and starlike, respectively.

For functions $f(z)$ and $g(z)$ analytic in $U$, we say that $f(z)$ is subordinate to $g(z)$ in $U$ and we write $f(z) \prec g(z)$ ($z \in U$), if there exists an analytic function $w(z)$ in $U$ such that

$$ |w(z)| \leq |z|, \quad f(z) = g(w(z)) \quad (z \in U). $$

Let $A_n$ denote the class of functions of the form

$$ f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \{1, 2, 3, \ldots\}), $$

which are analytic in the open unit disk $U = \{ z : |z| < 1 \}$. Let $S_n$ and $S^*_n$ denote the subclasses of $A_n$ whose members are univalent and starlike, respectively.

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1.1

Let $S_0(A, B, \gamma, \alpha)$ denote the class of functions of the form

$$ f(z) = z + \sum_{k=1}^{\infty} a_k z^k \quad (n \in \mathbb{N}) $$

which are analytic in the open unit disk $U = \{ z : |z| < 1 \}$. Let $S_n$ and $S^*_n$ denote the subclasses of $A_n$ whose members are univalent and starlike, respectively.

For functions $f(z)$ and $g(z)$ analytic in $U$, we say that $f(z)$ is subordinate to $g(z)$ in $U$ and we write $f(z) \prec g(z)$ ($z \in U$), if there exists an analytic function $w(z)$ in $U$ such that

$$ |w(z)| \leq |z|, \quad f(z) = g(w(z)) \quad (z \in U). $$

1.2
Furthermore, if the function $g(z)$ is univalent in $U$, then

$$f(z) < g(z) \quad (z \in U) \iff f(0) = g(0), \quad f(U) \subset g(U). \quad (1.3)$$

Throughout our present discussion, we assume that

$$n \in \mathbb{N}, \quad -1 \leq B < 1, \quad B < A, \quad \alpha > 0, \quad \beta < 1, \quad 0 < \gamma \leq 1. \quad (1.4)$$

We introduce the following subclass of $A_n$.

**Definition 1.1.** A function $f(z) \in A_n$ is said to be in the class $T_n(A, B, \gamma, \alpha)$ if it satisfies

$$f'(z) + azf''(z) < h(z) \quad (z \in U), \quad (1.5)$$

where

$$h(z) = \begin{cases} 
\left(\frac{1 + Az}{1 + Bz}\right)\gamma, & (A \leq 1; 0 < \gamma < 1), \\
\frac{1 + Az}{1 + Bz}, & (\gamma = 1).
\end{cases} \quad (1.6)$$

The classes

$$T_1(1 - 2\beta, -1, 1, 1) = R(\beta) \quad (\beta = 0 \text{ or } \beta < 1), \quad T_1(A, 0, 1, \alpha) = \tilde{R}(\alpha, A) \quad (A > 0) \quad (1.7)$$

have been studied by several authors (see [1–5]). Recently, Gao and Zhou [6] showed some mapping properties of the following subclass of $A_1$:

$$R(\beta, \alpha) = \{ f(z) \in A_1 : \text{Re}\{f'(z) + azf''(z)\} > \beta \ (z \in U) \}. \quad (1.8)$$

Note that

$$R(\beta, 1) = R(\beta), \quad T_1(1 - 2\beta, -1, 1, \alpha) = R(\beta, \alpha). \quad (1.9)$$

For further information of the above classes (with $\gamma = 1$) and related analytic function classes, see Srivastava et al. [7], Yang and Liu [8], Kim [9], and Kim and Srivastava [10].

