Research Article

Fuzzy Stability of a Functional Equation Deriving from Quadratic and Additive Mappings

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We investigate a fuzzy version of stability for the functional equation $f(2x+y) + f(2x-y) + 2f(x) - f(x+y) - f(x-y) - 2f(2x) = 0$ in the sense of Mirmostafae and Moslehian.

1. Introduction and Preliminaries

A classical question in the theory of functional equations is “when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?”. Such a problem, called a *stability problem of the functional equation*, was formulated by Ulam [1] in 1940. In the next year, Hyers [2] gave a partial solution of Ulam’s problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki [3] for additive mappings, and by Rassias [4] for linear mappings, to considering the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [5–15].

In 1984, Katsaras [16] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, some mathematicians have introduced several types of fuzzy norm in different points of view. In particular, Bag and Samanta [17], following Cheng and Mordeson [18], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [19]. In 2008, Mirmostafae and Moslehian [20] obtained a fuzzy version of stability for the *Cauchy functional equation*

$$f(x+y) - f(x) - f(y) = 0. \quad (1.1)$$

In the same year, they [21] proved a fuzzy version of stability for *the quadratic functional equation*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0. \quad (1.2)$$

We call a solution of (1.1) *an additive mapping* and a solution of (1.2) is called a *quadratic mapping*. Now, we consider the functional equation

$$f(2x+y) + f(2x-y) + 2f(x) - f(x+y) - f(x-y) - 2f(2x) = 0, \quad (1.3)$$

which is called a *functional equation deriving from quadratic and additive mappings*. We call a solution of (1.3) a *general quadratic mapping*. In 2008, Najati and Moghimi [22] obtained a stability of the functional equation (1.3) by taking and composing an additive mapping A and a quadratic mapping Q to prove the existence of a general quadratic mapping F which is close to the given mapping f . In their processing, A is approximate to the odd part $(f(x) - f(-x))/2$ of f , and Q is close to the even part $(f(x) + f(-x))/2 - f(0)$ of it, respectively.

In this paper, we get a general stability result of the functional equation deriving from quadratic and additive mappings (1.3) in the fuzzy normed linear space. To do it, we introduce a Cauchy sequence $\{J_n f(x)\}$, starting from a given mapping f , which converges to the desired mapping F in the fuzzy sense. As we mentioned before, in previous studies of stability problem of (1.3), they attempted to get stability theorems by handling the odd and even part of f , respectively. According to our proposal in this paper, we can take the desired approximate solution F at once. Therefore, this idea is a refinement with respect to the simplicity of the proof.

2. Fuzzy Stability of the Functional Equation (1.3)

We use the definition of a fuzzy normed space given in [17] to exhibit a reasonable fuzzy version of stability for the functional equation deriving from quadratic and additive mappings in the fuzzy normed linear space.

Definition 2.1 (see [17]). Let X be a real linear space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ (the so-called fuzzy subset) is said to be a *fuzzy norm on X* if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

- (N1) $N(x, c) = 0$ for $c \leq 0$,
- (N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$,
- (N3) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$,
- (N4) $N(x+y, s+t) \geq \min\{N(x, s), N(y, t)\}$,
- (N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t \rightarrow \infty} N(x, t) = 1$.

The pair (X, N) is called a *fuzzy normed linear space*. Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X . Then, $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence $\{x_n\}$* , and we denote it by $N - \lim_{n \rightarrow \infty} x_n = x$. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$, there exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is known that every convergent sequence in a fuzzy normed space

is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete*, and the fuzzy normed space is called a *fuzzy Banach space*.

Let (X, N) be a fuzzy normed space and (Y, N') a fuzzy Banach space. For a given mapping $f : X \rightarrow Y$, we use the abbreviation

$$Df(x, y) := f(2x + y) + f(2x - y) + 2f(x) - f(x + y) - f(x - y) - 2f(2x), \quad (2.1)$$

for all $x, y \in X$. Recall $Df \equiv 0$ means that f is a general quadratic mapping. For given $q > 0$, the mapping f is called a *fuzzy q -almost general quadratic mapping* if

$$N'(Df(x, y), t + s) \geq \min\{N(x, s^q), N(y, t^q)\}, \quad (2.2)$$

for all $x, y \in X \setminus \{0\}$ and all $s, t \in [0, \infty)$. Now, we get the general stability result in the fuzzy normed linear setting.

Theorem 2.2. *Let q be a positive real number with $q \neq 1/2, 1$. And let f be a fuzzy q -almost general quadratic mapping from a fuzzy normed space (X, N) into a fuzzy Banach space (Y, N') . Then, there is a unique general quadratic mapping $F : X \rightarrow Y$ such that*

$$N'(F(x) - f(x), t) \geq \sup_{0 < t' < t} N\left(x, \frac{t'^q}{((7+2^p+3^p+4^p)/(4-2^p|3^p|) + (5+2 \cdot 2^p+3^p)/(2|2-2^p|))^q}\right), \quad (2.3)$$

for each $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. We will prove the theorem in three cases, $q > 1$, $1/2 < q < 1$, and $0 < q < 1/2$.

