Research Article

General Cubic-Quartic Functional Equation

M. Eshaghi Gordji,^{1,2} M. Kamyar,^{1,2} and Th. M. Rassias³

¹ Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran

² Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Semnan, Iran

³ Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com

Received 25 June 2011; Accepted 22 August 2011

Academic Editor: Victor M. Perez Garcia

Copyright © 2011 M. Eshaghi Gordji et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We obtain the general solution and the generalized Hyers-Ulam stability of the general cubic-quartic functional equation for fixed integers k with $k \neq 0, \pm 1$: $f(x + ky) + f(x - ky) = k^2(f(x + y) + f(x - y)) + 2(1 - k^2)f(x) + ((k^4 - k^2)/4)(f(2y) - 8f(y)) + \tilde{f}(2x) - 16\tilde{f}(x)$, where $\tilde{f}(x) := f(x) + f(-x)$.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \rightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta \quad (1.1)$$

for all $x, y \in E$ and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta \quad (1.2)$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. In 1978, Rassias [3] proved the following theorem.

Theorem 1.1. *Let $f : E \rightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \quad (1.3)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p \quad (1.4)$$

for all $x \in E$. If $p < 0$ then inequality (1.3) holds for all $x, y \neq 0$ and (1.4) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in real t for each fixed $x \in E$, then T is linear.

In 1990, Rassias during the 27th International Symposium on Functional Equations asked the question whether such a Theorem can also be proved for all real values of p that are greater or equal to one. In 1991, Gajda [4], following the same approach as that of Rassias, provided an affirmative solution to this question for all real values of p that are strictly greater than one. The new concept of stability of the linear mapping that was inspired by Rassias' stability theorem is called Hyers-Ulam-Rassias stability of functional equations.

Jun and Kim [5] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \quad (1.5)$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5). The function $f(x) = x^3$ satisfies the functional equation (1.5), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables. The stability of the quartic functional equations was studied by Park and Bae [6], when

$$f(x+2y) + f(x-2y) = 4(f(x+y) + f(x-y)) + 24f(y) - 6f(x). \quad (1.6)$$

In fact, they proved that a function f between real vector spaces X and Y is a solution of (1.6) if and only if there exists a unique symmetric multi-additive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f(x) = Q(x, x, x, x)$ for all $x \in X$ (see also [7, 8]). It is straightforward to verify that the function $f(x) = x^4$ satisfies the functional equation (1.6), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem. (see [9–45]).

In 2008, Gordji et al. [17] provided the solution as well as the stability of a mixed type cubic-quartic functional equation. We only mention here the papers [19, 32, 33] concerning the stability of the mixed type functional equations.

In this paper, we deal with the following general cubic-quartic functional equation:

$$\begin{aligned} f(x+ky) + f(x-ky) &= k^2(f(x+y) + f(x-y)) + 2(1-k^2)f(x) + \frac{k^4-k^2}{4} \\ &\quad \times (f(2y) - 8f(y)) + \tilde{f}(2x) - 16\tilde{f}(x), \end{aligned} \quad (1.7)$$

where $\tilde{f}(x) := f(x) + f(-x)$.

Then it follows easily that the function $f(x) = ax^4 + bx^3$ satisfies (1.7). We investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.7).

2. General Solution

In this section, we establish the general solution of functional equation (1.7).

Theorem 2.1. *Let X, Y be vector spaces and let $f : X \rightarrow Y$ be a function. Then f satisfies (1.7) if and only if there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ and a unique function $C : X \times X \times X \rightarrow Y$ such that $f(x) = Q(x, x, x, x) + C(x, x, x)$ for all $x \in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables.*

Proof. Let f satisfies (1.7). We decompose f into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)) \quad (2.1)$$

for all $x \in X$. By (1.7), we have

$$\begin{aligned} f_e(x+ky) + f_e(x-ky) &= \frac{1}{2}[f(x+ky) + f(-x-ky) + f(x-ky) + f(-x+ky)] \\ &= \frac{1}{2}[f(x+ky) + f(x-ky)] + \frac{1}{2}[f((-x)+(-ky)) + f((-x)-(-ky))] \\ &= \frac{1}{2} \left[k^2(f(x+y) + f(x-y)) + 2(1-k^2)f(x) \right. \\ &\quad \left. + \frac{k^4-k^2}{4}(f(2y) - 8f(y)) + \tilde{f}(2x) - 16\tilde{f}(x) \right] \\ &\quad + \frac{1}{2} \left[k^2(f(-x-y) + f(-x+y)) + 2(1-k^2)f(-x) + \frac{k^4-k^2}{4} \right. \\ &\quad \left. \times (f(-2y) - 8f(-y)) + \tilde{f}(-2x) - 16\tilde{f}(-x) \right] \end{aligned}$$

$$\begin{aligned}
&= k^2 \left[\frac{1}{2} (f(x+y) + f(-(x+y))) \right] + k^2 \left[\frac{1}{2} (f(x-y) + f(-(x-y))) \right] \\
&\quad + 2(1-k^2) \left[\frac{1}{2} (f(x) + f(-x)) \right] + \frac{k^4-k^2}{4} \left[\frac{1}{2} (f(2y) + f(-2y)) \right] \\
&\quad - \frac{k^4-k^2}{4} \left[\frac{1}{2} (8f(y) + 8f(-y)) \right] + \left[\frac{1}{2} (\tilde{f}(2x) + \tilde{f}(-2x)) \right] \\
&\quad - 16 \left[\frac{1}{2} (\tilde{f}(x) + \tilde{f}(-x)) \right] \\
&= k^2 (f_e(x+y) + f_e(x-y)) + 2(1-k^2) f_e(x) \\
&\quad + \frac{k^4-k^2}{4} (f_e(2y) - 8f_e(y)) + \tilde{f}_e(2x) - 16\tilde{f}_e(x)
\end{aligned} \tag{2.2}$$