In this paper, we obtain sharp bounds on $\text{Re} f'(z)$, $\text{Re}(f(z)/z)$, $|f(z)|$, and coefficient estimates for functions $f(z)$ belonging to the class $T_n(A, B, \gamma, \alpha)$. Conditions for univalency and starlikeness, convolution properties, and the radius of convexity are also presented. One can see that the methods used in [6] do not work for the more general class $T_n(A, B, \gamma, \alpha)$ than $R(\beta, \alpha)$. 
2. The bounds on $Re f'(z)$, $Re (f(z)/z)$, and $|f(z)|$ in $T_n(A, B, \gamma, \alpha)$

In this section, we let

$$\lambda_m(A, B, \gamma) = \begin{cases} 
\sum_{j=0}^{m} \binom{\gamma}{j} \binom{-\gamma}{m-j} A^j B^{m-j}, & (A \leq 1; 0 < \gamma < 1), \\
(A - B)(-B)^{m-1}, & (\gamma = 1),
\end{cases}$$

(2.1)

where $m \in \mathbb{N}$ and

$$\binom{\gamma}{j} = \begin{cases} 
\frac{\gamma(\gamma-1) \cdots (\gamma-j+1)}{j!}, & (j = 1, 2, \ldots, m), \\
1, & (j = 0).
\end{cases}$$

(2.2)

With (2.1), it is easily seen that the function $h(z)$ given by (1.6) can be expressed as

$$h(z) = 1 + \sum_{m=1}^{\infty} \lambda_m(A, B, \gamma) z^m \quad (z \in U).$$

(2.3)

**Theorem 2.1.** Let $f(z) \in T_n(A, B, \gamma, \alpha)$. Then, for $|z| = r < 1$,

$$Re f'(z) \geq 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \lambda_m(A, B, \gamma)}{anm + 1} r^{nm},$$

$$Re f'(z) \leq 1 + \sum_{m=1}^{\infty} \frac{\lambda_m(A, B, \gamma)}{anm + 1} r^{nm}.$$  

(2.4)

The bounds in (2.4) are sharp for the function $f_n(z)$ defined by

$$f_n(z) = z + \sum_{m=1}^{\infty} \frac{\lambda_m(A, B, \gamma)}{(nm + 1)(anm + 1)} z^{nm+1} \quad (z \in U).$$

(2.5)

**Proof.** The analytic function $h(z)$ given by (1.6) is convex (univalent) in $U$ (cf. [11]) and satisfies $h(\overline{z}) = \overline{h(z)}$ $(z \in U)$. Thus, for $|\zeta| \leq \sigma (\zeta \in C \text{ and } \sigma < 1)$,

$$h(-\sigma) \leq \text{Re } h(\zeta) \leq h(\sigma).$$

(2.6)

Let $f(z) \in T_n(A, B, \gamma, \alpha)$. Then, we can write

$$f'(z) + azf''(z) = h(w(z)) \quad (z \in U),$$

(2.7)
where \( w(z) = w_n z^n + w_{n+1} z^{n+1} + \cdots \) is analytic and \( |w(z)| < 1 \) for \( z \in U \). By the Schwarz lemma, we know that \( |w(z)| \leq |z|^n \) \( (z \in U) \). It follows from (2.7) that

\[
\left( z^{1/\alpha} f'(z) \right)' = \frac{1}{\alpha} z^{(1/\alpha) - 1} h(w(z)), \tag{2.8}
\]

which leads to

\[
f'(z) = \frac{1}{\alpha} z^{-1/\alpha} \int_0^z \zeta^{(1/\alpha) - 1} h(w(\zeta)) d\zeta \tag{2.9}
\]

or to

\[
f'(z) = \frac{1}{\alpha} \int_0^1 t^{(1/\alpha) - 1} h(w(tz)) dt \quad (z \in U). \tag{2.10}
\]

Since

\[
|w(tz)| \leq (tr)^n \quad (|z| = r < 1; 0 \leq t \leq 1), \tag{2.11}
\]

we deduce from (2.6) and (2.10) that

\[
\frac{1}{\alpha} \int_0^1 t^{(1/\alpha) - 1} h(-(tr)^n) dt \leq \text{Re} f'(z) \leq \frac{1}{\alpha} \int_0^1 t^{(1/\alpha) - 1} h((tr)^n) dt. \tag{2.12}
\]

Now, by using (2.3) and (2.12), we can obtain (2.4).

