Research Article

Nearly Jordan $*$ -Homomorphisms between Unital C^* -Algebras

A. Ebadian,¹ S. Kaboli Gharetapeh,¹ and M. Eshaghi Gordji^{2,3}

¹ Department of Mathematics, Urmia University, Urmia, Iran

² Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran

³ Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Semnan, Iran

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com

Received 26 February 2011; Revised 7 April 2011; Accepted 10 April 2011

Academic Editor: Irena Lasiecka

Copyright © 2011 A. Ebadian et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let A, B be two unital C^* -algebras. We prove that every almost unital almost linear mapping $h : A \rightarrow B$ which satisfies $h(3^n uy + 3^n yu) = h(3^n u)h(y) + h(y)h(3^n u)$ for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \dots$, is a Jordan homomorphism. Also, for a unital C^* -algebra A of real rank zero, every almost unital almost linear continuous mapping $h : A \rightarrow B$ is a Jordan homomorphism when $h(3^n uy + 3^n yu) = h(3^n u)h(y) + h(y)h(3^n u)$ holds for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \dots$. Furthermore, we investigate the Hyers-Ulam-Aoki-Rassias stability of Jordan $*$ -homomorphisms between unital C^* -algebras by using the fixed points methods.

1. Introduction

The stability of functional equations was first introduced by Ulam [1] in 1940. More precisely, he proposed the following problem: given a group G_1 , a metric group (G_2, d) and a positive number ϵ , does there exist a $\delta > 0$ such that if a function $f : G_1 \rightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \rightarrow G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$. As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [2] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Hyers theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. This phenomenon of stability is called the Hyers-Ulam-Aoki-Rassias stability.

J. M. Rassias [5–7] established the stability of linear and nonlinear mappings with new control functions.

During the last decades, several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [8–10].

Bourgin is the first mathematician dealing with the stability of ring homomorphisms. The topic of approximate ring homomorphisms was studied by a number of mathematicians, see [11–19] and references therein.

Jun and Lee [20] proved the following: Let X and Y be Banach spaces. Denote by $\phi : X - \{0\} \times Y - \{0\} \rightarrow [0, \infty)$ a function such that $\tilde{\phi}(x, y) = \sum_{n=0}^{\infty} 3^{-n} \phi(3^n x, 3^n y) < \infty$ for all $x, y \in X - \{0\}$. Suppose that $f : X \rightarrow Y$ is a mapping satisfying

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \phi(x, y) \quad (1.1)$$

for all $x, y \in X - \{0\}$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{1}{3} (\tilde{\phi}(x, -x) + \tilde{\phi}(-x, 3x)) \quad (1.2)$$

for all $x \in X - \{0\}$.

Recently, C. Park and W. Park [21] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^* -algebra. Johnson (Theorem 7.2 of [22]) also investigated almost algebra $*$ -homomorphisms between Banach $*$ -algebras: Suppose that U and B are Banach $*$ -algebras which satisfy the conditions of (Theorem 3.1 of [22]). Then for each positive ϵ and K , there is a positive δ such that if $T \in L(U, B)$ with $\|T\| < K$, $\|T^\vee\| < \delta$ and $\|T(x^*)^* - T(x)\| < \delta \|x\|$ ($x \in U$), then there is a $*$ -homomorphism $T' : U \rightarrow B$ with $\|T - T'\| < \epsilon$. Here $L(U, B)$ is the space of bounded linear maps from U into B , and $T^\vee(x, y) = T(xy) - T(x)T(y)$ ($x, y \in U$). See [22] for details. Throughout this paper, let A be a unital C^* -algebra with unit e , and B a unital C^* -algebra. Let $U(A)$ be the set of unitary elements in A , $A_{sa} := \{x \in A \mid x = x^*\}$, and $I_1(A_{sa}) = \{v \in A_{sa} \mid \|v\| = 1, v \in Inv(A)\}$. In this paper, we prove that every almost unital almost linear mapping $h : A \rightarrow B$ is a Jordan homomorphism when $h(3^n uy + 3^n yu) = h(3^n u)h(y) + h(y)h(3^n u)$ holds for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \dots$, and that for a unital C^* -algebra A of real rank zero (see [23]), every almost unital almost linear continuous mapping $h : A \rightarrow B$ is a Jordan homomorphism when $h(3^n uy + 3^n yu) = h(3^n u)h(y) + h(y)h(3^n u)$ holds for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \dots$. Furthermore, we investigate the Hyers-Ulam-Aoki-Rassias stability of Jordan $*$ -homomorphisms between unital C^* -algebras by using the fixed point methods.