for all $x, y \in X$. This means that f_e satisfies (1.7), or

$$\begin{aligned}
f_e(x+ky) + f_e(x-ky) &= k^2 (f_e(x+y) + f_e(x-y)) + 2(1-k^2) f_e(x) \\
&\quad + \frac{k^4-k^2}{4} (f_e(2y) - 8f_e(y)) + \tilde{f}_e(2x) - 16\tilde{f}_e(x)
\end{aligned} \tag{1.5(e)}$$

for all $x, y \in X$. Applying the fact that the function f_e is even for all $x, y \in X$, (1.5(e)) can be written in the form

$$\begin{aligned}
f_e(x+ky) + f_e(x-ky) &= k^2 (f_e(x+y) + f_e(x-y)) + 2(1-k^2) f_e(x) \\
&\quad + \frac{k^4-k^2}{4} (f_e(2y) - 8f_e(y)) + 2f_e(2x) - 32f_e(x)
\end{aligned} \tag{2.3}$$

for all $x, y \in X$. Now be setting $x = y = 0$ in (2.3), we get $f_e(0) = 0$. Similarly, by setting $y = 0$ in (2.3), we obtain

$$f_e(2x) = 16f_e(x) \tag{2.4}$$

for all $x \in X$. Hence (2.3) can be written as

$$f_e(x+ky) + f_e(x-ky) = k^2 (f_e(x+y) + f_e(x-y)) + 2(1-k^2) f_e(x) + 2(k^4-k^2) f_e(y) \tag{2.5}$$

for all $x, y \in X$. By substituting x by $x + y$ in (2.5), we have

$$\begin{aligned} & f_e(x + (1+k)y) + f_e(x + (1-k)y) \\ &= k^2(f_e(x + 2y) + f_e(x)) + 2(1 - k^2)f_e(x + y) + 2(k^4 - k^2)f_e(y) \end{aligned} \quad (2.6)$$

for all $x, y \in X$. Substituting $-y$ for y in (2.6), we get by evenness of f

$$\begin{aligned} & f_e(x - (1+k)y) + f_e(x - (1-k)y) \\ &= k^2(f_e(x - 2y) + f_e(x)) + 2(1 - k^2)f_e(x - y) + 2(k^4 - k^2)f_e(y) \end{aligned} \quad (2.7)$$

for all $x, y \in X$. Adding (2.6) to (2.7), we obtain

$$\begin{aligned} & f_e(x + (1+k)y) + f_e(x + (1-k)y) + f_e(x - (1+k)y) + f_e(x - (1-k)y) \\ &= k^2(f_e(x + 2y) + f_e(x - 2y)) + 2k^2f_e(x) + 2(1 - k^2)(f_e(x + y) + f_e(x - y)) \\ &+ 4(k^4 - k^2)f_e(y) \end{aligned} \quad (2.8)$$

for all $x, y \in X$. By substituting x by $x - ky$ in (2.5), we have

$$\begin{aligned} f_e(x) + f_e(x - 2ky) &= k^2(f_e(x + (1-k)y) + f_e(x - (k+1)y)) + 2(1 - k^2)f_e(x - ky) \\ &+ 2(k^4 - k^2)f_e(y) \end{aligned} \quad (2.9)$$

for all $x, y \in X$. Substituting $-x$ for x in (2.9), we get by evenness of f_e

$$\begin{aligned} f_e(x) + f_e(x + 2ky) &= k^2(f_e(x + (k-1)y) + f_e(x + (k+1)y)) + 2(1 - k^2)f_e(x + ky) \\ &+ 2(k^4 - k^2)f_e(y) \end{aligned} \quad (2.10)$$

for all $x, y \in X$. Adding (2.9) to (2.10), we obtain

$$\begin{aligned} f_e(x + 2ky) + f_e(x - 2ky) &= k^2(f_e(x + (1-k)y) + f_e(x - (k+1)y) + f_e(x + (k-1)y) \\ &+ f_e(x + (k+1)y)) + 2(1 - k^2)(f_e(x - ky) + f_e(x + ky)) \\ &+ 4(k^4 - k^2)f_e(y) - 2f_e(x) \end{aligned} \quad (2.11)$$

for all $x, y \in X$. Now, by using (2.5), (2.8), and (2.11), we lead to

$$\begin{aligned} f_e(x+2ky) + f_e(x-2ky) &= k^4(f_e(x+2y) + f_e(x-2y)) \\ &\quad + 4k^2(1-k^2)(f_e(x+y) + f_e(x-y)) + 8(k^4-k^2)f_e(y) \\ &\quad + (6k^4-8k^2+2)f_e(x) \end{aligned} \quad (2.12)$$

for all $x, y \in X$. If we replace y by $2y$ in (2.5), we get

$$\begin{aligned} f_e(x+2ky) + f_e(x-2ky) &= k^2(f_e(x+2y) + f_e(x-2y)) + 2(1-k^2)f_e(x) \\ &\quad + 2(k^4-k^2)f_e(2y) \end{aligned} \quad (2.13)$$

for all $x, y \in X$. It follows from (2.12) and (2.13) that

$$\begin{aligned} &k^4(f_e(x+2y) + f_e(x-2y)) + 4k^2(1-k^2)(f_e(x+y) + f_e(x-y)) + 8(k^4-k^2)f_e(y) \\ &\quad + (6k^4-8k^2+2)f_e(x) \\ &= k^2(f_e(x+2y) + f_e(x-2y)) + 2(1-k^2)f_e(x) + 2(k^4-k^2)f_e(2y) \end{aligned} \quad (2.14)$$

for all $x, y \in X$. So we have

$$f_e(x+2y) + f_e(x-2y) = 4(f_e(x+y) + f_e(x-y)) + 24f_e(y) - 6f_e(x) \quad (2.15)$$

