Research Article

A Note on the Modified q -Bernoulli Numbers and Polynomials with Weight α

T. Kim,¹ D. V. Dolgy,² S. H. Lee,¹ B. Lee,³ and S. H. Rim⁴

¹ Division of General Education, Kwangwoon University, Seoul 139-701, Republic of Korea

² Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea

³ Department of Wireless Communications Engineering, Kwangwoon University, Seoul 139-701, Republic of Korea

⁴ Department of Mathematics Education, Kyungpook National University, Taegu 702-701, Republic of Korea

Correspondence should be addressed to T. Kim, tkkim@kw.ac.kr

Received 13 July 2011; Accepted 2 October 2011

Academic Editor: Alberto d'Onofrio

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A systemic study of some families of the modified q -Bernoulli numbers and polynomials with weight α is presented by using the p -adic q -integration $\mathbb{Z}_p$. The study of these numbers and polynomials yields an interesting q -analogue related to Bernoulli numbers and polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p -adic rational integers, the field of p -adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of q -extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q - 1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

The q -number $[x]_q$ is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad (1.1)$$

see [1–10].

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \quad (1.2)$$

have a limit $l = f'(a)$ as $(x, y) \rightarrow (a, a)$. c.f. [11].

For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable functions}\}$, the p -adic q -integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (1.3)$$

(see [3]).

From (1.3), we can easily derive the following:

$$q^n I_q(f_n) = I_q(f) + (q-1) \sum_{l=0}^{n-1} f(l) q^l + \frac{q-1}{\log q} \sum_{l=0}^{n-1} q^l f'(l), \quad (1.4)$$

where $f_n(x) = f(x+n)$, (see [5, 12]).

In [1, 2], Carlitz defined a set of numbers $B_{k,q}$ inductively by

$$B_{0,q} = 1, \quad (qB_q + 1)^k - B_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \quad (1.5)$$

with the usual convention about replacing B_q^k by $B_{k,q}$.

These numbers are the q -extension of ordinary Bernoulli numbers. But they do not remain finite when $q = 1$. So, Carlitz modified (1.5) as follows:

$$\beta_{0,q} = 1, \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \quad (1.6)$$

with the usual convention of replacing β^k by $\beta_{k,q}$.

In [1], Carlitz also considered the extension of Carlitz's q -Bernoulli numbers as follows:

$$\beta_{0,q}^h = \frac{h}{[h]_q}, \quad q^h (q\beta^h + 1)^k - \beta_{k,q}^h = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \quad (1.7)$$

with the usual convention of replacing $(\beta^h)^k$ by $\beta_{k,q}^h$.

In this paper, we construct the modified q -Bernoulli numbers with weight α , which are different Carlitz's q -Bernoulli numbers, by using p -adic q -integral equation. From (1.4),

we derive some interesting identities and relations on the modified q -Bernoulli numbers and polynomials.

2. The Modified q -Bernoulli Numbers and Polynomials with Weight α

In this section, we assume $\alpha \in \mathbb{Q}$. Now, we define the modified q -Bernoulli numbers with weight α ($= \tilde{B}_{n,q}^{(\alpha)}$) as follows:

$$\begin{aligned}\tilde{B}_{n,q}^{(\alpha)} &= \int_{\mathbb{Z}_p} q^{-x} [x]_{q^\alpha}^n d\mu_q(x) \\ &= \frac{1}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{\alpha l}{[\alpha l]_q}.\end{aligned}\tag{2.1}$$

Thus, by (2.1), we have

$$\tilde{B}_{n,q}^{(\alpha)} = \frac{1}{(1-q^\alpha)^n} \frac{q-1}{\log q} - n \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha m} [m]_{q^\alpha}^{n-1}.\tag{2.2}$$

Therefore, by (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, one has

$$\begin{aligned}\tilde{B}_{n,q}^{(\alpha)} &= \frac{\alpha}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{l}{[\alpha l]_q} \\ &= \frac{1}{(1-q^\alpha)^n} \frac{q-1}{\log q} - n \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha m} [m]_{q^\alpha}^{n-1}.\end{aligned}\tag{2.3}$$

Let us define the generating function of the modified q -Bernoulli numbers with weight α as follows:

$$F_q^{(\alpha)}(t) = \sum_{n=0}^{\infty} \tilde{B}_{n,q}^{(\alpha)} \frac{t^n}{n!}.\tag{2.4}$$

Then, by (2.3) and (2.4), we get

$$F_q^{(\alpha)}(t) = \frac{q-1}{\log q} e^{(1/(1-q^\alpha))t} - t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha m} e^{[m]_{q^\alpha} t}.\tag{2.5}$$