Furthermore, for the function \( f_n(z) \) defined by (2.5), we find that

\[
f'_n(z) = 1 + \sum_{m=1}^{\infty} \frac{\lambda_m(A, B, \gamma)}{anm + 1} z^{nm}, \tag{2.13}
\]

\[
f'_n(z) + \alpha z f''_n(z) = 1 + \sum_{m=1}^{\infty} \lambda_m(A, B, \gamma) z^{nm} = h(z^n) < h(z) \quad (z \in U). \tag{2.14}
\]

Hence, \( f_n(z) \in T_n(A, B, \gamma, \alpha) \) and from (2.13), we see that the bounds in (2.4) are the best possible.

Hereafter, we write

\[
T_n(A, B, 1, \alpha) = T_n(A, B, \alpha). \tag{2.15}
\]
Corollary 2.2. Let \( f(z) \in T_n(A, B, \alpha) \). Then, for \( z \in U \),

\[
\text{Re} \ f'(z) > 1 - (A - B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{anm + 1}, \tag{2.16}
\]

\[
\text{Re} \ f'(z) < 1 + (A - B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1}}{anm + 1} \quad (B \neq -1). \tag{2.17}
\]

The results are sharp.

Proof. For \( \gamma = 1 \), it follows from (2.12) (used in the proof of Theorem 2.1) that

\[
\text{Re} \ f'(z) > \frac{1}{\alpha} \int_0^1 t^{(1/\alpha) - 1} \left( \frac{1 - At^n}{1 - Bt^n} \right) dt,
\]

\[
\text{Re} \ f'(z) < \frac{1}{\alpha} \int_0^1 t^{(1/\alpha) - 1} \left( \frac{1 + At^n}{1 + Bt^n} \right) dt \quad (B \neq -1),
\]

for \( z \in U \). From these, we have the desired results. \( \square \)

The bounds in (2.16) and (2.17) are sharp for the function

\[
f_n(z) = z + (A - B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1}}{(nm + 1)(anm + 1)} z^{nm+1} \in T_n(A, B, \alpha). \tag{2.19}
\]

Theorem 2.3. Let \( f(z) \in T_n(A, B, \gamma, \alpha) \). Then, for \( |z| = r < 1 \),

\[
\text{Re} \ \frac{f(z)}{z} \geq 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \lambda_m(A, B, \gamma)}{(nm + 1)(anm + 1)} r^{nm},
\]

\[
\text{Re} \ \frac{f(z)}{z} \leq 1 + \sum_{m=1}^{\infty} \frac{\lambda_m(A, B, \gamma)}{(nm + 1)(anm + 1)} r^{nm}. \tag{2.20}
\]

The results are sharp.

Proof. Noting that

\[
f(z) = z \int_0^1 f'(uz) du, \quad \text{Re} \ \frac{f(z)}{z} = \int_0^1 \text{Re} \ f'(uz) du \quad (z \in U), \tag{2.21}
\]

an application of Theorem 2.1 yields (2.20). Furthermore, the results are sharp for the function \( f_n(z) \) defined by (2.5). \( \square \)
Corollary 2.4. Let \( f(z) \in T_n(A, B, \alpha) \). Then, for \( z \in U \),

\[
\begin{align*}
\text{Re} \left( \frac{f(z)}{z} \right) &> 1 - (A - B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{(nm + 1)(anm + 1)}, \\
\text{Re} \left( \frac{f(z)}{z} \right) &< 1 + (A - B) \sum_{m=1}^{\infty} \frac{(-B)^{m-1}}{(nm + 1)(anm + 1)}.
\end{align*}
\tag{2.22}
\]

The results are sharp for the function \( f_n(z) \) defined by (2.19).