for all $x, y \in X$. This means that f_e is a quartic function. Thus there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f_e(x) = Q(x, x, x, x)$ for all $x \in X$. On the other hand, we can show that f_o satisfies (1.7), or

$$\begin{aligned} f_o(x+ky) + f_o(x-ky) &= k^2(f_o(x+y) + f_o(x-y)) + 2(1-k^2)f_o(x) \\ &\quad + \frac{k^4-k^2}{4}(f_o(2y) - 8f_o(y)) + \tilde{f}_o(2x) - 16\tilde{f}_o(x) \end{aligned} \quad (1.5(o))$$

for all $x, y \in X$. By oddness of f_o for all $x, y \in X$, (1.5(o)) can be written as

$$\begin{aligned} f_o(x+ky) + f_o(x-ky) &= k^2(f_o(x+y) + f_o(x-y)) + 2(1-k^2)f_o(x) \\ &\quad + \frac{k^4-k^2}{4}(f_o(2y) - 8f_o(y)) \end{aligned} \quad (2.16)$$

for all $x, y \in X$. Now by setting $x = y = 0$ in (3.2), we get $f_o(0) = 0$, and by setting $x = 0$ in (2.16), we obtain

$$f_o(2y) = 8f_o(y) \quad (2.17)$$

for all $y \in X$. Hence (2.16) can be written as

$$f_o(x + ky) + f_o(x - ky) = k^2(f_o(x + y) + f_o(x - y)) + 2(1 - k^2)f_o(x) \quad (2.18)$$

for all $x, y \in X$. Replacing x by $x - y$ in (2.18), we obtain

$$f_o(x + (k - 1)y) + f_o(x - (k + 1)y) = k^2(f_o(x - 2y) + f_o(x)) + 2(1 - k^2)f_o(x - y) \quad (2.19)$$

for all $x, y \in X$. Substituting $-x$ for x in (2.19), we get by oddness of f_o

$$-f_o(x + (1 - k)y) - f_o(x + (k + 1)y) = k^2(-f_o(x + 2y) - f_o(x)) - 2(1 - k^2)f_o(x + y) \quad (2.20)$$

for all $x, y \in X$. If we subtract (2.19) from (2.20), we obtain

$$\begin{aligned} & f_o(x + (k - 1)y) + f_o(x - (k + 1)y) + f_o(x + (1 - k)y) + f_o(x + (k + 1)y) \\ &= k^2(f_o(x + 2y) + f_o(x - 2y)) + 2k^2f_o(x) + 2(1 - k^2)(f_o(x + y) + f_o(x - y)) \end{aligned} \quad (2.21)$$

for all $x, y \in X$. By substituting x by $x + ky$ in (2.18), we have

$$f_o(x) + f_o(x + 2ky) = k^2(f_o(x + (k + 1)y) + f_o(x + (k - 1)y)) + 2(1 - k^2)f_o(x + ky) \quad (2.22)$$

for all $x, y \in X$. Substituting $-y$ for y in (2.22), we get

$$f_o(x) + f_o(x - 2ky) = k^2(f_o(x - (k + 1)y) + f_o(x - (k - 1)y)) + 2(1 - k^2)f_o(x - ky) \quad (2.23)$$

for all $x, y \in X$. Adding (2.22) to (2.23), we obtain

$$\begin{aligned} f_o(x + 2ky) + f_o(x - 2ky) &= k^2(f_o(x + (k + 1)y) + f_o(x + (k - 1)y) + f_o(x - (k + 1)y) \\ &\quad + f_o(x - (k - 1)y)) + 2(1 - k^2)(f_o(x + ky) + f_o(x - ky)) \\ &\quad - 2f_o(x) \end{aligned} \quad (2.24)$$

for all $x, y \in X$. Now, by using (2.18), (2.21), and (2.24), we lead to

$$\begin{aligned} f_o(x + 2ky) + f_o(x - 2ky) &= 4k^2(1 - k^2)(f_o(x + y) + f_o(x - y)) \\ &\quad + (6k^4 - 8k^2 + 2)f_o(x) + k^4(f_o(x + 2y) + f_o(x - 2y)) \end{aligned} \quad (2.25)$$

for all $x, y \in X$. If we replace y by $2y$ in (2.18), we get

$$f_o(x + 2ky) + f_o(x - 2ky) = k^2(f_o(x + 2y) + f_o(x - 2y)) + 2(1 - k^2)f_o(x) \quad (2.26)$$

for all $x, y \in X$. If we compare (2.25) with (2.26), then we conclude that

$$f_o(x + 2y) + f_o(x - 2y) = 4(f_o(x + y) + f_o(x - y)) - 6f_o(x) \quad (2.27)$$

for all $x, y \in X$. Replacing x by $2x$ in (2.27), we get

$$f_o(2(x + y)) + f_o(2(x - y)) = 4(f_o(2x + y) + f_o(2x - y)) - 6f_o(2x) \quad (2.28)$$

for all $x, y \in X$. Finally, it follows from (2.17) and (2.28) that

$$8(f_o(x + y) + f_o(x - y)) = 4(f_o(2x + y) + f_o(2x - y)) - 48f_o(x) \quad (2.29)$$

for all $x, y \in X$. By multiplying both sides of (2.29) by $1/4$, we get

$$2(f_o(x + y) + f_o(x - y)) = (f_o(2x + y) + f_o(2x - y)) - 12f_o(x) \quad (2.30)$$

for all $x, y \in X$. This means that f_o is a cubic function and that there exists a unique function $C : X \times X \times X \rightarrow Y$ such that $f_o(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables. Thus for all $x \in X$, we have

$$f(x) = f_e(x) + f_o(x) = C(x, x, x) + Q(x, x, x, x). \quad (2.31)$$

The proof of the converse is trivially. □

The following corollary is an alternative result of above Theorem 2.1.

