Research Article

An Inequality of Meromorphic Vector Functions and Its Application

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Firstly, an inequality for vector-valued meromorphic functions is established which extend a corresponding inequality of Milloux for meromorphic scalar-valued function (1946). As an application, the relationship between the characteristic function of a vector-valued meromorphic function f and its derivative f' is studied, results are obtained to extend some related results for meromorphic scalar-valued function of Weitsman (1969) and Singh and Gopalakrishna (1971).

1. Introduction of Vector-Valued Meromorphic Function

In 1980s, Ziegler [1] established Nevanlinna's theory for the vector-valued meromorphic function in finite dimensional spaces. After Ziegler some works related to vector-valued meromorphic function were done in 1990s [2–4]. In this section, we shall introduce the following fundamental notations and results of vector-valued Nevanlinna theory which were quoted from Ziegler [1].

We denote by $\mathbb{C}^n$ the usual n dimensional complex Euclidean space with the coordinates $w = (w_1, w_2, \dots, w_n)$, the Hermitian scalar product

$$\langle v, w \rangle = v_1 \bar{w}_1 + v_2 \bar{w}_2 + \dots + v_n \bar{w}_n, \quad (v, w \in \mathbb{C}^n), \quad (1.1)$$

and the distance

$$\|v - w\| = +\langle v - w, v - w \rangle^{1/2}. \quad (1.2)$$

Let

$$w_1 = f_1(z), \quad w_2 = f_2(z), \dots, w_n = f_n(z) \quad (1.3)$$

be $n \geq 1$ complex valued functions of the complex variable z , which are meromorphic and not all constant in the Gaussian plane $\mathbb{C}^1 = \mathbb{C}$, or in a finite disc

$$\mathbb{C}_R = \{|z| < R\} \subset \mathbb{C}, \quad 0 < R < +\infty. \quad (1.4)$$

Thus in $\mathbb{C}_R$, $0 < R \leq +\infty$ (we put $\mathbb{C}_{+\infty} = \mathbb{C}$), a vector-valued meromorphic function

$$f(z) = (f_1(z), f_2(z), \dots, f_n(z)) \quad (1.5)$$

is given, which does not reduce to the constant zero vector $0 = (0, 0, \dots, 0)$. The j th derivative $j = 1, 2, \dots$ of $f(z)$ are defined by

$$f^{(j)}(z) = (f_1^{(j)}(z), f_2^{(j)}(z), \dots, f_n^{(j)}(z)). \quad (1.6)$$

For such a function, the notations "zero," "pole," and "multiplicity" are defined as in the scalar case $n = 1$ of only one meromorphic function $f_1(z)$. More explicitly, in the punctured vicinity of each point $z_0 \in \mathbb{C}_R$, the vector function $w = f(z)$ can be developed into a Laurent series

$$f(z) = c_{k_0}(z - z_0)^{k_0} + c_{k_0+1}(z - z_0)^{k_0+1} + \dots, \quad (1.7)$$

where the coefficients are vectors

$$c_k = (c_k^1, c_k^2, \dots, c_k^n) \in \mathbb{C}^n, \quad c_{k_0} \neq (0, 0, \dots, 0). \quad (1.8)$$

In order to introduce the Nevanlinna theory of vector-valued meromorphic function, we will denote by " ∞ " the ideal element of the Aleksandrov one-point compactification of $\mathbb{C}^n$ (the two real infinities will be denoted by $+\infty$ and $-\infty$, resp.). Now, if $k_0 \leq 0$ in the above Laurent expansion, then z_0 will be called a pole or an ∞ -point of $f(z)$ of multiplicity $-k_0$; in such a point z_0 at least one of the meromorphic component functions $f_j(z)$ has a pole of this multiplicity in the ordinary sense of function theory, so that in z_0 itself $f(z)$ is not defined. If $k_0 > 0$ in Laurent expansion, then z_0 is called a zero of $f(z)$ of multiplicity k_0 ; in such a point z_0 , all component functions $f_j(z)$ vanish, each with at least this multiplicity.