Note that a unital C^* -algebra is of real rank zero, if the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [23]). We denote the algebraic center of algebra A by $Z(A)$.

2. Jordan $*$ -Homomorphisms on Unital C^* -Algebras

By a following similar way as in [24], we obtain the next theorem.

Theorem 2.1. Let $f : A \rightarrow B$ be a mapping such that $f(0) = 0$ and that

$$f(3^n uy + 3^n yu) = f(3^n u)f(y) + f(y)f(3^n u) \quad (2.1)$$

for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \dots$. If there exists a function $\phi : (A - \{0\})^2 \times A \rightarrow [0, \infty)$ such that $\tilde{\phi}(x, y, z) = \sum_{n=0}^{\infty} 3^{-n} \phi(3^n x, 3^n y, 3^n z) < \infty$ for all $x, y \in A - \{0\}$ and all $z \in A$ and that

$$\left\| 2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y) + f(u^*) - f(u)^* \right\| \leq \phi(x, y, u), \quad (2.2)$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A, u \in (U(A) \cup \{0\})$. If $\lim_n (f(3^n e)/3^n) \in U(B) \cap Z(B)$, then the mapping $f : A \rightarrow B$ is a Jordan $*$ -homomorphism.

Proof. Put $u = 0, \mu = 1$ in (2.2), it follows from of [20, Theorem 1] that there exists a unique additive mapping $T : A \rightarrow B$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{3} \left(\tilde{\phi}(x, -x, 0) + \tilde{\phi}(-x, 3x, 0) \right) \quad (2.3)$$

for all $x \in A - \{0\}$. This mapping is given by

$$T(x) = \lim_n \frac{f(3^n x)}{3^n} \quad (2.4)$$

for all $x \in A$. By the same reasoning as the proof of [24, Theorem 1], T is $\mathbb{C}$ -linear and $*$ -preserving. It follows from (2.1) that

$$T(uy + yu) = \lim_n \frac{f(3^n uy + 3^n yu)}{3^n} = \lim_n \left[\frac{f(3^n u)}{3^n} f(y) + f(y) \frac{f(3^n u)}{3^n} \right] = T(u)f(y) + f(y)T(u) \quad (2.5)$$

for all $u \in U(A)$, all $y \in A$. Since T is additive, then by (2.5), we have

$$3^n T(uy + yu) = T(u(3^n y) + (3^n y)u) = T(u)f(3^n y) + f(3^n y)T(u) \quad (2.6)$$

for all $u \in U(A)$ and all $y \in A$. Hence,

$$T(uy + yu) = \lim_n \left[T(u) \frac{f(3^n y)}{3^n} + \frac{f(3^n y)}{3^n} T(u) \right] = T(u)T(y) + T(y)T(u) \quad (2.7)$$

for all $u \in U(A)$ and all $y \in A$. By the assumption, we have

$$T(e) = \lim_n \frac{f(3^n e)}{3^n} \in U(B) \cap Z(B) \quad (2.8)$$

hence, it follows by (2.5) and (2.7) that

$$2T(e)T(y) = T(e)T(y) + T(y)T(e) = T(ye + ey) = T(e)f(y) + f(y)T(e) = 2T(e)f(y) \quad (2.9)$$

for all $y \in A$. Since $T(e)$ is invertible, then $T(y) = f(y)$ for all $y \in A$. We have to show that f is Jordan homomorphism. To this end, let $x \in A$. By Theorem 4.1.7 of [25], x is a finite linear combination of unitary elements, that is, $x = \sum_{j=1}^n c_j u_j$ ($c_j \in \mathbb{C}, u_j \in U(A)$), and then it follows from (2.7) that