Corollary 2.2. *Let X, Y be vector spaces, and let $f : X \rightarrow Y$ be a function satisfying (1.7). Then the following assertions hold.*

- (a) *If f is even function, then f is quartic.*
- (b) *If f is odd function, then f is cubic.*

3. Stability

We will investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.7). In the following, let X be a real vector space and let Y be a Banach space. Given $f : X \rightarrow Y$, we define the difference operator $D_f : X \times X \rightarrow Y$ by

$$\begin{aligned} D_f(x, y) = & f(x + ky) + f(x - ky) - k^2(f(x + y) + f(x - y)) - 2(1 - k^2)f(x) \\ & - \frac{k^4 - k^2}{4}(f(2y) - 8f(y)) - \tilde{f}(2x) + 16\tilde{f}(x) \end{aligned} \quad (3.1)$$

for all $x, y \in X$.

Theorem 3.1. *Let $j \in \{-1, 1\}$ be fixed and let $\varphi : X \times X \rightarrow [0, \infty)$ be a function such that*

$$\sum_{i=(1+j)/2}^{\infty} k^{4ij} \varphi\left(\frac{x}{k^{ij}}, \frac{y}{k^{ij}}\right) < \infty \quad (3.2)$$

for all $x, y \in X$. Suppose that an even function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\| \leq \varphi(x, y) \quad (3.3)$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n \rightarrow \infty} k^{4nj} f\left(\frac{x}{k^{nj}}\right) \quad (3.4)$$

exists for all $x \in X$ and $Q : X \rightarrow Y$ is a unique quartic function satisfying

$$\|f(x) - Q(x)\| \leq \frac{1}{k^4} \tilde{\varphi}_e(x) \quad (3.5)$$

for all $x \in X$, where

$$\tilde{\varphi}_e(x) = \sum_{i=(1+j)/2}^{\infty} k^{4ij} \left[\frac{1}{2} \varphi\left(0, \frac{x}{k^{ij}}\right) + \frac{k^4 - k^2}{16} \varphi\left(\frac{x}{k^{ij}}, 0\right) \right]. \quad (3.6)$$

Proof. Let $j = 1$. It follows from (3.3) and using evenness of f that

$$\begin{aligned} & \left\| f(x + ky) + f(x - ky) - k^2(f(x + y) + f(x - y)) - 2(1 - k^2)f(x) \right. \\ & \quad \left. - \frac{k^4 - k^2}{4}(f(2y) - 8f(y)) - 2f(2x) + 32f(x) \right\| \leq \varphi(x, y) \end{aligned} \quad (3.7)$$

for all $x, y \in X$. Replacing x and y by 0 and x in (3.7), respectively, we see that

$$\left\| 2f(kx) + (2k^4 - 4k^2)f(x) + \frac{k^2 - k^4}{4}f(2x) \right\| \leq \varphi(0, x) \quad (3.8)$$

for all $x \in X$. If we divide both sides of (3.8) by 2, we get

$$\left\| f(kx) + (k^4 - 2k^2)f(x) + \frac{k^2 - k^4}{8}f(2x) \right\| \leq \frac{1}{2}\varphi(0, x) \quad (3.9)$$

for all $x \in X$. Putting $y = 0$ in (3.7), we obtain

$$\|2f(2x) - 32f(x)\| \leq \varphi(x, 0) \quad (3.10)$$

for all $x \in X$. If we multiply both sides of (3.10) by $(k^4 - k^2)/16$, then we have

$$\left\| \frac{k^4 - k^2}{8}f(2x) - 2(k^4 - k^2)f(x) \right\| \leq \frac{k^4 - k^2}{16}\varphi(x, 0) \quad (3.11)$$

for all $x \in X$. It follows from (3.9) and (3.11) that

$$\|f(kx) - k^4f(x)\| \leq \frac{1}{2}\varphi(0, x) + \frac{k^4 - k^2}{16}\varphi(x, 0) \quad (3.12)$$

for all $x \in X$. Let

$$\psi_e(x) = \frac{1}{2}\varphi(0, x) + \frac{k^4 - k^2}{16}\varphi(x, 0) \quad (3.13)$$

for all $x \in X$. Thus by (3.12), we get

$$\|f(kx) - k^4f(x)\| \leq \psi_e(x) \quad (3.14)$$

for all $x \in X$. If we replace x in (3.14) by x/k^{n+1} and multiply both sides of (3.14) by k^{4n} , we see that

$$\left\| k^{4(n+1)}f\left(\frac{x}{k^{n+1}}\right) - k^{4n}f\left(\frac{x}{k^n}\right) \right\| \leq k^{4n}\psi_e\left(\frac{x}{k^{n+1}}\right) \quad (3.15)$$

for all $x \in X$ and all nonnegative integers n . So

$$\begin{aligned} \left\| k^{4(n+1)} f\left(\frac{x}{k^{n+1}}\right) - k^{4m} f\left(\frac{x}{k^m}\right) \right\| &\leq \sum_{i=m}^n \left\| k^{4(i+1)} f\left(\frac{x}{k^{i+1}}\right) - k^{4i} f\left(\frac{x}{k^i}\right) \right\| \\ &\leq \sum_{i=m}^n k^{4i} \psi_e\left(\frac{x}{k^{i+1}}\right) \end{aligned} \quad (3.16)$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in X$. By (3.2), we infer that

$$\sum_{i=m}^n k^{4i} \psi_e\left(\frac{x}{k^{i+1}}\right) < \infty, \quad \lim_{n \rightarrow \infty} k^{4n} \psi_e\left(\frac{x}{k^{n+1}}\right) = 0 \quad (3.17)$$