In the viewpoint of (2.1), we define the modified q -Bernoulli numbers with weight α as follows:

$$\begin{aligned}\tilde{B}_{n,q}^{(\alpha)}(x) &= \int_{\mathbb{Z}_p} q^{-y} [x+y]_{q^\alpha}^n d\mu_q(y) \\ &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{B}_{l,q}^{(\alpha)} \\ &= \left([x]_{q^\alpha} + q^{\alpha x} \tilde{B}_q^{(\alpha)} \right)^n, \quad \text{for } n \in \mathbb{Z}_+, \end{aligned} \quad (2.6)$$

with the usual convention of replacing $(\tilde{B}_q^{(\alpha)})^n$ by $\tilde{B}_{n,q}^{(\alpha)}$.
From (2.6), we note that

$$\begin{aligned}\tilde{B}_{n,q}^{(\alpha)}(x) &= \frac{\alpha}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{l}{[l]_q} \\ &= \frac{1}{(1-q^\alpha)^n} \frac{q-1}{\log q} - n \frac{\alpha}{[\alpha]_q} q^{\alpha x} \sum_{m=0}^{\infty} q^{\alpha m} [m+x]_{q^\alpha}^{n-1}. \end{aligned} \quad (2.7)$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned}\tilde{B}_{n,q}^{(\alpha)}(x) &= \frac{\alpha}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{l}{[l]_q} \\ &= \frac{1}{(1-q^\alpha)^n} \frac{q-1}{\log q} - n \frac{\alpha}{[\alpha]_q} q^{\alpha x} \sum_{m=0}^{\infty} q^{\alpha m} [m+x]_{q^\alpha}^{n-1}. \end{aligned} \quad (2.8)$$

Let $F_q^{(\alpha)}(t, x) = \sum_{n=0}^{\infty} \tilde{B}_{n,q}^{(\alpha)}(x) (t^n/n!)$ be the generating function of the modified q -Bernoulli polynomials with weight α .

Then, by (2.7), we get

$$F_q^{(\alpha)}(t, x) = \frac{q-1}{\log q} e^{(1/(1-q^\alpha))t} - t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(m+x)} e^{[m+x]_{q^\alpha} t}. \quad (2.9)$$

Therefore, by (2.9), we obtain the following corollary

Corollary 2.3. *Let $F_q^{(\alpha)}(t, x) = \sum_{n=0}^{\infty} \tilde{B}_{n,q}^{(\alpha)}(x) (t^n/n!)$. Then one has*

$$F_q^{(\alpha)}(t, x) = \frac{q-1}{\log q} e^{(1/(1-q^\alpha))t} - t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(m+x)} e^{[m+x]_{q^\alpha} t}. \quad (2.10)$$

In particular, $F_q^{(\alpha)}(t, 0) = F_q^{(\alpha)}(t)$.

From Corollary 2.3, we can derive the following equation:

$$F_q^\alpha(t, 1) - F_q^\alpha(t) = t \frac{\alpha}{[\alpha]_q}. \quad (2.11)$$

By (2.5) and (2.11), we get

$$\tilde{B}_{0,q}^{(\alpha)} = \frac{q-1}{\log q}, \quad \tilde{B}_{n,q}^{(\alpha)}(1) - \tilde{B}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \quad (2.12)$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.4. For $n \in \mathbb{Z}_+$, one has

$$\tilde{B}_{0,q}^{(\alpha)} = \frac{q-1}{\log q}, \quad \tilde{B}_{n,q}^{(\alpha)}(1) - \tilde{B}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \quad (2.13)$$

By using (2.6), we obtain the following corollary.

Corollary 2.5. For $n \in \mathbb{Z}_+$, one has

$$\tilde{B}_{0,q}^{(\alpha)} = \frac{q-1}{\log q}, \quad \left(q^\alpha \tilde{B}_q^{(\alpha)} + 1 \right)^n - \tilde{B}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \quad (2.14)$$

with the usual convention of replacing $(\tilde{B}_q^{(\alpha)})^n$ by $\tilde{B}_{n,q}^{(\alpha)}$.

From (1.4), we can derive the following equation:

$$\int_{\mathbb{Z}_p} f(x+n) q^{-x} d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) q^{-x} d\mu_q(x) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} f'(l). \quad (2.15)$$

Thus, by (1.6), (2.6), and (2.15), we get

$$\tilde{B}_{m,q}^{(\alpha)}(n) - \tilde{B}_{m,q}^{(\alpha)} = \frac{\alpha}{[\alpha]_q} m \sum_{l=0}^{n-1} [l]_{q^\alpha}^{m-1} q^{\alpha l}, \quad n \in \mathbb{N}, \quad m \in \mathbb{Z}_+. \quad (2.16)$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.6. For $n \in \mathbb{N}$, $m \in \mathbb{Z}_+$, one has

$$\tilde{B}_{m,q}^{(\alpha)}(n) - \tilde{B}_{m,q}^{(\alpha)} = \frac{\alpha}{[\alpha]_q} m \sum_{l=0}^{n-1} [l]_{q^\alpha}^{m-1} q^{\alpha l}. \quad (2.17)$$

From (2.6), we note that

$$\begin{aligned}
 \tilde{B}_{n,q}^{(\alpha)}(x) &= \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n q^{-y} d\mu_q(y) \\
 &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{y=0}^{p^N-1} [x+y]_{q^\alpha}^n \\
 &= \frac{1-q}{1-q^d} \sum_{a=0}^{d-1} \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{q^d}} \sum_{y=0}^{p^N-1} [a+x+dy]_{q^\alpha}^n \\
 &= \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \int_{\mathbb{Z}_p} \left[\frac{a+x}{d} + y \right]_{q^{\alpha d}}^n q^{-dy} d\mu_{q^d}(y) \\
 &= \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \tilde{B}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right).
 \end{aligned} \tag{2.18}$$

Therefore, by (2.18), we obtain the following distribution relation for the modified q -Bernoulli polynomials with weight α .