Proof. For \( f(z) \in T_n(A, B, \alpha) \), it follows from (2.6) and (2.10) (with \( \gamma = 1 \)) that

\[
\frac{1}{\alpha} \int_0^1 f^{(1/\alpha)-1} \left( \frac{1 - A(u t)^n}{1 - B(u t)^n} \right) \,dt < \text{Re} f'(uz) < \frac{1}{\alpha} \int_0^1 f^{(1/\alpha)-1} \left( \frac{1 + A(u t)^n}{1 + B(u t)^n} \right) \,dt,
\tag{2.23}
\]

for \( z \in U \) and \( 0 < u \leq 1 \). Making use of (2.21) and (2.23), we can obtain (2.22). \( \square \)

Theorem 2.5. Let \( f(z) \in T_1(A, B, \alpha) \) and \( g(z) \in T_1(A_0, B_0, \alpha_0) \) \(( -1 \leq B_0 < 1, B_0 < A_0 \) and \( \alpha_0 > 0 \)). If

\[
(A_0 - B_0) \sum_{m=1}^{\infty} \frac{B^{m-1}_0}{(m + 1)(\alpha_0 m + 1)} \leq \frac{1}{2},
\tag{2.24}
\]

then \( (f \ast g)(z) \in T_1(A, B, \alpha) \), where the symbol \( \ast \) stands for the familiar Hadamard product (or convolution) of two analytic functions in \( U \).

Proof. Since \( g(z) \in T_1(A_0, B_0, \alpha_0)(-1 \leq B_0 < 1, B_0 < A_0 \) and \( \alpha_0 > 0 \)), it follows from Corollary 2.4 (with \( n = 1 \)) and (2.24) that

\[
\text{Re} \left( \frac{g(z)}{z} \right) > 1 - (A_0 - B_0) \sum_{m=1}^{\infty} \frac{B^{m-1}_0}{(m + 1)(\alpha_0 m + 1)} \geq \frac{1}{2} \quad (z \in U).
\tag{2.25}
\]

Thus, \( g(z)/z \) has the Herglotz representation

\[
\frac{g(z)}{z} = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in U),
\tag{2.26}
\]

where \( \mu(x) \) is a probability measure on the unit circle \( |x| = 1 \) and \( \int_{|x|=1} d\mu(x) = 1 \).

For \( f(z) \in T_1(A, B, \alpha) \), we have

\[
(f \ast g)'(z) + az(f \ast g)^n(z) = F(z) \ast \frac{g(z)}{z} \quad (z \in U),
\tag{2.27}
\]

(2.27)
where

\[ F(z) = f'(z) + azf''(z) < \frac{1 + Az}{1 + Bz} \quad (z \in U). \]  

(2.28)

In view of the function \((1 + Az)/(1 + Bz)\) is convex (univalent) in \(U\), we deduce from (2.26) to (2.28) that

\[ (f * g)'(z) + az(f * g)''(z) = \int_{|x|=1} F(xz)d\mu(x) < \frac{1 + Az}{1 + Bz} \quad (z \in U). \]  

(2.29)

This shows that \((f * g)(z) \in T_1(A, B, \alpha)\).

\[ \square \]

**Corollary 2.6.** Let \(f(z) \in T_1(A, B, \alpha)\), \(g(z) \in R(\beta, 1)\) and

\[ \beta \geq -\frac{\pi^2 - 9}{12 - \pi^2}. \]  

(3.30)

Then, \((f * g)(z) \in T_1(A, B, \alpha)\).

**Proof.** By taking \(A_0 = 1 - 2\beta, B_0 = -1\) and \(a_0 = 1\), (2.24) in Theorem 2.5 becomes

\[ 2(1 - \beta) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(m + 1)^2} = 2(1 - \beta) \left(1 - \frac{\pi^2}{12}\right) \leq \frac{1}{2}, \]  

(3.31)

that is,

\[ \beta \geq -\frac{\pi^2 - 9}{12 - \pi^2}. \]  

(3.32)

Hence, the desired result follows as a special case from Theorem 2.5.

\[ \square \]

**Remark 2.7.** R. Singh and S. Singh [4, Theorem 3] proved that, if \(f(z)\) and \(g(z)\) belong to \(R(0, 1)\), then \((f \ast g)(z) \in R(0, 1)\). Obviously, for

\[ -\frac{\pi^2 - 9}{12 - \pi^2} \leq \beta < 0, \]  

(3.33)

Corollary 2.6 generalizes and improves Theorem 3 in [4].