Case 1. Let $q > 1$. We define a mapping $J_n f : X \rightarrow Y$ by

$$J_n f(x) = \frac{1}{2} (4^{-n} (f(2^n x) + f(-2^n x) - 2f(0)) + 2^{-n} (f(2^n x) - f(-2^n x))) + f(0), \quad (2.4)$$

for all $x \in X$. Then, $J_0 f(x) = f(x)$, $J_j f(0) = f(0)$, and

$$\begin{aligned} J_j f(x) - J_{j+1} f(x) &= \frac{Df(2^j x/3, 2^j x/3)}{4^{j+1}} - \frac{Df(2^j x/3, 2^{j+1} x/3)}{2 \cdot 4^{j+1}} - \frac{Df(2^j x/3, 2^j x)}{2 \cdot 4^{j+1}} \\ &\quad - \frac{Df(2^j x/3, 2^{j+2} x/3)}{2 \cdot 4^{j+1}} + \frac{Df(-2^j x/3, -2^j x/3)}{4^{j+1}} - \frac{Df(-2^j x/3, -2^{j+1} x/3)}{2 \cdot 4^{j+1}} \end{aligned}$$

$$\begin{aligned}
& - \frac{Df(-2^j x/3, -2^j x)}{2 \cdot 4^{j+1}} - \frac{Df(-2^j x/3, -2^{j+2} x/3)}{2 \cdot 4^{j+1}} + \frac{Df(2^{j+1} x, 2^j x)}{2^{j+2}} \\
& - \frac{Df(2^j x, 3 \cdot 2^j x)}{2^{j+2}} + \frac{Df(2^j x, 2^j x)}{2^{j+2}} + \frac{Df(2^j x, -2^{j+1} x)}{2^{j+2}},
\end{aligned} \tag{2.5}$$

for all $x \in X \setminus \{0\}$ and $j \geq 0$. Together with (N3), (N4), and (2.2), this equation implies that if $n + m > m \geq 0$, then

$$\begin{aligned}
& N' \left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{7 + 2^p + 3^p + 4^p}{4 \cdot 3^p} \left(\frac{2^p}{4} \right)^j + \frac{5 + 2 \cdot 2^p + 3^p}{4} \left(\frac{2^p}{2} \right)^j \right) t^p \right) \\
& \geq \min \bigcup_{j=m}^{n+m-1} \left\{ N' \left(J_j f(x) - J_{j+1} f(x), \left(\frac{7 + 2^p + 3^p + 4^p}{4^{j+1} \cdot 3^p} + \frac{5 + 2 \cdot 2^p + 3^p}{2^{j+2}} \right) 2^{jp} t^p \right) \right\} \\
& \geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N' \left(\frac{Df(2^j x/3, 2^j x/3)}{4^{j+1}}, \frac{2^{jp} t^p}{2 \cdot 4^j \cdot 3^p} \right), \right. \right. \\
& \quad N' \left(-\frac{Df(2^j x/3, 2^{j+1} x/3)}{2 \cdot 4^{j+1}}, \frac{2^{jp}(1 + 2^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& \quad N' \left(-\frac{Df(2^j x/3, 2^j x)}{2 \cdot 4^{j+1}}, \frac{2^{jp}(1 + 3^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& \quad N' \left(-\frac{Df(2^j x/3, 2^{j+2} x/3)}{2 \cdot 4^{j+1}}, \frac{2^{jp}(1 + 4^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& \quad N' \left(\frac{Df(-2^j x/3, -2^j x/3)}{4^{j+1}}, \frac{2^{jp} t^p}{2 \cdot 4^j \cdot 3^p} \right), \\
& \quad N' \left(-\frac{Df(-2^j x/3, -2^{j+1} x/3)}{2 \cdot 4^{j+1}}, \frac{2^{jp}(1 + 2^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& \quad N' \left(-\frac{Df(-2^j x/3, -2^j x)}{2 \cdot 4^{j+1}}, \frac{2^{jp}(1 + 3^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& \quad N' \left(-\frac{Df(-2^j x/3, -2^{j+2} x/3)}{2 \cdot 4^{j+1}}, \frac{2^{jp}(1 + 4^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& \quad N' \left(\frac{Df(2^{j+1} x, 2^j x)}{2^{j+2}}, \frac{2^{jp}(1 + 2^p) t^p}{2^{j+2}} \right), \\
& \quad N' \left(-\frac{Df(2^j x, 3 \cdot 2^j x)}{2^{j+2}}, \frac{2^{jp}(1 + 3^p) t^p}{2^{j+2}} \right), \\
& \quad N' \left(\frac{Df(2^j x, 2^j x)}{2^{j+2}}, \frac{2^{jp} t^p}{2^{j+1}} \right), \\
& \quad \left. N' \left(\frac{Df(2^j x, -2^{j+1} x)}{2^{j+2}}, \frac{2^{jp}(1 + 2^p) t^p}{2^{j+2}} \right) \right\} \bigg\}
\end{aligned}$$

$$\begin{aligned}
&\geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N(2^j x, 2^j t), N(2^{j+1} x, 2^{j+1} t), N(3 \cdot 2^j x, 3 \cdot 2^j t), \right. \right. \\
&\quad \left. \left. N\left(\frac{2^j x}{3}, \frac{2^j t}{3}\right), N\left(\frac{2^{j+1} x}{3}, \frac{2^{j+1} t}{3}\right), N\left(\frac{2^{j+2} x}{3}, \frac{2^{j+2} t}{3}\right) \right\} \right\} \\
&= N(x, t),
\end{aligned} \tag{2.6}$$

for all $x \in X \setminus \{0\}$ and $t > 0$. Let $\varepsilon > 0$ be given. Since $\lim_{t \rightarrow \infty} N(x, t) = 1$, there is $t_0 > 0$ such that