for all $x \in X$. It follows from (3.16) and (3.17) that the sequence $\{k^{4n} f(x/k^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{k^{4n} f(x/k^n)\}$ converges for all $x \in X$. So one can define a mapping $Q : X \rightarrow Y$ by (3.4) for all $x \in X$. Letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (3.16), we obtain (3.5). It follows from (3.4), (3.15), and (3.17) that

$$\left\| Q(x) - k^4 Q\left(\frac{x}{k}\right) \right\| = \lim_{n \rightarrow \infty} \left\| k^{4n} f\left(\frac{x}{k^n}\right) - k^{4(n+1)} f\left(\frac{x}{k^{n+1}}\right) \right\| \leq \lim_{n \rightarrow \infty} k^{4n} \psi_e\left(\frac{x}{k^{n+1}}\right) = 0 \quad (3.18)$$

for all $x \in X$. So

$$Q(kx) = k^4 Q(x) \quad (3.19)$$

for all $x \in X$. On the other hand, it follows from (3.2), (3.3), and (3.4) that

$$\|D_Q(x, y)\| = \lim_{n \rightarrow \infty} k^{4n} \left\| D_f\left(\frac{x}{k^n}, \frac{y}{k^n}\right) \right\| \leq \lim_{n \rightarrow \infty} k^{4n} \varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0 \quad (3.20)$$

for all $x, y \in X$. Therefore, by Corollary 2.2, the function $Q : X \rightarrow Y$ is quartic.

To prove the uniqueness of Q , let $Q' : X \rightarrow Y$ be a another quartic function satisfying (3.5). Since

$$\lim_{n \rightarrow \infty} k^{4n} \sum_{i=1}^{\infty} k^{4i} \varphi\left(\frac{x}{k^{n+i}}, \frac{y}{k^{n+i}}\right) = \lim_{n \rightarrow \infty} \sum_{i=n+1}^{\infty} k^{4i} \varphi\left(\frac{x}{k^i}, \frac{y}{k^i}\right) = 0 \quad (3.21)$$

for all $x, y \in X$, hence

$$\lim_{n \rightarrow \infty} k^{4n} \tilde{\psi}_e\left(\frac{x}{k^n}\right) = 0 \quad (3.22)$$

for all $x \in X$. So it follows from (3.5) and (3.22) that

$$\|Q(x) - Q'(x)\| = \lim_{n \rightarrow \infty} k^{4n} \left\| f\left(\frac{x}{k^n}\right) - Q'\left(\frac{x}{k^n}\right) \right\| \leq \lim_{n \rightarrow \infty} \frac{k^{4n}}{k^4} \tilde{\varphi}_e\left(\frac{x}{k^n}\right) = 0 \quad (3.23)$$

for all $x \in X$. Hence $Q = Q'$.

For $j = -1$, the proof of the theorem is similar. $\square$

Theorem 3.2. Let $j \in \{-1, 1\}$ be fixed, and let $\varphi : X \times X \rightarrow [0, \infty)$ be a function such that

$$\tilde{\varphi}_c(x, y) = \sum_{i=(1+j)/2}^{\infty} 2^{3ij} \varphi\left(\frac{x}{2^{ij}}, \frac{y}{2^{ij}}\right) < \infty \quad (3.24)$$

for all $x, y \in X$. Suppose that an odd function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality (3.3). Then the limit

$$C(x) := \lim_{n \rightarrow \infty} 2^{3nj} f\left(\frac{x}{2^{nj}}\right) \quad (3.25)$$

exists for all $x \in X$ and $C : X \rightarrow Y$ is a unique cubic function satisfying

$$\|f(x) - C(x)\| \leq \frac{1}{2(k^4 - k^2)} \tilde{\varphi}_c(0, x) \quad (3.26)$$

for all $x \in X$.

Proof. Let $j = 1$. It follows from (3.3) and using oddness of f that

$$\begin{aligned} & \left\| f(x + ky) + f(x - ky) - k^2(f(x + y) + f(x - y)) - 2(1 - k^2)f(x) \right. \\ & \quad \left. - \frac{k^4 - k^2}{4}(f(2y) - 8f(y)) \right\| \leq \varphi(x, y) \end{aligned} \quad (3.27)$$

for all $x, y \in X$. Replacing x and y by 0 and x in (3.27), respectively, we see that

$$\left\| \frac{k^4 - k^2}{4}(f(2x) - 8f(x)) \right\| \leq \varphi(0, x) \quad (3.28)$$

for all $x \in X$. If we multiply both sides of (3.28) by $4/(k^4 - k^2)$, we get

$$\|f(2x) - 8f(x)\| \leq \frac{4}{k^4 - k^2} \varphi(0, x) \quad (3.29)$$

for all $x \in X$. If we replace x in (3.29) by $x/2^{n+1}$ and multiply both sides of (3.29) by 2^{3n} , we see that

$$\left\| 2^{3(n+1)} f\left(\frac{x}{2^{n+1}}\right) - 2^{3n} f\left(\frac{x}{2^n}\right) \right\| \leq 2^{3n} \frac{4}{k^4 - k^2} \varphi\left(0, \frac{x}{2^{n+1}}\right) \quad (3.30)$$

for all $x \in X$ and all nonnegative integers n . So

$$\begin{aligned} \left\| 2^{3(n+1)} f\left(\frac{x}{2^{n+1}}\right) - 2^{3m} f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{i=m}^n \left\| 2^{3(i+1)} f\left(\frac{x}{2^{i+1}}\right) - 2^{3i} f\left(\frac{x}{2^i}\right) \right\| \\ &\leq \frac{4}{k^4 - k^2} \sum_{i=m}^n 2^{3i} \varphi\left(0, \frac{x}{2^{i+1}}\right) \end{aligned} \quad (3.31)$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in X$. By (3.24), we infer that