Let $n(r, f)$ or $n(r, \infty)$ denote the number of poles of $f(z)$ in $|z| \leq r$ and $n(r, a)$ denote the number of a -points of $f(z)$ in $|z| \leq r$, counting with multiplicities. Define the volume function associated with vector-valued meromorphic function $f(z)$,

$$\begin{aligned} V(r, \infty) &= V(r, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi)\| dx \wedge dy, \quad \xi = x + iy \\ V(r, a) &= V\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi) - a\| dx \wedge dy, \quad \xi = x + iy \end{aligned} \quad (1.9)$$

and the counting function of finite or infinite a -points by

$$\begin{aligned} N(r, f) &= n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt, \\ N(r, \infty) &= n(0, \infty) \log r + \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} dt, \\ N(r, a) &= n(0, a) \log r + \int_0^r \frac{n(t, a) - n(0, a)}{t} dt, \end{aligned} \quad (1.10)$$

respectively. Next, we define

$$\begin{aligned} m(r, \infty) &= m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \|f(re^{i\theta})\| d\theta, \\ m(r, a) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\|f(re^{i\theta}) - a\|} d\theta, \\ T(r, f) &= m(r, f) + N(r, f). \end{aligned} \quad (1.11)$$

Let $\bar{n}(r, f)$ or $\bar{n}(r, \infty)$ denote the number of poles of $f(z)$ in $|z| \leq r$ and $\bar{n}(r, a)$ denote the number of a -points of $f(z)$ in $|z| \leq r$, ignoring multiplicities. Define the counting function of finite or infinite a -points by

$$\begin{aligned} \bar{N}(r, f) &= \bar{n}(0, f) \log r + \int_0^r \frac{\bar{n}(t, f) - \bar{n}(0, f)}{t} dt, \\ \bar{N}(r, \infty) &= \bar{n}(0, \infty) \log r + \int_0^r \frac{\bar{n}(t, \infty) - \bar{n}(0, \infty)}{t} dt, \\ \bar{N}(r, a) &= \bar{n}(0, a) \log r + \int_0^r \frac{\bar{n}(t, a) - \bar{n}(0, a)}{t} dt, \end{aligned} \quad (1.12)$$

respectively.

If $f(z)$ is a vector-valued meromorphic function in the whole complex plane, then the order and the lower order of $f(z)$ are defined by

$$\begin{aligned}\lambda(f) &= \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}, \\ \mu(f) &= \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.\end{aligned}\tag{1.13}$$

We call the vector-valued meromorphic function f admissible if

$$\limsup_{r \rightarrow +\infty} \frac{T(r, f)}{\log r} = +\infty.\tag{1.14}$$

Definition 1.1. For a meromorphic function $f(z)$ (vector-valued or scalar-valued), we denote by $S(r, f)$ any quantity such that

$$S(r, f) = o(T(r, f)), \quad r \longrightarrow +\infty\tag{1.15}$$

without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of r of finite linear measure.

Definition 1.1 quoted from [2]. In [1], Ziegler established the following first main theorem, logarithmic derivative lemma, and deficient values theorem for meromorphic vector function.

Theorem A. Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be a meromorphic vector function in $\mathbb{C}_R$. Then for $0 < r < R \leq +\infty$, $a \in \mathbb{C}^n$, $f(z) \neq a$, then

$$T(r, f) = V(r, a) + N(r, a) + m(r, a) + O(1).\tag{1.16}$$

Theorem B. Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be a nonconstant meromorphic vector function in $\mathbb{C}$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{\|f'(re^{i\theta})\|}{\|f(re^{i\theta}) - a\|} d\theta = S(r, f), \quad a \in \mathbb{C}^n.\tag{1.17}$$