$$N(x, t_0) \geq 1 - \varepsilon. \tag{2.7}$$

We observe that for some $\tilde{t} > t_0$, the series $\sum_{j=0}^{\infty} ((7 + 2^p + 3^p + 4^p)/(4^{j+1} \cdot 3^p) + (5 + 2 \cdot 2^p + 3^p)/(2^{j+2})) 2^{jp} \tilde{t}^p$ converges for $p = 1/q < 1$. It guarantees that for an arbitrary given $c > 0$, there exists $n_0 \geq 0$ such that

$$\sum_{j=m}^{n+m-1} \left(\frac{7 + 2^p + 3^p + 4^p}{4^{j+1} \cdot 3^p} + \frac{5 + 2 \cdot 2^p + 3^p}{2^{j+2}} \right) 2^{jp} \tilde{t}^p < c, \tag{2.8}$$

for each $m \geq n_0$ and $n > 0$. By (N5) and (2.6) we have

$$\begin{aligned}
&N'(J_m f(x) - J_{n+m} f(x), c) \\
&\geq N' \left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{7 + 2^p + 3^p + 4^p}{4^{j+1} \cdot 3^p} + \frac{5 + 2 \cdot 2^p + 3^p}{2^{j+2}} \right) 2^{jp} \tilde{t}^p \right) \\
&\geq N(x, \tilde{t}) \geq N(x, t_0) \geq 1 - \varepsilon,
\end{aligned} \tag{2.9}$$

for all $x \in X \setminus \{0\}$. Recall $J_n f(0) = f(0)$ for all $n > 0$. Thus, $\{J_n f(x)\}$ becomes a Cauchy sequence for all $x \in X$. Since (Y, N') is complete, we can define a mapping $F : X \rightarrow Y$ by

$$F(x) := N' - \lim_{n \rightarrow \infty} J_n f(x), \tag{2.10}$$

for all $x \in X$. Moreover, if we put $m = 0$ in (2.6), we have

$$\begin{aligned}
&N'(f(x) - J_n f(x), t) \\
&\geq N \left(x, \frac{t^q}{\left(\sum_{j=0}^{n-1} ((7 + 2^p + 3^p + 4^p)/(4^{j+1} \cdot 3^p) + (5 + 2 \cdot 2^p + 3^p)/2^{j+2}) 2^{jp} \right)^q} \right),
\end{aligned} \tag{2.11}$$

for all $x \in X$. Next, we will show that F is a general quadratic mapping. Using (N4), we have

$$\begin{aligned}
& N'(DF(x, y), t) \\
& \geq \min \left\{ N' \left(F(2x + y) - J_n f(2x + y), \frac{t}{16} \right), \right. \\
& \quad N' \left(-F(x + y) + J_n f(x + y), \frac{t}{16} \right), N' \left(-F(x - y) + J_n f(x - y), \frac{t}{16} \right), \quad (2.12) \\
& \quad N' \left(2F(x) - 2J_n f(x), \frac{t}{8} \right), N' \left(-2F(2x) + 2J_n f(2x), \frac{t}{8} \right), \\
& \quad \left. N' \left(F(2x - y) + J_n f(2x - y), \frac{t}{16} \right), N' \left(DJ_n f(x, y), \frac{t}{2} \right) \right\},
\end{aligned}$$

for all $x, y \in X \setminus \{0\}$ and $n \in \mathbb{N}$. The first six terms on the right hand side of (2.12) tend to 1 as $n \rightarrow \infty$ by the definition of F and (N2), and the last term holds

$$\begin{aligned}
N' \left(DJ_n f(x, y), \frac{t}{2} \right) & \geq \min \left\{ N' \left(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right), N' \left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right), \right. \\
& \quad \left. N' \left(\frac{Df(2^n x, 2^n y)}{2 \cdot 2^n}, \frac{t}{8} \right), N' \left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 2^n}, \frac{t}{8} \right) \right\}, \quad (2.13)
\end{aligned}$$

for all $x, y \in X \setminus \{0\}$. By (N3) and (2.2), we obtain

$$\begin{aligned}
N' \left(\frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 4^n}, \frac{t}{8} \right) & = N' \left(Df(\pm 2^n x, \pm 2^n y), \frac{2 \cdot 4^n t}{8} \right) \\
& \geq \min \left\{ N \left(\pm 2^n x, \left(\frac{4^n t}{8} \right)^q \right), N \left(\pm 2^n y, \left(\frac{4^n t}{8} \right)^q \right) \right\} \\
& \geq \min \left\{ N \left(x, 2^{(2q-1)n-3q} t^q \right), N \left(y, 2^{(2q-1)n-3q} t^q \right) \right\}, \quad (2.14) \\
N' \left(\frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 2^n}, \frac{t}{8} \right) & \geq \min \left\{ N \left(x, 2^{(q-1)n-3q} t^q \right), N \left(y, 2^{(q-1)n-3q} t^q \right) \right\},
\end{aligned}$$

for all $x, y \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Since $q > 1$, together with (N5), we can deduce that the last term of (2.12) also tends to 1 as $n \rightarrow \infty$. It follows from (2.12) that

$$N'(DF(x, y), t) = 1, \quad (2.15)$$

for all $x, y \in X \setminus \{0\}$ and $t > 0$. Since $DF(0, 0) = 0$, $DF(x, 0) = 0$ and $DF(0, y) = 0$ for all $x, y \in X \setminus \{0\}$, this means that $DF(x, y) = 0$ for all $x, y \in X$ by (N2).