**Theorem 2.8.** Let \(f(z) \in T_n(A, B, \gamma, \alpha)\) and \(AB \leq 1\). Then, for \(|z| = r < 1\),

\[ |f(z)| \leq r + \sum_{m=1}^{\infty} \frac{\lambda_m(A, B, \gamma)}{(nm + 1)(anm + 1)} r^{nm+1}. \]  

(3.34)

The result is sharp, with the extremal function \(f_n(z)\) defined by (2.5).
Proof. It is well known that for $\zeta \in \mathbb{C}$ and $|\zeta| \leq \sigma < 1$,

$$\frac{1 + A\zeta - 1 - AB\sigma^2}{1 + B\zeta - 1 - B^2\sigma^2} \leq \frac{(A - B)\sigma}{1 - B^2\sigma^2} \tag{2.35}$$

Since $AB \leq 1$, we have $1 - AB\sigma^2 > 0$ and so (2.35) leads to

$$\frac{1 + A\zeta}{1 + B\zeta} \leq \left( \frac{1 - AB\sigma^2}{1 - B^2\sigma^2} + \frac{(A - B)\sigma}{1 - B^2\sigma^2} \right)^{\gamma} = \left( \frac{1 + A\sigma}{1 + B\sigma} \right)^{\gamma} \leq \left( |\zeta| \leq \sigma < 1 \right). \tag{2.36}$$

By virtue of (1.6), (2.10), and (2.36), we have

$$|f'(uz)| \leq \frac{1}{\alpha} \int_0^1 t^{(1/\alpha) - 1} |h(w(utz))| dt \leq \frac{1}{\alpha} \int_0^1 t^{(1/\alpha) - 1} h((ut|z|)^n) dt, \tag{2.37}$$

for $z \in U$ and $0 \leq u \leq 1$. Now, by using (2.3), (2.21) and (2.37), we can obtain (2.34).

**Theorem 2.9.** Let

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in T_n(A, B, \gamma, \alpha). \tag{2.38}$$

Then,

$$|a_k| \leq \frac{\gamma(A - B)}{k(\alpha(k - 1) + 1)} \quad (k \geq n + 1). \tag{2.39}$$

The result is sharp for each $k \geq n + 1$.

Proof. It is known (cf. [12]) that, if

$$\varphi(z) = \sum_{k=1}^{\infty} b_k z^k < \varphi(z) \quad (z \in U), \tag{2.40}$$

where $\varphi(z)$ is analytic in $U$ and $\varphi(z) = z + \cdots$ is analytic and convex univalent in $U$, then $|b_k| \leq 1 (k \in \mathbb{N})$.

By (2.38), we have

$$f'(z) + azf''(z) - 1 = \sum_{k=n+1}^{\infty} \frac{k(\alpha(k - 1) + 1)}{\gamma(A - B)} a_k z^{k-1} < \varphi(z) \quad (z \in U), \tag{2.41}$$

where

$$\varphi(z) = \frac{h(z) - 1}{\gamma(A - B)} = z + \cdots \tag{2.42}$$
and $h(z)$ is given by (1.6). Since the function $\varphi(z)$ is analytic and convex univalent in $U$, it follows from (2.41) that

$$\frac{k(a(k-1)+1)}{\gamma(A-B)}|a_k| \leq 1 \quad (k \geq n+1),$$

(2.43)

which gives (2.39).

Next, we consider the function

$$f_{k-1}(z) = z + \sum_{m=1}^{\infty} \frac{\lambda_m(A,B,\gamma)}{(m(k-1)+1)(am(k-1)+1)} z^{m(k-1)+1} \quad (z \in U; k \geq n+1).$$

(2.44)

It is easy to verify that

$$f_{k-1}'(z) + azf_{k-1}''(z) = h(z^{k-1}) < h(z) \quad (z \in U),$$

$$f_{k-1}(z) = z + \frac{\gamma(A-B)}{k(a(k-1)+1)} z^k + \cdots.$$  

(2.45)

The proof of Theorem 2.9 is completed. \(\square\)

### 3. The Univalency and Starlikeness of $T_n(A,B,\alpha)$

**Theorem 3.1.** $T_n(A,B,\alpha) \subset S_n$ if and only if

$$(A-B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{amn+1} \leq 1.$$  

(3.1)

**Proof.** Let $f(z) \in T_n(A,B,\alpha)$ and (3.1) be satisfied. Then, by (2.16) in Corollary 2.2, we see that $\text{Re} f'(z) > 0 (z \in U)$. Thus, $f(z)$ is close-to-convex and univalent in $U$.