Theorem 2.7. For $d \in \mathbb{N}, n \in \mathbb{Z}_+$, one has

$$\tilde{B}_{n,q}^{(\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \tilde{B}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right). \tag{2.19}$$

To derive the relation of reflection symmetry of the modified q -Bernoulli polynomials with weight α , we evaluate the following p -adic q -integral on $\mathbb{Z}_p$:

$$\begin{aligned}
 \tilde{B}_{n,q^{-1}}^{(\alpha)}(1-x) &= \int_{\mathbb{Z}_p} [1-x+x_1]_{q^{-\alpha}}^n q^{x_1} d\mu_{q^{-1}}(x_1) \\
 &= \frac{1}{(1-q^{-\alpha})^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x - 1} \frac{\alpha l}{[\alpha l]_q} \\
 &= \frac{(-1)^n}{q} \frac{q^{\alpha n}}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\alpha l}{[\alpha l]_q} \\
 &= q^{\alpha n - 1} (-1)^n \tilde{B}_{n,q}^{(\alpha)}(x).
 \end{aligned} \tag{2.20}$$

Therefore, by (2.20), we obtain the following reflection symmetry relation of the modified q -Bernoulli polynomials with weight α .

Theorem 2.8. For $n \in \mathbb{Z}_+$, one has

$$\tilde{B}_{n,q^{-1}}^{(\alpha)}(1-x) = q^{\alpha n - 1} (-1)^n \tilde{B}_{n,q}^{(\alpha)}(x). \tag{2.21}$$

From (1.3), we note that

$$\begin{aligned} \frac{1}{q} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n q^{-x} d\mu_q(x) &= (-1)^n q^{\alpha n-1} \int_{\mathbb{Z}_p} [x-1]_{q^\alpha}^n q^{-x} d\mu_q(x) \\ &= (-1)^n q^{\alpha n-1} \tilde{B}_{n,q}^{(\alpha)}(-1) \\ &= \tilde{B}_{n,q^{-1}}^{(\alpha)}(2), \end{aligned} \quad (2.22)$$

and, by (2.6), we get

$$\begin{aligned} \tilde{B}_{n,q}^{(\alpha)}(2) &= \left(q^{2\alpha} \tilde{B}_q^{(\alpha)} + [2]_{q^\alpha} \right)^n = \left(q^\alpha \left(q^\alpha \tilde{B}_q^{(\alpha)} + 1 \right) + 1 \right)^n \\ &= \sum_{l=0}^n \binom{n}{l} q^{\alpha l} \left(q^\alpha \tilde{B}_q^{(\alpha)} + 1 \right)^l \\ &= \tilde{B}_{0,q}^{(\alpha)} + n q^\alpha \left(q^\alpha \tilde{B}_q^{(\alpha)} + 1 \right)^1 + \sum_{l=2}^n \binom{n}{l} q^{\alpha l} \left(q^\alpha \tilde{B}_q^{(\alpha)} + 1 \right)^l \\ &= \frac{(q-1)}{\log q} + n q^\alpha \left(\frac{\alpha}{[\alpha]_q} + \tilde{B}_{1,q}^{(\alpha)} \right) + \sum_{l=2}^n \binom{n}{l} q^{\alpha l} \tilde{B}_{l,q}^{(\alpha)} \\ &= n q^\alpha \frac{\alpha}{[\alpha]_q} + \sum_{l=0}^n \binom{n}{l} q^{\alpha l} \tilde{B}_{l,q}^{(\alpha)}. \end{aligned} \quad (2.23)$$

Let $n \in \mathbb{N}$ with $n \geq 2$. Then, by (2.12) and (2.23), we obtain the following theorem.

Theorem 2.9. For $n \in \mathbb{N}$ with $n \geq 2$, one has

$$\tilde{B}_{n,q}^{(\alpha)}(2) - n q^\alpha \frac{\alpha}{[\alpha]_q} = \left(q^\alpha \tilde{B}_q^{(\alpha)} + 1 \right)^n = \tilde{B}_{n,q}^{(\alpha)}. \quad (2.24)$$

In particular,

$$\frac{1}{q} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n q^{-x} d\mu_q(x) = \tilde{B}_{n,q^{-1}}^{(\alpha)}(2) = \frac{n}{q} \frac{\alpha}{[\alpha]_q} + \tilde{B}_{n,q^{-1}}^{(\alpha)}. \quad (2.25)$$

Acknowledgments

The authors express their sincere gratitude to the referees for their valuable suggestions and comments. This paper is supported in part by the Research Grant of Kwangwoon University in 2011.

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