$$\sum_{i=m}^n 2^{3i} \varphi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) < \infty, \quad \lim_{n \rightarrow \infty} 2^{3n} \varphi\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) = 0 \quad (3.32)$$

for all $x, y \in X$. It follows from (3.31) and (3.32) that the sequence $\{2^{3n} f(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^{3n} f(x/2^n)\}$ converges for all $x \in X$. So one can define a mapping $C : X \rightarrow Y$ by (3.25) for all $x \in X$. Letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (3.31), we obtain (3.26). It follows from (3.25), (3.30), and (3.32) that

$$\left\| C(x) - 2^3 C\left(\frac{x}{2}\right) \right\| = \lim_{n \rightarrow \infty} \left\| 2^{3n} f\left(\frac{x}{2^n}\right) - 2^{3(n+1)} f\left(\frac{x}{2^{n+1}}\right) \right\| \leq \lim_{n \rightarrow \infty} \frac{4}{k^4 - k^2} 2^{3n} \varphi\left(0, \frac{x}{2^{n+1}}\right) = 0 \quad (3.33)$$

for all $x \in X$. So

$$C(2x) = 2^3 C(x) \quad (3.34)$$

for all $x \in X$. On the other hand, it follows from (3.3), (3.24), and (3.25) that

$$\|D_c(x, y)\| = \lim_{n \rightarrow \infty} 2^{3n} \left\| D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \leq \lim_{n \rightarrow \infty} 2^{3n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad (3.35)$$

for all $x, y \in X$. Therefore by Corollary 2.2, the function $C : X \rightarrow Y$ is cubic.

To prove the uniqueness of C , let $C' : X \rightarrow Y$ be another cubic function satisfying (3.26). Since

$$\lim_{n \rightarrow \infty} 2^{3n} \sum_{i=1}^{\infty} 2^{3i} \varphi\left(\frac{x}{2^{n+i}}, \frac{y}{2^{n+i}}\right) = \lim_{n \rightarrow \infty} \sum_{i=n+1}^{\infty} 2^{3i} \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) = 0 \quad (3.36)$$

for all $x, y \in X$, hence

$$\lim_{n \rightarrow \infty} 2^{3n} \tilde{\psi}_c \left(\frac{x}{2^n}, \frac{y}{2^n} \right) = 0 \quad (3.37)$$

for all $x, y \in X$. So it follows from (3.26) and (3.37) that

$$\|C(x) - C'(x)\| = \lim_{n \rightarrow \infty} 2^{3n} \left\| f \left(\frac{x}{2^n} \right) - C' \left(\frac{x}{2^n} \right) \right\| \leq \lim_{n \rightarrow \infty} \frac{1}{2(k^4 - k^2)} 2^{3n} \tilde{\psi}_c \left(0, \frac{x}{2^n} \right) = 0 \quad (3.38)$$

for all $x \in X$. Hence $C = C'$.

For $j = -1$, the proof of the theorem is similar. $\square$

Theorem 3.3. Let $j \in \{1, -1\}$ be fixed. Suppose that a function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality (3.3). If the upper bound $\phi : X \times X \rightarrow [0, \infty)$ is a mapping such that

$$\sum_{i=(1+j)/2}^{\infty} \left[\left(\frac{1+j}{2} \right) k^{4ij} \varphi \left(\frac{x}{k^{ij}}, \frac{y}{k^{ij}} \right) + \left(\frac{1-j}{2} \right) 2^{3ij} \varphi \left(\frac{x}{2^{ij}}, \frac{y}{2^{ij}} \right) \right] < \infty, \quad (3.39)$$

for all $x, y \in X$, then there exists a unique quartic function $Q : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \frac{1}{2k^4} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)] + \frac{1}{4(k^4 - k^2)} [\tilde{\psi}_c(0, x) + \tilde{\psi}_c(0, -x)] \quad (3.40)$$

for all $x \in X$, where

$$\begin{aligned} \tilde{\psi}_e(x) &= \sum_{i=(1+j)/2}^{\infty} k^{4ij} \left[\frac{1}{2} \varphi \left(0, \frac{x}{k^{ij}} \right) + \frac{k^4 - k^2}{16} \varphi \left(\frac{x}{k^{ij}}, 0 \right) \right], \\ \tilde{\psi}_c(x, y) &= \sum_{i=(1+j)/2}^{\infty} 2^{3nj} \varphi \left(\frac{x}{2^{ij}}, \frac{y}{2^{ij}} \right). \end{aligned} \quad (3.41)$$

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$. Then $f_e(0) = 0$ and f_e is even function satisfying $\|D_{f_e}(x, y)\| \leq (1/2)[\phi(x, y) + \phi(-x, -y)]$ for all $x, y \in X$. By Theorem 3.1, there exists a unique quartic function $Q : X \rightarrow Y$ satisfying

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2k^4} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)] \quad (3.42)$$

for all $x \in X$, where

$$\tilde{\psi}_e(x) = \sum_{i=(1+j)/2}^{\infty} k^{4ij} \left[\frac{1}{2} \varphi \left(0, \frac{x}{k^{ij}} \right) + \frac{k^4 - k^2}{16} \varphi \left(\frac{x}{k^{ij}}, 0 \right) \right] \quad (3.43)$$

for all $x \in X$. Let now $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$. Then $f_e(0) = 0$ and f_o is an odd function satisfying $\|D_{f_o}(x, y)\| \leq (1/2)[\phi(x, y) + \phi(-x, -y)]$ for all $x, y \in X$. Hence, in view of Theorem 3.2, there exists a unique cubic function $C : X \rightarrow Y$ such that

$$\|f_o(x) - Q(x)\| \leq \frac{1}{4(k^4 - k^2)} [\tilde{\psi}_c(0, x) + \tilde{\psi}_c(0, -x)] \quad (3.44)$$