By the second main theorem, Ziegler [1] studies the following deficiency theorem for meromorphic vector function. For any vector $a \in \mathbb{C}^n$, we define the number $\delta(a) = \delta(a, f)$ by putting

$$\begin{aligned}\delta(a) &= \delta(a, f) = \liminf_{r \rightarrow +\infty} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r \rightarrow +\infty} \frac{V(r, a) + N(r, a)}{T(r, f)}, \\ \delta(\infty) &= \delta(\infty, f) = \liminf_{r \rightarrow +\infty} \frac{m(r, f)}{T(r, f)} = 1 - \limsup_{r \rightarrow +\infty} \frac{N(r, f)}{T(r, f)},\end{aligned}\tag{1.18}$$

and $\Theta(a) = \Theta(a, f)$ by putting

$$\begin{aligned}\Theta(a) &= \Theta(a, f) = 1 - \limsup_{r \rightarrow +\infty} \frac{V(r, a) + \overline{N}(r, a)}{T(r, f)}, \\ \Theta(\infty) &= \Theta(\infty, f) = 1 - \limsup_{r \rightarrow +\infty} \frac{\overline{N}(r, f)}{T(r, f)},\end{aligned}\tag{1.19}$$

Theorem C. Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be an admissible meromorphic vector function in $\mathbb{C}$. Then the set $\{a \in \mathbb{C}^n \cup \{\infty\}, \delta(a) > 0\}$ is at most countable and summing over all such points we have

$$\sum_a \delta(a) \leq \sum_a \Theta(a) \leq 2.\tag{1.20}$$

2. A Fundamental Inequality of Meromorphic Vector Function

For meromorphic scalar-valued function $f(z)$, Milloux [5] has proved the following theorem.

Theorem D. If $f(z)$ is a nonconstant meromorphic scalar-valued function in Gaussian complex plane $\mathbb{C}$ and if a_i , $i = 1, 2, \dots, q$, are distinct elements of $\mathbb{C}$ (where q is any positive integer), then

$$qT(r, f) \leq T(r, f') + \sum_{i=1}^q \overline{N}(r, a_i) + S(r, f).\tag{2.1}$$

For some alternative proofs of Theorem D, see [6] or [7]. It is natural to consider whether there exists a similar results if meromorphic scalar-valued function $f(z)$ is replaced by meromorphic vector-valued function $f(z)$. In this section, the main contribution is to extend Theorem D to vector-valued meromorphic function by referring the method of [1, 7].

Theorem 2.1. Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be an admissible meromorphic vector function in $\mathbb{C}$ and if $a^{[j]}$, $j = 1, 2, \dots, q$, are distinct elements of $\mathbb{C}^n$ (where q is any positive integer), then

$$qT(r, f) \leq T(r, f') + \sum_{j=1}^q \left(\overline{N}(r, a^{[j]}) + V(r, a^{[j]}) \right) + S(r, f).\tag{2.2}$$

Proof. Put

$$F(z) = \sum_{j=1}^q \frac{1}{\|f(z) - a^{[j]}\|}.\tag{2.3}$$

We can get

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ F(re^{i\theta}) d\theta \leq m(r, 0, f') + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\{ F(re^{i\theta}) \|f'(re^{i\theta})\| \right\} d\theta.\tag{2.4}$$

Put

$$\delta = \min_{i \neq j} \|a^{[i]} - a^{[j]}\|. \quad (2.5)$$

Let for the moment $\mu \in \{1, 2, \dots, q\}$ be fixed. Then we get in every point where

$$\|f(z) - a^{[\mu]}\| < \frac{\delta}{2q} \leq \frac{\delta}{4}, \quad (2.6)$$

the inequality

$$\|f(z) - a^{[v]}\| \geq \|a^{[\mu]} - a^{[v]}\| - \|f(z) - a^{[\mu]}\| \geq \frac{3\delta}{4}, \quad (2.7)$$