$$\begin{aligned} f(xy + yx) &= T(xy + yx) = T\left(\sum_{j=1}^n c_j u_j y + \sum_{j=1}^n c_j y u_j\right) = \sum_{j=1}^n c_j T(u_j y + y u_j) \\ &= \sum_{j=1}^n c_j (T(u_j y) + T(y u_j)) = \sum_{j=1}^n c_j (T(u_j)T(y) + T(y)T(u_j)) \\ &= T\left(\sum_{j=1}^n c_j u_j\right)T(y) + T(y)T\left(\sum_{j=1}^n c_j u_j\right) = T(x)T(y) + T(y)T(x) \\ &= f(x)f(y) + f(y)f(x) \end{aligned} \quad (2.10)$$

for all $y \in A$. And this completes the proof of theorem. $\square$

Corollary 2.2. Let $p \in (0, 1)$, $\theta \in [0, \infty)$ be real numbers. Let $f : A \rightarrow B$ be a mapping such that $f(0) = 0$ and that

$$f(3^n u y + 3^n y u) = f(3^n u)f(y) + f(y)f(3^n u) \quad (2.11)$$

for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \dots$. Suppose that

$$\left\| 2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y) + f(z^*) - f(z)^* \right\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \quad (2.12)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. If $\lim_n (f(3^n e)/3^n) \in U(B) \cap Z(B)$, then the mapping $f : A \rightarrow B$ is a Jordan $*$ -homomorphism.

Proof. Setting $\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in A$. Then by Theorem 2.1, we get the desired result. $\square$

Theorem 2.3. Let A be a C^* -algebra of real rank zero. Let $f : A \rightarrow B$ be a continuous mapping such that $f(0) = 0$ and that

$$f(3^n u y + 3^n y u) = f(3^n u)f(y) + f(y)f(3^n u) \quad (2.13)$$

for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \dots$. Suppose that there exists a function $\phi : (A - \{0\})^2 \times A \rightarrow [0, \infty)$ satisfying (2.2) and $\tilde{\phi}(x, y, z) < \infty$ for all $x, y \in A - \{0\}$ and all $z \in A$. If $\lim_n (f(3^n e)/3^n) \in U(B) \cap Z(B)$, then the mapping $f : A \rightarrow B$ is a Jordan $*$ -homomorphism.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique involutive $\mathbb{C}$ -linear mapping $T : A \rightarrow B$ satisfying (2.3). It follows from (2.13) that

$$T(uy + yu) = \lim_n \frac{f(3^n uy + 3^n yu)}{3^n} = \lim_n \left[\frac{f(3^n u)}{3^n} f(y) + f(y) \frac{f(3^n u)}{3^n} \right] = T(u)f(y) + f(y)T(u) \quad (2.14)$$

for all $u \in I_1(A_{sa})$, and all $y \in A$. By additivity of T and (2.14), we obtain that

$$3^n T(uy + yu) = T(u(3^n y) + (3^n y)u) = T(u)f(3^n y) + f(3^n y)T(u) \quad (2.15)$$

for all $u \in I_1(A_{sa})$ and all $y \in A$. Hence,

$$T(uy + yu) = \lim_n \left[T(u) \frac{f(3^n y)}{3^n} + \frac{f(3^n y)}{3^n} T(u) \right] = T(u)T(y) + T(y)T(u) \quad (2.16)$$

for all $u \in I_1(A_{sa})$ and all $y \in A$. By the assumption, we have

$$T(e) = \lim_n \frac{f(3^n e)}{3^n} \in U(B) \cap Z(B). \quad (2.17)$$

Similar to the proof of Theorem 2.1, it follows from (2.14) and (2.16) that $T = f$ on A . So T is continuous. On the other hand, A is real rank zero. One can easily show that $I_1(A_{sa})$ is dense in $\{x \in A_{sa} : \|x\| = 1\}$. Let $v \in \{x \in A_{sa} : \|x\| = 1\}$. Then there exists a sequence $\{z_n\}$ in $I_1(A_{sa})$ such that $\lim_n z_n = v$. Since T is continuous, it follows from (2.16) that

$$\begin{aligned} T(vy + yv) &= T\left(\lim_n (z_n y + y z_n)\right) = \lim_n T(z_n y + y z_n) \\ &= \lim_n T(z_n)T(y) + \lim_n T(y)T(z_n) \\ &= T\left(\lim_n z_n\right)T(y) + T(y)T\left(\lim_n z_n\right) \\ &= T(v)T(y) + T(y)T(v) \end{aligned} \quad (2.18)$$

for all $y \in A$. Now, let $x \in A$. Then we have $x = x_1 + ix_2$, where $x_1 := (x + x^*)/2$ and $x_2 = (x - x^*)/2i$ are self adjoint.