Now, we approximate the difference between f and F in a fuzzy sense. For an arbitrary fixed $x \in X$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since F is the limit of $\{J_n f(x)\}$, there is $n \in \mathbb{N}$ such that $N'(F(x) - J_n f(x), t - t') \geq 1 - \varepsilon$. By (2.11), we have

$$\begin{aligned}
 & N'(F(x) - f(x), t) \\
 & \geq \min\{N'(F(x) - J_n f(x), t - t'), N'(J_n f(x) - f(x), t')\} \\
 & \geq \min\left\{1 - \varepsilon, N\left(x, \frac{t'^q}{\left(\sum_{j=0}^{n-1} ((7+2^p+3^p+4^p)/(4^{j+1} \cdot 3^p) + (5+2 \cdot 2^p+3^p)/2^{j+2}) 2^{jp}\right)^q}\right)\right\} \\
 & \geq \min\left\{1 - \varepsilon, N\left(x, \frac{t'^q}{((7+2^p+3^p+4^p)/(4-2^p)3^p + (5+2 \cdot 2^p+3^p)/2(2-2^p))^q}\right)\right\}.
 \end{aligned} \tag{2.16}$$

Because $0 < \varepsilon < 1$ is arbitrary and $F(0) = f(0)$, we get (2.3) in this case.

Finally, to prove the uniqueness of F , let $F' : X \rightarrow Y$ be another general quadratic mapping satisfying (2.3). Then, by (2.5), we get

$$\begin{aligned}
 F(x) - J_n F(x) &= \sum_{j=0}^{n-1} (J_j F(x) - J_{j+1} F(x)) = 0, \\
 F'(x) - J_n F'(x) &= \sum_{j=0}^{n-1} (J_j F'(x) - J_{j+1} F'(x)) = 0,
 \end{aligned} \tag{2.17}$$

for all $x \in X$ and $n \in \mathbb{N}$. Together with (N4) and (2.3), this implies that

$$\begin{aligned}
 & N'(F(x) - F'(x), t) = N'(J_n F(x) - J_n F'(x), t) \\
 & \geq \min\left\{N'\left(J_n F(x) - J_n f(x), \frac{t}{2}\right), N'\left(J_n f(x) - J_n F'(x), \frac{t}{2}\right)\right\} \\
 & \geq \min\left\{N'\left(\frac{(F-f)(2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right),\right. \\
 & \quad \left.N'\left(\frac{(F-f)(-2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right),\right. \\
 & \quad \left.N'\left(\frac{(F-f)(2^n x)}{2 \cdot 2^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(2^n x)}{2 \cdot 2^n}, \frac{t}{8}\right),\right\}
 \end{aligned}$$

$$\begin{aligned}
& N' \left(\frac{(F-f)(-2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right), N' \left(\frac{(f-F')(-2^n x)}{2 \cdot 2^n}, \frac{t}{8} \right) \Big\} \\
& \geq \sup_{t' < t} N \left(x, \frac{2^{(q-1)n-2q} t'^q}{((7+2^p+3^p+4^p)/(4-2^p)3^p + (5+2 \cdot 2^p+3^p)/2(2-2^p))^q} \right),
\end{aligned} \tag{2.18}$$

for all $x \in X$ and $n \in \mathbb{N}$. Observe that for $q = 1/p$, the last term of the above inequality tends to 1 as $n \rightarrow \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$, and so we get

$$F(x) = F'(x), \tag{2.19}$$

for all $x \in X$ by (N2).

Case 2. Let $1/2 < q < 1$, and let $J_n f : X \rightarrow Y$ be a mapping defined by

$$J_n f(x) = \frac{1}{2} \left(4^{-n} (f(2^n x) + f(-2^n x) - 2f(0)) + 2^n \left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right) \right) \right) + f(0), \tag{2.20}$$

for all $x \in X$. Then, we have $J_0 f(x) = f(x)$, $J_j f(0) = f(0)$, and

$$\begin{aligned}
J_j f(x) - J_{j+1} f(x) &= \frac{Df(2^j x/3, 2^j x/3)}{4^{j+1}} - \frac{Df(2^j x/3, 2^{j+1} x/3)}{2 \cdot 4^{j+1}} - \frac{Df(2^j x/3, 2^j x)}{2 \cdot 4^{j+1}} \\
&\quad - \frac{Df(2^j x/3, 2^{j+2} x/3)}{2 \cdot 4^{j+1}} + \frac{Df(-2^j x/3, -2^j x/3)}{4^{j+1}} - \frac{Df(-2^j x/3, -2^{j+1} x/3)}{2 \cdot 4^{j+1}} \\
&\quad - \frac{Df(-2^j x/3, -2^j x)}{2 \cdot 4^{j+1}} - \frac{Df(-2^j x/3, -2^{j+2} x/3)}{2 \cdot 4^{j+1}} \\
&\quad - 2^{j-1} \left(Df\left(\frac{x}{2^j}, \frac{x}{2^{j+1}}\right) - Df\left(\frac{x}{2^{j+1}}, \frac{3x}{2^{j+1}}\right) + Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right. \\
&\quad \left. + Df\left(\frac{x}{2^{j+1}}, \frac{-x}{2^j}\right) \right),
\end{aligned} \tag{2.21}$$