for $\mu \neq v$. Therefore, the set of points on $\partial\mathbb{C}_r$ which is determined by (2.6) is either empty or any two such sets for different μ have empty intersection. In any case,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log^+ F(re^{i\theta}) d\theta &\geq \frac{1}{2\pi} \sum_{\mu=1}^q \int_{\|f(z)-a^{[\mu]}\| < \delta/2q, |z|=r} \log^+ F(re^{i\theta}) d\theta \\ &\geq \frac{1}{2\pi} \sum_{\mu=1}^q \int_{\|f(z)-a^{[\mu]}\| < \delta/2q, |z|=r} \log^+ \frac{1}{\|f(re^{i\theta}) - a^{[\mu]}\|} d\theta. \end{aligned} \quad (2.8)$$

Because of

$$\begin{aligned} &\frac{1}{2\pi} \int_{\|f(z)-a^{[\mu]}\| < \delta/2q, |z|=r} \log^+ \frac{1}{\|f(re^{i\theta}) - a^{[\mu]}\|} d\theta \\ &= m(r, a^{[\mu]}) - \frac{1}{2\pi} \int_{\|f(z)-a^{[\mu]}\| \geq \delta/2q, |z|=r} \log^+ \frac{1}{\|f(re^{i\theta}) - a^{[\mu]}\|} d\theta \\ &\geq m(r, a^{[\mu]}) - \log^+ \frac{2q}{\delta}, \end{aligned} \quad (2.9)$$

it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ F(re^{i\theta}) d\theta \geq \sum_{\mu=1}^q m(r, a^{[\mu]}) - \log^+ \frac{2q}{\delta}, \quad (2.10)$$

so that by (2.4)

$$\sum_{\mu=1}^q m(r, a^{[\mu]}) \leq m(r, 0, f') + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \{F(re^{i\theta}) \|f'(re^{i\theta})\|\} d\theta + \log^+ \frac{2q}{\delta}. \quad (2.11)$$

Thus by Theorem B, we have

$$\sum_{\mu=1}^q m(r, a^{[\mu]}) \leq m(r, 0, f') + S(r, f). \quad (2.12)$$

It follows from Theorem A that

$$m(r, 0, f') + N(r, 0, f') + V(r, 0, f') = T(r, f') + O(1). \quad (2.13)$$

Thus from (2.12) and (2.13), we deduce

$$\sum_{\mu=1}^q m(r, a^{[\mu]}) \leq T(r, f') - N(r, 0, f') + S(r, f). \quad (2.14)$$

Adding $\sum_{\mu=1}^q N(r, a^{[\mu]})$ to both sides,

$$\begin{aligned} \sum_{\mu=1}^q T\left(r, \frac{1}{f - a^{[\mu]}}\right) &\leq T(r, f') + \sum_{\mu=1}^q N(r, a^{[\mu]}) - N(r, 0, f') + S(r, f) \\ &= T(r, f') + \sum_{\mu=1}^q \overline{N}(r, a^{[\mu]}) - N_0(r, 0, f') + S(r, f), \end{aligned} \quad (2.15)$$

where $N_0(r, 0, f')$ is formed with the zeros of f' which are not zeros of any of $f - a^{[i]}$, ($i = 1, 2, \dots, q$). Since $N_0(r, 0, f') \geq 0$, we have

$$\sum_{\mu=1}^q T\left(r, \frac{1}{f - a^{[\mu]}}\right) \leq T(r, f') + \sum_{\mu=1}^q \overline{N}(r, a^{[\mu]}) + S(r, f). \quad (2.16)$$

Since

$$T\left(r, \frac{1}{f - a^{[\mu]}}\right) + V(r, a^{[\mu]}) = T(r, f) + O(1), \quad (2.17)$$

it follows that

$$qT(r, f) \leq T(r, f') + \sum_{j=1}^q \left(\overline{N}(r, a^{[j]}) + V(r, a^{[j]}) \right) + S(r, f). \quad (2.18)$$

□

3. Characteristic Function of Derivative of Meromorphic Vector Function

Let $f(z)$ be a meromorphic scalar-valued function in $\mathbb{C}$. The characteristic function of derivative of $f(z)$ with $\sum_a \delta(a) = 2$ has been studied by Edrei [8], Shan and Singh [9], Singh and Gopalakrishna [7], Singh and Kulkarni [10] and Weitsman [11]. For example, Edrei [8] and Weitsman [11] have proved the following theorem.