On the other hand, if

$$(A-B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{amn+1} > 1,$$

(3.2)

then the function $f_n(z)$ defined by (2.19) satisfies $f_n'(0) = 1 > 0$ and

$$f_n'(re^{\pi i/n}) = 1 - (A-B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{amn+1} r^{mn} \rightarrow 1 - (A-B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{amn+1} < 0$$

as $r \to 1$. Hence, there exists a point $z_n = r_ne^{\pi i/n} (0 < r_n < 1)$ such that $f_n'(z_n) = 0$. This implies that $f_n(z)$ is not univalent in $U$ and so the theorem is proved. \(\square\)
Theorem 3.2. Let (3.1) in Theorem 3.1 be satisfied. If $\alpha \geq 1$ and

$$(\alpha - 1) \left( 1 - (A - B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{anm + 1} \right) + \frac{n\alpha}{2} \left( 1 - (A - B) \sum_{m=1}^{\infty} \frac{B^{m-1}}{(nm + 1)(anm + 1)} \right) \geq \frac{A - 1}{1 - B},$$

then $T_n(A, B, \alpha) \subset S_n^*$. 

Proof. We first show that

$$\sum_{m=1}^{\infty} \frac{B^{m-1}}{anm + 1} \geq \sum_{m=1}^{\infty} \frac{B^{m-1}}{(nm + 1)(anm + 1)} \quad (\alpha \geq 1).$$

Equation (3.5) is obvious when $B \geq 0$. For $0 > B \geq -1$, we have

$$\sum_{m=1}^{\infty} \frac{B^{m-1}}{anm + 1} - \sum_{m=1}^{\infty} \frac{B^{m-1}}{(nm + 1)(anm + 1)} = \mu_1 - \mu_2 + \mu_3 - \mu_4 + \cdots + (-1)^{m-1} \mu_m + \cdots,$$

where

$$\mu_m = \frac{nm|B|^{m-1}}{(nm + 1)(anm + 1)} > 0. \quad (3.7)$$

Since $|B| \leq 1$ and

$$\frac{d}{dx} \left( \frac{x}{(x + 1)(ax + 1)} \right) = \frac{1 - ax^2}{(x + 1)^2(ax + 1)^2} \leq 0 \quad (x \geq 1; \alpha \geq 1), \quad (3.8)$$

$\{\mu_m\}$ is a monotonically decreasing sequence. Therefore, the inequality (3.5) follows from (3.6).

Let $f(z) \in T_n(A, B, \alpha)$. Then,

$$\text{Re} \{ f'(z) + azf''(z) \} > \frac{1 - A}{1 - B} \quad (z \in U). \quad (3.9)$$

Define $p(z)$ in $U$ by

$$p(z) = \frac{zf'(z)}{f(z)}. \quad (3.10)$$
Remark 3.3. In proof of Theorem 3.2 is completed.

Let \( z \) diverges.

4. The Radius of Convexity

**Theorem 4.1.** Let \( f(z) \) belong to the class \( T_n(\gamma) \) defined by

\[
T_n(\gamma) = T_n(1,-1,\gamma,0) = \left\{ f(z) \in A_n : f'(z) < \left( \frac{1 + z}{1 - z} \right)^\gamma, \ (z \in U) \right\},
\]

then \( R(\beta,\alpha) \in S_1^* \) for \( \beta \geq \beta_1 \). However, this result is not true because the series in (3.15) diverges.
0 < \delta \leq 1 and 0 \leq \rho < 1. Then,

$$\text{Re}\left\{(1 - \delta)\left(f'(z)\right)^{1/\gamma} + \delta \left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \rho \quad (|z| < r_n(\gamma, \delta, \rho)),$$

(4.2)

where $r_n(\gamma, \delta, \rho)$ is the root in $(0,1)$ of the equation

$$(1 - 2\delta + \rho)r^{2n} - 2(1 - \delta + n\delta\gamma)r^n + 1 - \rho = 0.$$

(4.3)

The result is sharp.