for all $x \in X$, where

$$\tilde{\psi}_c(x, y) = \sum_{i=(1+j)/2}^{\infty} 2^{3nj} \varphi\left(\frac{x}{2^{ij}}, \frac{y}{2^{ij}}\right) \quad (3.45)$$

for all $x, y \in X$. On the other hand, we have $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. Then by combining (3.42) and (3.44), it follows that

$$\begin{aligned} \|f(x) - C(x) - Q(x)\| &\leq \|f_e(x) - Q(x)\| + \|f_o(x) - C(x)\| \\ &\leq \frac{1}{2k^4} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)] + \frac{1}{4(k^4 - k^2)} [\tilde{\psi}_c(0, x) + \tilde{\psi}_c(0, -x)] \end{aligned} \quad (3.46)$$

for all $x \in X$. □

We are going to investigate the Hyers-Ulam-Rassias stability problem for functional equation (1.7).

Corollary 3.4. *Let $p \in (-\infty, 3) \cup (4, +\infty)$, $\theta > 0$. Suppose $f : X \rightarrow Y$ satisfies $f(0) = 0$ and inequality*

$$\|D_f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad (3.47)$$

for all $x, y \in X$. Then there exist a unique quartic function $Q : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ satisfying

$$\begin{aligned} &\|f(x) - Q(x) - C(x)\| \\ &\leq \begin{cases} \theta\|x\|^p \left(\frac{1}{k^4} \left(\left(\frac{1}{2} + \frac{k^4 - k^2}{16} \right) \left(\frac{1}{k^{p-4} - 1} \right) \right) + \frac{1}{2(k^4 - k^2)} \left(\frac{1}{2^{p-3} - 1} \right) \right), & p > 4, \\ \theta\|x\|^p \left(\frac{1}{k^4} \left(\left(\frac{1}{2} + \frac{k^4 - k^2}{16} \right) \left(\frac{1}{1 - k^{p-4}} \right) \right) + \frac{1}{2(k^4 - k^2)} \left(\frac{1}{1 - 2^{p-3}} \right) \right), & p < 3, \end{cases} \end{aligned} \quad (3.48)$$

for all $x \in X$.

Proof. In Theorem 3.3, put $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. □

Similarly, one can solve Ulam stability problem for functional equation (1.7) when the norm of the Cauchy difference is controlled by the mixed type product-sum function

$$(x, y) \mapsto \theta(\|x\|_X^u \|y\|_X^v + \|x\|^p + \|y\|^p). \quad (3.49)$$

Corollary 3.5. *Let u, v, p be real numbers such that $u + v, p \in (-\infty, 3) \cup (4, +\infty)$ and $\theta > 0$. Suppose $f : X \rightarrow Y$ satisfies $f(0) = 0$ and inequality*

$$\|D_f(x, y)\| \leq \theta(\|x\|_X^u \|y\|_X^v + \|x\|^p + \|y\|^p) \quad (3.50)$$

for all $x, y \in X$. Then there exist a unique quartic function $Q : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ satisfying

$$\begin{aligned} & \|f(x) - Q(x) - C(x)\| \\ & \leq \begin{cases} \theta \|x\|^p \left(\frac{1}{k^4} \left(\left(\frac{1}{2} + \frac{k^4 - k^2}{16} \right) \left(\frac{1}{k^{p-4} - 1} \right) \right) + \frac{1}{2(k^4 - k^2)} \left(\frac{1}{2^{p-3} - 1} \right) \right), & p > 4, \\ \theta \|x\|^p \left(\frac{1}{k^4} \left(\left(\frac{1}{2} + \frac{k^4 - k^2}{16} \right) \left(\frac{1}{1 - k^{p-4}} \right) \right) + \frac{1}{2(k^4 - k^2)} \left(\frac{1}{1 - 2^{p-3}} \right) \right), & p < 3, \end{cases} \end{aligned} \quad (3.51)$$

for all $x \in X$.

Applying Corollary 3.4, one can obtain the stability of the functional equation (1.7) in the following form.

Corollary 3.6. *Let ϵ be a positive real number. Suppose $f : X \rightarrow Y$ satisfies $f(0) = 0$ and $\|D_f(x, y)\| \leq \epsilon$ for all $x, y \in X$. Then there exists a unique quartic function $Q : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ satisfying*

$$\|f(x) - Q(x) - C(x)\| \leq \epsilon \left(\frac{1}{k^4} \left(\left(\frac{1}{2} + \frac{k^4 - k^2}{16} \right) \left(\frac{k^4}{k^4 - 1} \right) \right) + \frac{1}{2(k^4 - k^2)} \left(\frac{8}{8 - 1} \right) \right) \quad (3.52)$$

for all $x \in X$.

References

- [1] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, Wiley, New York, NY, USA, Science edition, 1940.
- [2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [3] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
- [4] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