for all $x \in X$ and $j \geq 0$. If $n + m > m \geq 0$, then we have

$$\begin{aligned}
& N' \left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{7+2^p+3^p+4^p}{4 \cdot 3^p} \left(\frac{2^p}{4} \right)^j + \frac{5+2 \cdot 2^p+3^p}{2 \cdot 2^p} \left(\frac{2}{2^p} \right)^j \right) t^p \right) \\
& \geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N' \left(\frac{Df(2^j x/3, 2^j x/3)}{4^{j+1}}, \frac{2 \cdot 2^{jp} t^p}{4^{j+1} \cdot 3^p} \right), \right. \right.
\end{aligned}$$

$$\begin{aligned}
& N' \left(\frac{Df(-2^j x/3, -2^j x/3)}{4^{j+1}}, \frac{2 \cdot 2^j t^p}{4^{j+1} \cdot 3^p} \right), \\
& N' \left(-\frac{Df(2^j x/3, 2^{j+1} x/3)}{2 \cdot 4^{j+1}}, \frac{2^j p (1 + 2^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& N' \left(-\frac{Df(2^j x/3, 2^j x)}{2 \cdot 4^{j+1}}, \frac{2^j p (1 + 3^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& N' \left(-\frac{Df(2^j x/3, 2^{j+2} x/3)}{2 \cdot 4^{j+1}}, \frac{2^j p (1 + 4^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& N' \left(-\frac{Df(-2^j x/3, -2^{j+1} x/3)}{2 \cdot 4^{j+1}}, \frac{2^j p (1 + 2^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& N' \left(-\frac{Df(-2^j x/3, -2^j x)}{2 \cdot 4^{j+1}}, \frac{2^j p (1 + 3^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& N' \left(-\frac{Df(-2^j x/3, -2^{j+2} x/3)}{2 \cdot 4^{j+1}}, \frac{2^j p (1 + 4^p) t^p}{2 \cdot 4^{j+1} \cdot 3^p} \right), \\
& N' \left(-2^{j-1} Df \left(\frac{x}{2^j}, \frac{x}{2^{j+1}} \right), \frac{2^{j-1} (1 + 2^p) t^p}{2^{(j+1)p}} \right), \\
& N' \left(2^{j-1} Df \left(\frac{x}{2^{j+1}}, \frac{3x}{2^{j+1}} \right), \frac{2^{j-1} (1 + 3^p) t^p}{2^{(j+1)p}} \right), \\
& N' \left(-2^{j-1} Df \left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}} \right), \frac{2^j t^p}{2^{(j+1)p}} \right), \\
& N' \left(-2^{j-1} Df \left(\frac{x}{2^{j+1}}, -\frac{x}{2^j} \right), \frac{2^{j-1} (1 + 2^p) t^p}{2^{(j+1)p}} \right) \Bigg\} \\
& \geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N \left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}} \right), N \left(\frac{x}{2^j}, \frac{t}{2^j} \right), N \left(\frac{3x}{2^{j+1}}, \frac{3t}{2^{j+1}} \right), \right. \right. \\
& \quad \left. \left. N \left(\frac{2^j x}{3}, \frac{2^j t}{3} \right), N \left(\frac{2^{j+1} x}{3}, \frac{2^{j+1} t}{3} \right), N \left(\frac{2^{j+2} x}{3}, \frac{2^{j+2} t}{3} \right) \right\} \right\} \\
& = N(x, t),
\end{aligned} \tag{2.22}$$

for all $x \in X$ and $t > 0$. In the similar argument following (2.6) of the previous case, we can define the limit $F(x) := N' - \lim_{n \rightarrow \infty} J_n f(x)$ of the Cauchy sequence $\{J_n f(x)\}$ in the Banach fuzzy space Y . Moreover, putting $m = 0$ in the above inequality, we have

$$\begin{aligned}
& N'(f(x) - J_n f(x), t) \\
& \geq N \left(x, \frac{t^q}{\left(\sum_{j=0}^{n-1} \left((7 + 2^p + 3^p + 4^p) / (4 \cdot 3^p) (2^p / 4)^j + (5 + 2 \cdot 2^p + 3^p) / (2 \cdot 2^p) (2 / 2^p)^j \right) \right)^q} \right),
\end{aligned} \tag{2.23}$$

for all $x \in X$ and $t > 0$. To prove that F is a general quadratic mapping, we have enough to show that the last term of (2.12) in Case 1 tends to 1 as $n \rightarrow \infty$. By (N3) and (2.2), we get

$$\begin{aligned}
N'\left(DJ_n f(x, y), \frac{t}{2}\right) &\geq \min\left\{N'\left(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8}\right), \right. \\
&\quad \left. N'\left(2^{n-1} Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{t}{8}\right), N'\left(2^{n-1} Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right), \frac{t}{8}\right)\right\} \quad (2.24) \\
&\geq \min\left\{N\left(x, 2^{(2q-1)n-4q} t^q\right), N\left(y, 2^{(2q-1)n-4q} t^q\right), \right. \\
&\quad \left. N\left(x, 2^{(1-q)n-4q} t^q\right), N\left(y, 2^{(1-q)n-4q} t^q\right)\right\},
\end{aligned}$$