First consider $x_1 = 0$, $x_2 \neq 0$. Since T is $\mathbb{C}$ -linear, it follows from (2.18) that

$$\begin{aligned}
 f(xy + yx) &= T(xy + yx) = T(ix_2y + y(ix_2)) = T\left(i\|x_2\|\frac{x_2}{\|x_2\|}y + y\left(i\|x_2\|\frac{x_2}{\|x_2\|}\right)\right) \\
 &= T\left(i\|x_2\|\frac{x_2}{\|x_2\|}\right)T(y) + T(y)T\left(i\|x_2\|\frac{x_2}{\|x_2\|}\right) \\
 &= T(ix_2)T(y) + T(y)T(ix_2) = T(x)T(y) + T(y)T(x) \\
 &= f(x)f(y) + f(y)f(x)
 \end{aligned} \tag{2.19}$$

for all $y \in A$.

If $x_2 = 0$, $x_1 \neq 0$, then by (2.18), we have

$$\begin{aligned}
 f(xy + yx) &= T(xy + yx) = T(x_1y + y(x_1)) = T\left(\|x_1\|\frac{x_1}{\|x_1\|}y + y\left(\|x_1\|\frac{x_1}{\|x_1\|}\right)\right) \\
 &= T\left(\|x_1\|\frac{x_1}{\|x_1\|}\right)T(y) + T(y)T\left(\|x_1\|\frac{x_1}{\|x_1\|}\right) = T(x_1)T(y) + T(y)T(x_1) \\
 &= T(x)T(y) + T(y)T(x) = f(x)f(y) + f(y)f(x)
 \end{aligned} \tag{2.20}$$

for all $y \in A$.

Finally, consider the case that $x_1 \neq 0$, $x_2 \neq 0$. Then it follows from (2.18) that

$$\begin{aligned}
 f(xy + yx) &= T(xy + yx) = T(x_1y + ix_2y + yx_1 + y(ix_2)) \\
 &= T\left(\|x_1\|\frac{x_1}{\|x_1\|}y + y\left(\|x_1\|\frac{x_1}{\|x_1\|}\right)\right) + T\left(i\|x_2\|\frac{x_2}{\|x_2\|}y + y\left(i\|x_2\|\frac{x_2}{\|x_2\|}\right)\right) \\
 &= \|x_1\|\left[T\left(\frac{x_1}{\|x_1\|}\right)T(y) + T(y)T\left(\frac{x_1}{\|x_1\|}\right)\right] + i\|x_2\|\left[T\left(\frac{x_2}{\|x_2\|}\right)T(y) + T(y)T\left(\frac{x_2}{\|x_2\|}\right)\right] \\
 &= [T(x_1) + T(ix_2)]T(y) + T(y)[T(x_1) + T(ix_2)] \\
 &= T(x)T(y) + T(y)T(x) = f(x)f(y) + f(y)f(x)
 \end{aligned} \tag{2.21}$$

for all $y \in A$. Hence, $f(xy + yx) = f(x)f(y) + f(y)f(x)$ for all $x, y \in A$, and f is Jordan $*$ -homomorphism. $\square$

Corollary 2.4. Let A be a C^* -algebra of rank zero. Let $p \in (0, 1)$, $\theta \in [0, \infty)$ be real numbers. Let $f : A \rightarrow B$ be a mapping such that $f(0) = 0$ and that

$$f(3^n uy + 3^n yu) = f(3^n u)f(y) + f(y)f(3^n u) \tag{2.22}$$

for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \dots$. Suppose that

$$\left\| 2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y) + f(z^*) - f(z)^* \right\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \quad (2.23)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. If $\lim_n f(3^n e)/3^n \in \mathcal{U}(B) \cap \mathcal{Z}(B)$, then the mapping $f : A \rightarrow B$ is a Jordan $*$ -homomorphism.