- [5] K.-W. Jun and H.-M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 267–278, 2002.
- [6] W.-G. Park and J.-H. Bae, "On a bi-quadratic functional equation and its stability," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 62, no. 4, pp. 643–654, 2005.
- [7] J. K. Chung and P. K. Sahoo, "On the general solution of a quartic functional equation," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 4, pp. 565–576, 2003.
- [8] S. H. Lee, S. M. Im, and I. S. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.
- [9] Y. J. Cho, M. E. Gordji, and S. Zolfaghari, "Solutions and stability of generalized mixed type QC functional equations in random normed spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 403101, 17 pages, 2010.
- [10] A. Charifi, B. Bouikhalene, and E. Elqorachi, "Hyers-Ulam-Rassias stability of a generalized Pexider functional equation," *Banach Journal of Mathematical Analysis*, vol. 1, no. 2, pp. 176–185, 2007.
- [11] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.
- [12] M. E. Gordji, "Stability of a functional equation deriving from quartic and additive functions," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 3, pp. 491–502, 2010.
- [13] M. E. Gordji, "Stability of an additive-quadratic functional equation of two variables in F-spaces," *Journal of Nonlinear Science and Its Applications*, vol. 2, no. 4, pp. 251–259, 2009.
- [14] M. E. Gordji and M. B. Savadkouhi, "Stability of cubic and quartic functional equations in non-Archimedean spaces," *Acta Applicandae Mathematicae*, vol. 110, no. 3, pp. 1321–1329, 2010.
- [15] M. E. Gordji and M. B. Savadkouhi, "Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1198–1202, 2010.
- [16] M. E. Gordji and H. Khodaei, "On the generalized Hyers-Ulam-Rassias stability of quadratic functional equations," *Abstract and Applied Analysis*, vol. 2009, Article ID 923476, 11 pages, 2009.
- [17] M. E. Gordji, A. Ebadian, and S. Zolfaghari, "Stability of a functional equation deriving from cubic and quartic functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 801904, 17 pages, 2008.
- [18] M. E. Gordji and M. Bavand Savadkouhi, "On approximate cubic homomorphisms," *Advances in Difference Equations*, vol. 2009, Article ID 618463, 11 pages, 2009.
- [19] M. E. Gordji and H. Khodaei, "Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 71, no. 11, pp. 5629–5643, 2009.
- [20] R. Farokhzad and S. A. R. Hosseinioun, "Perturbations of Jordan higher derivations in Banach ternary algebras: an alternative fixed point approach," *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 1, pp. 42–53, 2010.
- [21] P. Găvruta, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
- [22] P. Găvruta and L. Găvruta, "A new method for the generalized Hyers-Ulam-Rassias stability," *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 1, pp. 11–18, 2010.
- [23] M. E. Gordji, S. Zolfaghari, J. M. Rassias, and M. B. Savadkouhi, "Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 417473, 14 pages, 2009.
- [24] M. E. Gordji, S. Kaboli-Gharetafeh, C. Park, and S. Zolfaghari, "Stability of an additive-cubic-quartic functional equation," *Advances in Difference Equations*, vol. 2009, Article ID 395693, 20 pages, 2009.
- [25] M. E. Gordji, S. Kaboli Gharetapeh, J. M. Rassias, and S. Zolfaghari, "Solution and stability of a mixed type additive, quadratic and cubic functional equation," *Advances in Difference Equations*, vol. 2009, Article ID 826130, 17 pages, 2009.
- [26] M. E. Gordji, M. B. Savadkouhi, J. M. Rassias, and S. Zolfaghari, "Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 417473, 14 pages, 2009.
- [27] M. E. Gordji and M. B. Savadkouhi, "Stability of mixed type cubic and quartic functional equations in random normed spaces," *Journal of Inequalities and Applications*, vol. 2009, Article ID 527462, 9 pages, 2009.
- [28] M. E. Gordji, S. Abbaszadeh, and C. Park, "On the stability of generalized mixed type quadratic and quartic functional equation in quasi-Banach spaces," *Journal of Inequalities and Applications*, vol. 2009, Article ID 153084, 26 pages, 2009.
- [29] M. E. Gordji, M. B. Savadkouhi, and C. Park, "Quadratic-quartic functional equations in RN-spaces," *Journal of Inequalities and Applications*, vol. 2009, Article ID 868423, 14 pages, 2009.

- [30] M. E. Gordji and M. B. Savadkouhi, "Approximation of generalized homomorphisms in quasi-Banach algebras," *Analele stiintifice ale Universitatii Ovidius Constanta*, vol. 17, no. 2, pp. 203–213, 2009.
- [31] M. E. Gordji, M. B. Savadkouhi, and M. Bidkham, "Stability of a mixed type additive and quadratic functional equation in non-Archimedean spaces," *Journal of Computational Analysis and Applications*, vol. 12, no. 2, pp. 454–462, 2010.
- [32] A. Gilányi, "On Hyers-Ulam stability of monomial functional equations," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 68, pp. 321–328, 1998.
- [33] A. Gilányi, "Hyers-Ulam stability of monomial functional equations on a general domain," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 96, no. 19, pp. 10588–10590, 1999.
- [34] A. Grabiec, "The generalized Hyers-Ulam stability of a class of functional equations," *Publicationes Mathematicae Debrecen*, vol. 48, no. 3-4, pp. 217–235, 1996.
- [35] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications, no. 34, Birkhäuser, Basel, Switzerland, 1998.
- [36] G. Isac and T. M. Rassias, "On the Hyers-Ulam stability of φ -additive mappings," *Journal of Approximation Theory*, vol. 72, no. 2, pp. 131–137, 1993.
- [37] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.
- [38] M. S. Moslehian and G. Sadeghi, "Stability of two types of cubic functional equations in non-Archimedean spaces," *Real Analysis Exchange*, vol. 33, no. 2, pp. 375–383, 2008.
- [39] A. Najati, "On the stability of a quartic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 569–574, 2008.
- [40] C.-G. Park, "On the stability of the orthogonally quartic functional equation," *Bulletin of the Iranian Mathematical Society*, vol. 31, no. 1, pp. 63–70, 2005.
- [41] C.-G. Park, "Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras," *Banach Journal of Mathematical Analysis*, vol. 1, no. 1, pp. 23–32, 2007.
- [42] T. M. Rassias, Ed., *Functional Equations and Inequalities*, vol. 518 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
- [43] T. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
- [44] T. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.
- [45] L. Tan and S. Xiang, "On the Aleksandrov-Rassias problem and the Hyers-Ulam-Rassias stability problem," *Banach Journal of Mathematical Analysis*, vol. 1, no. 1, pp. 11–22, 2007.