Proof. For $f(z) \in T_n(\gamma)$, we can write

$$(f'(z))^{1/\gamma} = \frac{1 + z^n\varphi(z)}{1 - z^n\varphi(z)},$$

(4.4)

where $\varphi(z)$ is analytic and $|\varphi(z)| \leq 1$ in $U$. Differentiating both sides of (4.4) logarithmically, we arrive at

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{2n\gamma z^n\varphi(z)}{1 - (z^n\varphi(z))^2} + \frac{2\gamma z^{n+1}\varphi'(z)}{1 - (z^n\varphi(z))^2} \quad (z \in U).$$

(4.5)

Put $|z| = r < 1$ and $(f'(z))^{1/\gamma} = u + iv$ $(u, v \in \mathbb{R})$. Then, (4.4) implies that

$$z^n\varphi(z) = \frac{u - 1 + iv}{u + 1 + iv},$$

(4.6)

$$\frac{1 - r^n}{1 + r^n} \leq u \leq \frac{1 + r^n}{1 - r^n}.$$  

(4.7)

With the help of the Carathéodory inequality

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - r^2},$$

(4.8)
it follows from (4.5) and (4.6) that

\[
\text{Re} \left\{ (1 - \delta) \left( f'(z) \right)^{1/\gamma} + \delta \left( 1 + \frac{z^n f'(z)}{f'(z)} \right) \right\} \\
\geq (1 - \delta)u + \delta + 2n\delta \gamma \text{ Re} \left\{ \frac{z^n \varphi(z)}{1 - (z^n \varphi(z))^2} \right\} - 2\delta \gamma \left| \frac{z^{n+1} \varphi'(z)}{1 - (z^n \varphi(z))^2} \right| \\
\geq (1 - \delta)u + \delta + \frac{n\delta \gamma}{2} \left( u - \frac{u}{u^2 + v^2} \right) + \frac{\delta \gamma}{2} \frac{(u - 1)^2 + v^2 - r^{2n} \left( (u + 1)^2 + v^2 \right)}{r^{n-1}(1 - r^2)(u^2 + v^2)^{1/2}} \\
= F_n(u, v) \ \text{(say),}
\]

where \(0 < r < 1, 0 < \delta \leq 1\) and

\[
\frac{\partial}{\partial v} F_n(u, v) = \delta \gamma v G_n(u, v),
\]

because of (4.6) and (4.7). In view of (4.10) and (4.11), we see that

\[
F_n(u, v) \geq F_n(u, 0) \\
= (1 - \delta)u + \delta + \frac{n\delta \gamma}{2} \left( u - \frac{1}{u} \right) + \frac{\delta \gamma}{2r^{n-1}(1 - r^2)} \\
\times \left\{ (1 - r^{2n}) \left( u + \frac{1}{u} \right) - 2 \left( 1 + r^{2n} \right) \right\}.
\]

Let us now calculate the minimum value of \(F_n(u, 0)\) on the closed interval \([(1-r^n)/(1+r^n), (1+r^n)/(1-r^n)]\). Noting that

\[
\frac{1 - r^{2n}}{r^{n-1}(1 - r^2)} \geq n \ \text{(see [8])}
\]
and (4.7), we deduce from (4.12) that