Proof. Setting $\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in A$. Then by Theorem 2.3, we get the desired result. $\square$

3. Stability of Jordan $*$ -Homomorphisms: A Fixed Point Approach

We investigate the generalized Hyers-Ulam-Aoki-Rassias stability of Jordan $*$ -homomorphisms on unital C^* -algebras by using the alternative fixed point.

Recently, Cădariu and Radu applied the fixed point method to the investigation of the functional equations. (See also [18, 26–43]).

Theorem 3.1. Let $f : A \rightarrow B$ be a mapping with $f(0) = 0$ for which there exists a function $\phi : A^5 \rightarrow [0, \infty)$ satisfying

$$\begin{aligned} & \left\| f\left(\frac{\mu x + \mu y + \mu z}{3}\right) + f\left(\frac{\mu x - 2\mu y + \mu z}{3}\right) + f\left(\frac{\mu x + \mu y - 2\mu z}{3}\right) - \mu f(x) + f(uv + uv) \right. \\ & \quad \left. - f(v)f(u) - f(u)f(v) + f(w^*) - f(w)^* \right\| \leq \phi(x, y, z, u, v, w), \end{aligned} \quad (3.1)$$

for all $\mu \in \mathbb{T}$, and all $x, y, z, u, v \in A, w \in \mathcal{U}(A) \cup \{0\}$. If there exists an $L < 1$ such that $\phi(x, y, z, u, v, w) \leq 3L\phi(x/3, y/3, z/3, u/3, v/3, w/3)$ for all $x, y, z, u, v, w \in A$, then there exists a unique Jordan $*$ -homomorphism $h : A \rightarrow B$ such that

$$\|f(x) - h(x)\| \leq \frac{L}{1-L} \phi(x, 0, 0, 0, 0, 0) \quad (3.2)$$

for all $x \in A$.

Proof. It follows from $\phi(x, y, z, u, v, w) \leq 3L\phi(x/3, y/3, z/3, u/3, v/3, w/3)$ that

$$\lim_j 3^{-j} \phi(3^j x, 3^j y, 3^j z, 3^j u, 3^j v, 3^j w) = 0 \quad (3.3)$$

for all $x, y, z, u, v, w \in A$.

Put $y = z = w = u = 0$ and $\mu = 1$ in (3.1) to obtain

$$\left\| 3f\left(\frac{x}{3}\right) - f(x) \right\| \leq \phi(x, 0, 0, 0, 0, 0) \quad (3.4)$$

for all $x \in A$. Hence,

$$\left\| \frac{1}{3}f(3x) - f(x) \right\| \leq \frac{1}{3}\phi(3x, 0, 0, 0, 0, 0) \leq L\phi(x, 0, 0, 0, 0, 0) \quad (3.5)$$

for all $x \in A$.

Consider the set $X := \{g \mid g : A \rightarrow B\}$ and introduce the generalized metric on X :

$$d(h, g) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0, 0, 0, 0, 0) \forall x \in A\}. \quad (3.6)$$

It is easy to show that (X, d) is complete. Now we define the linear mapping $J : X \rightarrow X$ by

$$J(h)(x) = \frac{1}{3}h(3x) \quad (3.7)$$

for all $x \in A$. By Theorem 3.1 of [44],

$$d(J(g), J(h)) \leq Ld(g, h) \quad (3.8)$$

for all $g, h \in X$.

It follows from (3.5) that

$$d(f, J(f)) \leq L. \quad (3.9)$$

Now, from the fixed point alternative [45], J has a unique fixed point in the set $X_1 := \{h \in X : d(f, h) < \infty\}$. Let h be the fixed point of J . h is the unique mapping with

$$h(3x) = 3h(x) \quad (3.10)$$

for all $x \in A$ satisfying there exists $C \in (0, \infty)$ such that

$$\|h(x) - f(x)\| \leq C\phi(x, 0, 0, 0, 0, 0) \quad (3.11)$$

for all $x \in A$. On the other hand, we have $\lim_n d(J^n(f), h) = 0$. It follows that