for all $x, y \in X \setminus \{0\}$ and $t > 0$. Observe that all the terms on the right hand side of the above inequality tend to 1 as $n \rightarrow \infty$, since $1/2 < q < 1$. Hence, together with the similar argument after (2.12), we can say that $DF(x, y) = 0$ for all $x, y \in X$. Recall that in Case 1, (2.3) follows from (2.11). By the same reasoning, we get (2.3) from (2.23) in this case. Now, to prove the uniqueness of F , let F' be another general quadratic mapping satisfying (2.3). Then, together with (N4), (2.3), and (2.17), we have

$$\begin{aligned}
&N'(F(x) - F'(x), t) \\
&= N'(J_n F(x) - J_n F'(x), t) \\
&\geq \min\left\{N'\left(J_n F(x) - J_n f(x), \frac{t}{2}\right), N'\left(J_n f(x) - J_n F'(x), \frac{t}{2}\right)\right\} \\
&\geq \min\left\{N'\left(\frac{(F-f)(2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right), \right. \\
&\quad N'\left(\frac{(F-f)(-2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-2^n x)}{2 \cdot 4^n}, \frac{t}{8}\right), \quad (2.25) \\
&\quad N'\left(2^{n-1} \left((F-f)\left(\frac{x}{2^n}\right)\right), \frac{t}{8}\right), N'\left(2^{n-1} \left((f-F')\left(\frac{x}{2^n}\right)\right), \frac{t}{8}\right), \\
&\quad \left. N'\left(2^{n-1} \left((F-f)\left(\frac{-x}{2^n}\right)\right), \frac{t}{8}\right), N'\left(2^{n-1} \left((f-F')\left(\frac{-x}{2^n}\right)\right), \frac{t}{8}\right)\right\} \\
&\geq \min\left\{N\left(x, \frac{2^{(2q-1)n-2q} t^q}{((7+2^p+3^p+4^p)/(4-2^p)3^p+(5+2 \cdot 2^p+3^p)/2(2^p-2))^q}\right), \right. \\
&\quad \left. N\left(x, \frac{2^{(1-q)n-2q} t^q}{((7+2^p+3^p+4^p)/(4-2^p)3^p+((5+2 \cdot 2^p+3^p)/2(2^p-2))^q)}\right)\right\},
\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} 2^{(2q-1)n-2q} = \lim_{n \rightarrow \infty} 2^{(1-q)n-2q} = \infty$ in this case, both terms on the right-hand side of the above inequality tend to 1 as $n \rightarrow \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$, and so $F(x) = F'(x)$ for all $x \in X$ by (N2).

Case 3. Finally, we take $0 < q < 1/2$ and define $J_n f : X \rightarrow Y$ by

$$J_n f(x) = \frac{1}{2} \left(4^n (f(2^{-n}x) + f(-2^{-n}x) - 2f(0)) + 2^n \left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right) \right) \right) + f(0), \quad (2.26)$$

for all $x \in X$. Then, we have $J_0 f(x) = f(x)$, $J_j f(0) = f(0)$, and

$$\begin{aligned} J_j f(x) - J_{j+1} f(x) &= -4^j Df\left(\frac{x}{3 \cdot 2^{j+1}}, \frac{x}{3 \cdot 2^{j+1}}\right) - 4^j Df\left(\frac{-x}{3 \cdot 2^{j+1}}, \frac{-x}{3 \cdot 2^{j+1}}\right) \\ &\quad + \frac{4^j}{2} Df\left(\frac{x}{3 \cdot 2^{j+1}}, \frac{x}{3 \cdot 2^j}\right) + \frac{4^j}{2} Df\left(\frac{x}{3 \cdot 2^{j+1}}, \frac{x}{2^{j+1}}\right) \\ &\quad + \frac{4^j}{2} Df\left(\frac{x}{3 \cdot 2^{j+1}}, \frac{x}{3 \cdot 2^{j-1}}\right) + \frac{4^j}{2} Df\left(\frac{-x}{3 \cdot 2^{j+1}}, \frac{-x}{3 \cdot 2^j}\right) \\ &\quad + \frac{4^j}{2} Df\left(\frac{-x}{3 \cdot 2^{j+1}}, \frac{-x}{2^{j+1}}\right) + \frac{4^j}{2} Df\left(\frac{-x}{3 \cdot 2^{j+1}}, \frac{-x}{3 \cdot 2^{j-1}}\right) \\ &\quad - 2^{j-1} \left(Df\left(\frac{x}{2^j}, \frac{x}{2^{j+1}}\right) - Df\left(\frac{x}{2^{j+1}}, \frac{3x}{2^{j+1}}\right) + Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right. \\ &\quad \left. + Df\left(\frac{x}{2^{j+1}}, \frac{-x}{2^j}\right) \right), \end{aligned} \quad (2.27)$$