\[
\frac{d}{du} F_n(u, 0) = 1 - \delta + \frac{\delta \gamma}{2} \left[ \left( \frac{1 - r^{2n}}{r^{n-1} (1 - r^2)} + n \right) - \frac{1}{u^2} \left( \frac{1 - r^{2n}}{r^{n-1} (1 - r^2)} - n \right) \right] \\
\geq 1 - \delta + \frac{\delta \gamma}{2} \left[ \left( \frac{1 - r^{2n}}{r^{n-1} (1 - r^2)} + n \right) - \left( \frac{1 + r^n}{1 - r^n} \right)^2 \left( \frac{1 - r^{2n}}{r^{n-1} (1 - r^2)} - n \right) \right] \\
= 1 - \delta + \frac{2 \delta \gamma I_n(r)}{(1 - r^n)^2},
\]

where

\[
I_n(r) = \frac{n}{2} \left( 1 + r^{2n} \right) - r \left( 1 + r^2 + \ldots + r^{2n-2} \right).
\]

Also

\[
I'_n(r) = n^2 r^{2n-1} - \left( 1 + 3r^2 + \ldots + (2n-1)r^{2n-2} \right)
\]

and \( I'_1(r) = r - 1 < 0 \). Suppose that \( I'_n(r) < 0 \). Then,

\[
I'_{n+1}(r) = (n + 1)^2 r^{2n+1} - (2n + 1) r^{2n} - \left( 1 + 3r^2 + \ldots + (2n-1)r^{2n-2} \right) \\
< n^2 r^{2n} - \left( 1 + 3r^2 + \ldots + (2n-1)r^{2n-2} \right) < I'_n(r) < 0.
\]

Hence, by virtue of the mathematical induction, we have \( I'_n(r) < 0 \) for all \( n \in \mathbb{N} \) and \( 0 \leq r < 1 \). This implies that

\[
I_n(r) > I_n(1) = 0 \quad (n \in \mathbb{N}; \ 0 \leq r < 1).
\]

In view of (4.14) and (4.18), we see that

\[
\frac{d}{du} F_n(u, 0) > 0 \quad \left( \frac{1 - r^n}{1 + r^n} \leq u \leq \frac{1 + r^n}{1 - r^n} \right).
\]
Further it follows from (4.9), (4.12), and (4.19) that

\[
\operatorname{Re}\left\{ (1 - \delta) \left( f'(z) \right)^{1/\gamma} + \delta \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} - \rho \\
\geq F_n \left( \frac{1 - r^n}{1 + r^n}, 0 \right) - \rho \\
= (1 - \delta) \frac{1 - r^n}{1 + r^n} + \delta \frac{1 - 2n\delta r^n - r^{2n}}{1 - r^{2n}} - \rho \\
= \frac{J_n(r)}{1 - r^{2n}}
\]

where 0 \leq \rho < 1 and

\[
J_n(r) = (1 - 2\delta + \rho)r^{2n} - 2(1 - \delta + n\delta \gamma)r^n + 1 - \rho.
\]

Note that \( J_n(0) = 1 - \rho > 0 \) and \( J_n(1) = -2n\delta \gamma < 0 \). If we let \( r_n(y, \delta, \rho) \) denote the root in \( (0, 1) \) of the equation \( J_n(r) = 0 \), then (4.20) yields the desired result (4.2).

To see that the bound \( r_n(y, \delta, \rho) \) is the best possible, we consider the function

\[
f(z) = \int_0^z \left( \frac{1 - t^n}{1 + t^n} \right)^{\gamma} dt \in T_n(y).
\]

It is clear that for \( z = r \in (r_n(y, \delta, \rho), 1) \),

\[
(1 - \delta) \left( f'(r) \right)^{1/\gamma} + \delta \left( 1 + \frac{rf''(r)}{f'(r)} \right) - \rho = \frac{J_n(r)}{1 - r^{2n}} < 0,
\]

which shows that the bound \( r_n(y, \delta, \rho) \) cannot be increased.

Setting \( \delta = 1 \), Theorem 4.1 reduces to the following result.

**Corollary 4.2.** Let \( f(z) \in T_n(y) \) and 0 \leq \rho < 1. Then, \( f(z) \) is convex of order \( \rho \) in

\[
|z| < \left[ \frac{\left( n\gamma \right)^2 + (1 - \rho)^2}{1 - \rho} - n\gamma \right]^{1/n}.
\]

The result is sharp.

**References**


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