$$\lim_n \frac{1}{3^n} f(3^n x) = h(x) \quad (3.12)$$

for all $x \in A$. It follows from $d(f, h) \leq (1/(1-L))d(f, J(f))$, that

$$d(f, h) \leq \frac{L}{1-L}. \quad (3.13)$$

This implies the inequality (3.2). It follows from (3.1), (3.3), and (3.12) that

$$\begin{aligned}
& \left\| h\left(\frac{x+y+z}{3}\right) + h\left(\frac{x-2y+z}{3}\right) + h\left(\frac{x+y-2z}{3}\right) - h(x) \right\| \\
&= \lim_n \frac{1}{3^n} \left\| f\left(3^{n-1}(x+y+z)\right) + f\left(3^{n-1}(x-2y+z)\right) + f\left(3^{n-1}(x+y-2z)\right) - f(3^n x) \right\| \\
&\leq \lim_n \frac{1}{3^n} \phi(3^n x, 3^n y, 3^n z, 0, 0, 0) \\
&= 0
\end{aligned} \tag{3.14}$$

for all $x, y, z \in A$. So

$$h\left(\frac{x+y+z}{3}\right) + h\left(\frac{x-2y+z}{3}\right) + h\left(\frac{x+y-2z}{3}\right) = h(x) \tag{3.15}$$

for all $x, y, z \in A$. Put $w = (x+y+z)/3$, $t = (x-2y+z)/3$ and $s = (x+y-2z)/3$ in above equation, we get $h(w+t+s) = h(w) + h(t) + h(s)$ for all $w, t, s \in A$. Hence, h is Cauchy additive. By putting $y = z = x$, $v = w = 0$ in (3.1), we have

$$\|f(\mu x) - \mu f(x)\| \leq \phi(x, x, x, 0, 0, 0) \tag{3.16}$$

for all $\mu \in \mathbb{T}$ and all $x \in A$. It follows that

$$\|h(\mu x) - \mu h(x)\| = \lim_m \frac{1}{3^m} \|f(\mu 3^m x) - \mu f(3^m x)\| \leq \lim_m \frac{1}{3^m} \phi(3^m x, 3^m x, 3^m x, 0, 0, 0) = 0 \tag{3.17}$$

for all $\mu \in \mathbb{T}$, and all $x \in A$. One can show that the mapping $h : A \rightarrow B$ is $\mathbb{C}$ -linear. By putting $x = y = z = u = v = 0$ in (3.1), it follows that

$$\begin{aligned}
\|h(w^*) - (h(w))^*\| &= \lim_m \left\| \frac{1}{3^m} f((3^m w)^*) - \frac{1}{3^m} (f(3^m w))^* \right\| \\
&\leq \lim_m \frac{1}{3^m} \phi(0, 0, 0, 0, 0, 3^m w) \\
&= 0
\end{aligned} \tag{3.18}$$

for all $w \in U(A)$. By the same reasoning as the proof of Theorem 2.1, we can show that $h : A \rightarrow B$ is $*$ -preserving.

Since h is $\mathbb{C}$ -linear, by putting $x = y = z = w = 0$ in (3.1), it follows that

$$\begin{aligned}
 & \|h(uv + vu) - h(u)h(v) - h(v)h(u)\| \\
 &= \lim_m \left\| \frac{1}{9^m} f(9^m(uv + vu)) - \frac{1}{9^m} [f(3^m u)f(3^m v) + f(3^m v)f(3^m u)] \right\| \\
 &\leq \lim_m \frac{1}{9^m} \phi(0, 0, 0, 3^m u, 3^m v, 0) \leq \lim_m \frac{1}{3^m} \phi(0, 0, 0, 3^m u, 3^m v, 0) \\
 &= 0
 \end{aligned} \tag{3.19}$$

for all $u, v \in A$. Thus $h : A \rightarrow B$ is Jordan $*$ -homomorphism satisfying (3.2), as desired. $\square$

We prove the following Hyers-Ulam-Aoki-Rassias stability problem for Jordan $*$ -homomorphisms on unital C^* -algebras.