which implies that if $n + m > m \geq 0$, then

$$\begin{aligned} &N' \left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{7 + 2^p + 3^p + 4^p}{2^p \cdot 3^p} \left(\frac{4}{2^p} \right)^j + \frac{5 + 2 \cdot 2^p + 3^p}{2 \cdot 2^p} \left(\frac{2}{2^p} \right)^j \right) t^p \right) \\ &\geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N' \left(-4^j Df\left(\frac{x}{3 \cdot 2^{j+1}}, \frac{x}{3 \cdot 2^{j+1}}\right), \frac{2 \cdot 4^j t^p}{2^{(j+1)p} \cdot 3^p} \right), \right. \right. \\ &\quad N' \left(-4^j Df\left(\frac{-x}{3 \cdot 2^{j+1}}, \frac{-x}{3 \cdot 2^{j+1}}\right), \frac{2 \cdot 4^j t^p}{2^{(j+1)p} \cdot 3^p} \right), \\ &\quad N' \left(\frac{4^j Df(x/(3 \cdot 2^{j+1}), x/(3 \cdot 2^j))}{2}, \frac{4^j (1 + 2^p) t^p}{2 \cdot 2^{(j+1)p} \cdot 3^p} \right), \\ &\quad N' \left(\frac{4^j Df(x/(3 \cdot 2^{j+1}), x/2^{j+1})}{2}, \frac{4^j (1 + 3^p) t^p}{2 \cdot 2^{(j+1)p} \cdot 3^p} \right), \\ &\quad N' \left(\frac{4^j Df(x/(3 \cdot 2^{j+1}), x/(3 \cdot 2^{j-1}))}{2}, \frac{4^j (1 + 4^p) t^p}{2 \cdot 2^{(j+1)p} \cdot 3^p} \right), \\ &\quad \left. N' \left(\frac{4^j Df(-x/(3 \cdot 2^{j+1}), -x/(3 \cdot 2^j))}{2}, \frac{4^j (1 + 2^p) t^p}{2 \cdot 2^{(j+1)p} \cdot 3^p} \right) \right\} \end{aligned}$$

$$\begin{aligned}
& N' \left(\frac{4^j Df(-x/(3 \cdot 2^{j+1}), -x/2^{j+1})}{2}, \frac{4^j(1+3^p)t^p}{2 \cdot 2^{(j+1)p} \cdot 3^p} \right), \\
& N' \left(\frac{4^j Df(-x/(3 \cdot 2^{j+1}), -x/(3 \cdot 2^{j-1}))}{2}, \frac{4^j(1+4^p)t^p}{2 \cdot 2^{(j+1)p} \cdot 3^p} \right), \\
& N' \left(-2^{j-1} Df\left(\frac{x}{2^j}, \frac{x}{2^{j+1}}\right), \frac{2^{j-1}(1+2^p)t^p}{2^{(j+1)p}} \right), \\
& N' \left(2^{j-1} Df\left(\frac{x}{2^{j+1}}, \frac{3x}{2^{j+1}}\right), \frac{2^{j-1}(1+3^p)t^p}{2^{(j+1)p}} \right), \\
& N' \left(-2^{j-1} Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), \frac{2^j t^p}{2^{(j+1)p}} \right), \\
& N' \left(-2^{j-1} Df\left(\frac{x}{2^{j+1}}, -\frac{x}{2^j}\right), \frac{2^{j-1}(1+2^p)t^p}{2^{(j+1)p}} \right) \Bigg\} \\
& \geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right), N\left(\frac{x}{2^j}, \frac{t}{2^j}\right), N\left(\frac{3x}{2^{j+1}}, \frac{3t}{2^{j+1}}\right), N\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right), \right. \right. \\
& \quad \left. \left. N\left(\frac{x}{3 \cdot 2^{j+1}}, \frac{t}{3 \cdot 2^{j+1}}\right), N\left(\frac{x}{3 \cdot 2^j}, \frac{t}{3 \cdot 2^j}\right), N\left(\frac{x}{3 \cdot 2^{j-1}}, \frac{t}{3 \cdot 2^{j-1}}\right) \right\} \right\} \\
& = N(x, t),
\end{aligned} \tag{2.28}$$

for all $x \in X \setminus \{0\}$ and $t > 0$. Similar to the previous cases, it leads us to define the mapping $F : X \rightarrow Y$ by $F(x) := N' - \lim_{n \rightarrow \infty} J_n f(x)$. Putting $m = 0$ in the above inequality, we have

$$\begin{aligned}
& N'(f(x) - J_n f(x), t) \\
& \geq N \left(x, \frac{t^q}{\left(\sum_{j=0}^{n-1} \left((7 + 2^p + 3^p + 4^p) / (2^p \cdot 3^p) (4/2^p)^j + (5 + 2 \cdot 2^p + 3^p) / (2 \cdot 2^p) (2/2^p)^j \right) \right)^q} \right),
\end{aligned} \tag{2.29}$$

for all $x \in X$. Notice that

$$\begin{aligned}
N' \left(DJ_n f(x, y), \frac{t}{2} \right) & \geq \min \left\{ N' \left(\frac{4^n}{2} Df \left(\frac{x}{2^n}, \frac{y}{2^n} \right), \frac{t}{8} \right), N' \left(\frac{4^n}{2} Df \left(\frac{-x}{2^n}, \frac{-y}{2^n} \right), \frac{t}{8} \right), \right. \\
& \quad \left. N' \left(2^{n-1} Df \left(\frac{x}{2^n}, \frac{y}{2^n} \right), \frac{t}{8} \right), N' \left(2^{n-1} Df \left(\frac{-x}{2^n}, \frac{-y}{2^n} \right), \frac{t}{8} \right) \right\} \\
& \geq \min \left\{ N \left(x, 2^{(1-2q)n-3q} t^q \right), N \left(y, 2^{(1-2q)n-3q} t^q \right) \right\},
\end{aligned} \tag{2.30}$$

for all $x, y \in X \setminus \{0\}$ and $t > 0$. Since $0 < q < 1/2$, both terms on the right-hand side tend to 1 as $n \rightarrow \infty$, which implies that the last term of (2.12) tends to 1 as $n \rightarrow \infty$. Therefore, we can say that $DF \equiv 0$. Moreover, using the similar argument after (2.12) in Case 1, we get