Theorem E. *Let $f(z)$ be a transcendental meromorphic scalar-valued function of finite order and assume $\sum_{a \in \mathbb{C}} \delta(a) = \eta \geq 1$ and $\delta(\infty) = 2 - \eta$. Then*

$$T(r, f') \sim \eta T(r, f), \quad r \rightarrow +\infty. \quad (3.1)$$

If $\sum_a \delta(a) = 2$ is replaced by $\sum_a \Theta(a) = 2$, Singh and Gopalakrishna [7] and Singh and Kulkarni [10] have proved the following theorem.

Theorem F. *Let $f(z)$ be a transcendental meromorphic scalar-valued function of finite order and assume $\sum_a \Theta(a) = 2$. Then*

$$\begin{aligned} \lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} &= 2 - \Theta(\infty), \\ \lim_{r \rightarrow +\infty} \frac{\overline{N}(r, a)}{T(r, f)} &= 1 - \Theta(a) \end{aligned} \quad (3.2)$$

for every $a \in \mathbb{C} \cup \{\infty\}$.

It is natural to consider whether there exists a similar results if meromorphic scalar-valued function $f(z)$ is replaced by meromorphic vector-valued function $f(z)$. In this section, the main purpose is to extend the above theorems to vector-valued meromorphic function by referring the method of [1, 7].

Theorem 3.1. *Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_a \Theta(a) = 2$. Then*

$$\begin{aligned} \lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} &= 2 - \Theta(\infty), \\ \lim_{r \rightarrow +\infty} \frac{\overline{N}(r, f)}{T(r, f)} &= 1 - \Theta(\infty), \quad \lim_{r \rightarrow +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a) \end{aligned} \quad (3.3)$$

for every $a \in \mathbb{C}^n$.

Proof. Now, basic estimates in vector-valued Nevanlinna theory [1] or [4] yields

$$\begin{aligned}
 T(r, f') &= m(r, f') + N(r, f') \\
 &= m\left(r, \frac{ff'}{f}\right) + N(r, f') \\
 &\leq m\left(r, \frac{f'}{f}\right) + m(r, f) + N(r, f) + \overline{N}(r, f) \\
 &\leq T(r, f) + \overline{N}(r, f) + m\left(r, \frac{f'}{f}\right).
 \end{aligned} \tag{3.4}$$

By Theorem B and the above inequality, we have

$$\limsup_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} \leq 2 - \Theta(\infty). \tag{3.5}$$

Let $\{a^{[j]}\}$ be a sequence of distinct vector in $\mathbb{C}^n$ containing all the vector of $\delta(a^{[j]}) > 0$. From Theorem 2.1, for any positive integer q , we have

$$qT(r, f) \leq T(r, f') + \sum_{j=1}^q \left(\overline{N}(r, a^{[j]}) + V(r, a^{[j]}) \right) + S(r, f). \tag{3.6}$$

Hence

$$\begin{aligned}
 q &\leq \liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} + \sum_{j=1}^q \limsup_{r \rightarrow +\infty} \frac{\overline{N}(r, a^{[j]}) + V(r, a^{[j]})}{T(r, f)} + \limsup_{r \rightarrow +\infty} \frac{S(r, f)}{T(r, f)} \\
 &= \liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} + \sum_{j=1}^q \left\{ 1 - \Theta(a^{[j]}) \right\} + \limsup_{r \rightarrow +\infty} \frac{S(r, f)}{T(r, f)}.
 \end{aligned} \tag{3.7}$$