Corollary 3.2. *Let $p \in (0, 1)$, $\theta \in [0, \infty)$ be real numbers. Suppose $f : A \rightarrow A$ satisfies*

$$\begin{aligned}
 & \left\| f\left(\frac{\mu x + \mu y + \mu z}{3}\right) + f\left(\frac{\mu x - 2\mu y + \mu z}{3}\right) + f\left(\frac{\mu x + \mu y - 2\mu z}{3}\right) - \mu f(x) + f(uv + uv) \right. \\
 & \quad \left. - f(v)f(u) - f(u)f(v) + f(w^*) - f(w)^* \right\| \\
 & \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|u\|^p + \|v\|^p + \|w\|^p),
 \end{aligned} \tag{3.20}$$

for all $\mu \in \mathbb{T}$ and all $x, y, z, u, v \in A, w \in \mathcal{U}(A) \cup \{0\}$. Then there exists a unique Jordan $*$ -homomorphism $h : A \rightarrow B$ such that

$$\|f(x) - h(x)\| \leq \frac{3^p \theta}{3 - 3^p} \|x\|^p \tag{3.21}$$

for all $x \in A$.

Proof. Setting $\phi(x, y, z, u, v, w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|u\|^p + \|v\|^p + \|w\|^p)$ all $x, y, z, u, v, w \in A$. Then by $L = 3^{p-1}$ in Theorem 3.2, one can prove the result. $\square$

References

- [1] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, Wiley, New York, NY, USA, 1940.
- [2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
- [4] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, pp. 297–300, 1978.
- [5] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.

- [6] J. M. Rassias and M. J. Rassias, "Asymptotic behavior of alternative Jensen and Jensen type functional equations," *Bulletin des Sciences Mathématiques*, vol. 129, no. 7, pp. 545–558, 2005.
- [7] J. M. Rassias and M. J. Rassias, "On the Ulam stability of Jensen and Jensen type mappings on restricted domains," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 2, pp. 516–524, 2003.
- [8] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.
- [9] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, Switzerland, 1998.
- [10] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.
- [11] R. Badora, "On approximate ring homomorphisms," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 2, pp. 589–597, 2002.
- [12] J. Baker, J. Lawrence, and F. Zorzitto, "The stability of the equation $f(x + y) = f(x)f(y)$," *Proceedings of the American Mathematical Society*, vol. 74, no. 2, pp. 242–246, 1979.
- [13] D. G. Bourgin, "Approximately isometric and multiplicative transformations on continuous function rings," *Duke Mathematical Journal*, vol. 16, pp. 385–397, 1949.
- [14] D. H. Hyers and T. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.
- [15] T. Miura, S. E. Takahasi, and G. Hirasawa, "Hyers-Ulam-Rassias stability of Jordan homomorphisms on Banach algebras," *Journal of Inequalities and Applications*, no. 4, pp. 435–441, 2005.
- [16] C. Park, "Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras," *Bulletin des Sciences Mathématiques*, vol. 132, no. 2, pp. 87–96, 2008.
- [17] C. Park, "Homomorphisms between Poisson JC^* -algebras," *Bulletin of the Brazilian Mathematical Society. New Series. Boletim da Sociedade Brasileira de Matemática*, vol. 36, no. 1, pp. 79–97, 2005.
- [18] C. Park and J. M. Rassias, "Stability of the Jensen-type functional equation in C^* -algebras: a fixed point approach," *Abstract and Applied Analysis*, Article ID 360432, 17 pages, 2009.
- [19] T. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
- [20] K. W. Jun and Y. H. Lee, "A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation," *Journal of Mathematical Analysis and Applications*, vol. 238, no. 1, pp. 305–315, 1999.
- [21] C. Park and W. Park, "On the Jensen's equation in Banach modules," *Taiwanese Journal of Mathematics*, vol. 6, no. 4, pp. 523–531, 2002.
- [22] B. E. Johnson, "Approximately multiplicative maps between Banach algebras," *Journal of the London Mathematical Society*, vol. 37, no. 2, pp. 294–316, 1988.
- [23] L. G. Brown and G. K. Pedersen, " C^* -algebras of real rank zero," *Journal of Functional Analysis*, vol. 99, no. 1, pp. 131–149, 1991.
- [24] C. Park, D. H. Boo, and J. S. An, "Homomorphisms between C^* -algebras and linear derivations on C^* -algebras," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1415–1424, 2008.
- [25] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras: Elementary Theory*, vol. 100 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1983.
- [26] L. Cădariu and V. Radu, "The fixed points method for the stability of some functional equations," *Carpathian Journal of Mathematics*, vol. 23, no. 1-2, pp. 63–72, 2007.
- [27] L. Cădariu and V. Radu, "Fixed points and the stability of quadratic functional equations," *Analele Universitatii de Vest din Timisoara*, vol. 41, no. 1, pp. 25–48, 2003.
- [28] L. Cădariu and V. Radu, "Fixed points and the stability of Jensen's functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 4, 2003.
- [29] M. Eshaghi Gordji, "Jordan $*$ -homomorphisms between unital C^* -algebras: a fixed point approach," *Fixed Point Theory*. In press.
- [30] P. Găvruta and L. Găvruta, "A new method for the generalized Hyers-Ulam-Rassias stability," *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 2, pp. 11–18, 2010.
- [31] A. Ebadian, N. Ghobadipour, and M. Eshaghi Gordji, "A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^* -ternary algebras," *Journal of Mathematical Physics*, vol. 51, no. 10, 2010.
- [32] M. Eshaghi Gordji, "Nearly ring homomorphisms and nearly ring derivations on non-Archimedean Banach algebras," *Abstract and Applied Analysis*, Article ID 393247, 12 pages, 2010.