(2.3) from (2.29) in this case. To prove the uniqueness of F , let $F' : X \rightarrow Y$ be another general quadratic mapping satisfying (2.3). Then, by (2.17), we get

$$\begin{aligned}
& N'(F(x) - F'(x), t) \\
& \geq \min \left\{ N' \left(J_n F(x) - J_n f(x), \frac{t}{2} \right), N' \left(J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\} \\
& \geq \min \left\{ N' \left(\frac{4^n}{2} \left((F - f) \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(\frac{4^n}{2} \left((f - F') \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), \right. \\
& \quad N' \left(\frac{4^n}{2} \left((F - f) \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(\frac{4^n}{2} \left((f - F') \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right), \\
& \quad N' \left(2^{n-1} \left((F - f) \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(2^{n-1} \left((f - F') \left(\frac{x}{2^n} \right) \right), \frac{t}{8} \right), \\
& \quad \left. N' \left(2^{n-1} \left((F - f) \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right), N' \left(2^{n-1} \left((f - F') \left(-\frac{x}{2^n} \right) \right), \frac{t}{8} \right) \right\} \\
& \geq \sup_{t' < t} N \left(x, \frac{2^{(1-2q)n-2q} t'^q}{((7 + 2^p + 3^p + 4^p)/(2^p - 4)3^p + (5 + 2 \cdot 2^p + 3^p)/2(2^p - 2))^q} \right),
\end{aligned} \tag{2.31}$$

for all $x \in X$ and $n \in \mathbb{N}$. Observe that for $0 < q < 1/2$, the last term tends to 1 as $n \rightarrow \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and $F(x) = F'(x)$ for all $x \in X$ by (N2). This completes the proof. $\square$

Remark 2.3. Consider a mapping $f : X \rightarrow Y$ satisfying (2.2) for all $x, y \in X \setminus \{0\}$ and a real number $q < 0$. Take any $t > 0$. If we choose a real number s with $0 < 2s < t$, then we have

$$N'(Df(x, y), t) \geq N'(Df(x, y), 2s) \geq \min\{N(x, s^q), N(y, s^q)\}, \tag{2.32}$$

for all $x, y \in X \setminus \{0\}$. Since $q < 0$, we have $\lim_{s \rightarrow 0^+} s^q = \infty$. This implies that

$$\lim_{s \rightarrow 0^+} N(x, s^q) = \lim_{s \rightarrow 0^+} N(y, s^q) = 1, \tag{2.33}$$

and so

$$N'(Df(x, y), t) = 1, \tag{2.34}$$

for all $t > 0$ and $x, y \in X \setminus \{0\}$. Since $DF(0, 0) = 0$, $DF(x, 0) = 0$, and $DF(0, y) = 0$ for all $x, y \in X \setminus \{0\}$, this means that $DF(x, y) = 0$ for all $x, y \in X$ by (N2). In other words, f is itself a general quadratic mapping if f is a fuzzy q -almost general quadratic mapping for the case $q < 0$.

We can use Theorem 2.2 to get a classical result in the framework of normed spaces. Let $(X, \|\cdot\|)$ be a normed linear space. Then, we can define a fuzzy norm N_X on X by

$$N_X(x, t) = \begin{cases} 0, & t \leq \|x\|, \\ 1, & t > \|x\|, \end{cases} \quad (2.35)$$

where $x \in X$ and $t \in \mathbb{R}$ [21]. Suppose that $f : X \rightarrow Y$ is a mapping into a Banach space $(Y, \|\cdot\|)$ such that

$$\|Df(x, y)\| \leq \|x\|^p + \|y\|^p, \quad (2.36)$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$. Let N_Y be a fuzzy norm on Y . Then, we get

$$N_Y(Df(x, y), t + s) = \begin{cases} 0, & t + s \leq \|Df(x, y)\|, \\ 1, & t + s > \|Df(x, y)\|, \end{cases} \quad (2.37)$$

for all $x, y \in X$ and $s, t \in \mathbb{R}$. Consider the case $N_Y(Df(x, y), t + s) = 0$. This implies that

$$\|x\|^p + \|y\|^p \geq \|Df(x, y)\| \geq t + s, \quad (2.38)$$

and so, either $\|x\|^p \geq t$ or $\|y\|^p \geq s$ in this case. Hence, for $q = 1/p$, we have

$$\min\{N_X(x, s^q), N_X(y, t^q)\} = 0, \quad (2.39)$$

for all $x, y \in X$ and $s, t > 0$. Therefore, in every case,

$$N_Y(Df(x, y), t + s) \geq \min\{N_X(x, s^q), N_X(y, t^q)\} \quad (2.40)$$

holds. It means that f is a fuzzy q -almost general quadratic mapping, and by Theorem 2.2, we get the following stability result.

Corollary 2.4. *Let $(X, \|\cdot\|)$ be a normed linear space, and let $(Y, \|\cdot\|)$ be a Banach space. If $f : X \rightarrow Y$ satisfies*

$$\|Df(x, y)\| \leq \|x\|^p + \|y\|^p, \quad (2.41)$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$, then there is a unique general quadratic mapping $F : X \rightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \left(\frac{2(7 + 2^p + 3^p + 4^p)}{3^p|4 - 2^p|} + \frac{5 + 2 \cdot 2 + 3^p}{|2 - 2^p|} \right) \|x\|^p, \quad (2.42)$$

for all $x \in X$.

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