Thus

$$\liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} \geq \sum_{j=1}^q \Theta(a^{[j]}). \tag{3.8}$$

Since q was arbitrary, we have

$$2 - \Theta(\infty) = \sum_{a \in \mathbb{C}^n} \Theta(a) \leq \liminf_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)}. \tag{3.9}$$

This and (3.5) yield

$$\lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \Theta(\infty). \quad (3.10)$$

Let $a \in \mathbb{C}^n \cup \{\infty\}$ and $\{a^{[i]}\}_{i=1}^{+\infty}$ an infinite sequence of distinct elements of $\mathbb{C}^n \cup \{\infty\}$ which includes every $b \in \mathbb{C}^n \cup \{\infty\}$ satisfying $b \neq a$ and $\Theta(b) > 0$. Then

$$\sum_{i=1}^{+\infty} \Theta(a^{[i]}) = \sum_{b \in \mathbb{C}^n \cup \{\infty\}, b \neq a} \Theta(b) = 2 - \Theta(a). \quad (3.11)$$

Let q be any integer ≥ 3 . From Generalized Second Main Theorem (see [1], Page 126), we have

$$(q-2)T(r, f) = \sum_{i=1}^{q-1} \left(\overline{N}(r, a^{[i]}) + V(r, a^{[i]}) \right) + \overline{N}(r, f) + S(r, f). \quad (3.12)$$

Hence

$$q-2 \leq \sum_{i=1}^{q-1} \left\{ 1 - \Theta(a^{[i]}) \right\} + \liminf_{r \rightarrow +\infty} \frac{\overline{N}(r, f)}{T(r, f)}. \quad (3.13)$$

Thus

$$\sum_{i=1}^{q-1} \Theta(a^{[i]}) - 1 \leq \liminf_{r \rightarrow +\infty} \frac{\overline{N}(r, f)}{T(r, f)}. \quad (3.14)$$

Since this holds for all $q \geq 3$, letting $q \rightarrow +\infty$ and combining (3.11), we get

$$1 - \Theta(\infty) = \sum_{i=1}^{+\infty} \Theta(a^{[i]}) - 1 \leq \liminf_{r \rightarrow +\infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow +\infty} \frac{\overline{N}(r, f)}{T(r, f)} = 1 - \Theta(\infty). \quad (3.15)$$

So

$$\lim_{r \rightarrow +\infty} \frac{\overline{N}(r, f)}{T(r, f)} = 1 - \Theta(\infty). \quad (3.16)$$

For every $a \in \mathbb{C}^n$, Let q be any integer ≥ 3 . From Generalized Second Main Theorem (see [1], Page 126), we have

$$(q-2)T(r, f) = \sum_{i=1}^{q-2} \left(\overline{N}(r, a^{[i]}) + V(r, a^{[i]}) \right) + \left(\overline{N}(r, a) + V(r, a) \right) + \overline{N}(r, f) + S(r, f). \quad (3.17)$$

Hence

$$q - 2 \leq \sum_{i=1}^{q-2} \left\{ 1 - \Theta(a^{[i]}) \right\} + (1 - \Theta(\infty)) + \liminf_{r \rightarrow +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)}. \quad (3.18)$$

Thus

$$\sum_{i=1}^{q-2} \Theta(a^{[i]}) + \Theta(\infty) - 1 \leq \liminf_{r \rightarrow +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)}. \quad (3.19)$$

Since this holds for all $q \geq 3$, letting $q \rightarrow +\infty$ and combining (3.11), we get

$$\begin{aligned} 1 - \Theta(a) &= \sum_{i=1}^{+\infty} \Theta(a^{[i]}) - 1 \leq \liminf_{r \rightarrow +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a). \end{aligned} \quad (3.20)$$