- [33] M. Eshaghi Gordji and Z. Alizadeh, "Stability and superstability of ring homomorphisms on non-archimedean banach algebras," *Abstract and Applied Analysis*, vol. 2011, Article ID 123656, 10 pages, 2011.
- [34] M. Eshaghi Gordji, M. Bavand Savadkouhi, M. Bidkham, C. Park, and J. R. Lee, "Nearly Partial Derivations on Banach Ternary Algebras," *Journal of Mathematics and Statistics*, vol. 6, no. 4, pp. 454–461, 2010.
- [35] M. Eshaghi Gordji, A. Bodaghi, and I. A. Alias, "On the stability of quadratic double centralizers and quadratic multipliers: a fixed point approach," *Journal of Inequalities and Applications*, vol. 2011, Article ID 957541, 12 pages, 2011.
- [36] M. Eshaghi Gordji, M. B. Ghaemi, S. Kaboli Gharetapeh, S. Shams, and A. Ebadian, "On the stability of J^* -derivations," *Journal of Geometry and Physics*, vol. 60, no. 3, pp. 454–459, 2010.
- [37] M. Eshaghi Gordji and H. Khodaei, *Stability of Functional Equations*, LAP- Lambert Academic Publishing, Saarbrücken, Germany, 2010.
- [38] M. Eshaghi Gordji and H. Khodaei, "The fixed point method for fuzzy approximation of a functional equation associated with inner product spaces," *Discrete Dynamics in Nature and Society*, Article ID 140767, 15 pages, 2010.
- [39] M. Eshaghi Gordji, H. Khodaei, and R. Khodabakhsh, "General quartic-cubic-quadratic functional equation in non-Archimedean normed spaces," *University "Politehnica" of Bucharest, Scientific Bulletin Series A*, vol. 72, no. 3, pp. 69–84, 2010.
- [40] M. Eshaghi Gordji and A. Najati, "Approximately J^* -homomorphisms: a fixed point approach," *Journal of Geometry and Physics*, vol. 60, no. 5, pp. 809–814, 2010.
- [41] R. Farokhzad and S. A. R. Hosseinioun, "Perturbations of Jordan higher derivations in Banach ternary algebras: an alternative fixed point approach," *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 1, pp. 42–53, 2010.
- [42] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory*, vol. 4, no. 1, pp. 91–96, 2003.
- [43] I. A. Rus, *Principles and Applications of Fixed Point Theory*, Editura Dacia, Cluj-Napoca, Romania, 1979.
- [44] L. Cădariu and V. Radu, "On the stability of the Cauchy functional equation: a fixed point approach," *Grazer Mathematische Berichte*, vol. 346, pp. 43–52, 2004.
- [45] B. Margolis and J. B. Diaz, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 126, pp. 305–309, 1968.