So

$$\lim_{r \rightarrow +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a). \quad (3.21)$$

□

From Theorem 3.1, we have the following corollary

Corollary 3.2. *Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_{a \in \mathbb{C}^n} \Theta(a) = 2$. Then*

$$T(r, f') \sim 2T(r, f), \quad r \rightarrow +\infty. \quad (3.22)$$

Corollary 3.3. *Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_{a \in \mathbb{C}^n} \Theta(a) = \eta \geq 1$ and $\delta(\infty) = 2 - \eta$. Then*

$$T(r, f') \sim \eta T(r, f), \quad r \rightarrow +\infty. \quad (3.23)$$

Corollary 3.4. *Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_a \delta(a) = 2$. Then*

$$\begin{aligned} \lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} &= 2 - \delta(\infty), \\ \lim_{r \rightarrow +\infty} \frac{N(r, f)}{T(r, f)} &= 1 - \delta(\infty), \quad \lim_{r \rightarrow +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} = 1 - \delta(a) \end{aligned} \quad (3.24)$$

for every $a \in \mathbb{C}^n$.

Proof. Since $\delta(a) \leq \Theta(a)$ for every $a \in \mathbb{C}^n \cup \{\infty\}$ and Theorem C, it follows that, if $\sum_a \delta(a) = 2$, then $\sum_a \Theta(a) = 2$ and $\delta(a) = \Theta(a)$ for every $a \in \mathbb{C}^n \cup \{\infty\}$. Hence

$$\lim_{r \rightarrow +\infty} \frac{T(r, f')}{T(r, f)} = 2 - \delta(\infty) \quad (3.25)$$

follows by Theorem 3.1.

Now, for every $a \in \mathbb{C}^n$,

$$\lim_{r \rightarrow +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} = 1 - \Theta(a) = 1 - \delta(a). \quad (3.26)$$

Further

$$\overline{N}(r, a) \leq N(r, a). \quad (3.27)$$

Hence

$$\begin{aligned} 1 - \delta(a) &\leq \lim_{r \rightarrow +\infty} \frac{\overline{N}(r, a) + V(r, a)}{T(r, f)} \\ &\leq \liminf_{r \rightarrow +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} \\ &\leq \limsup_{r \rightarrow +\infty} \frac{N(r, a) + V(r, a)}{T(r, f)} \\ &= 1 - \delta(a). \end{aligned} \quad (3.28)$$

Similarly,

$$\lim_{r \rightarrow +\infty} \frac{\overline{N}(r, f)}{T(r, f)} = 1 - \Theta(\infty) = 1 - \delta(\infty). \quad (3.29)$$

Further

$$\overline{N}(r, \infty) \leq N(r, \infty). \quad (3.30)$$

Hence

$$1 - \delta(\infty) \leq \lim_{r \rightarrow +\infty} \frac{\overline{N}(r, \infty)}{T(r, f)} \leq \liminf_{r \rightarrow +\infty} \frac{N(r, \infty)}{T(r, f)} \leq \limsup_{r \rightarrow +\infty} \frac{N(r, \infty)}{T(r, f)} = 1 - \delta(\infty). \quad (3.31)$$

□

From Corollary 3.4, we have the following corollary.

Corollary 3.5. Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_{a \in \mathbb{C}^n} \delta(a) = 2$. Then

$$T(r, f') \sim 2T(r, f), \quad r \longrightarrow +\infty. \quad (3.32)$$

Corollary 3.6. Let $f(z) = (f_1(z), f_2(z), \dots, f_n(z))$ be an admissible meromorphic vector function of finite order in $\mathbb{C}$ and assume $\sum_{a \in \mathbb{C}^n} \delta(a) = \eta \geq 1$ and $\delta(\infty) = 2 - \eta$. Then

$$T(r, f') \sim \eta T(r, f), \quad r \longrightarrow +\infty. \quad (3.33)$$

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