Research Article

Bounds of Solutions of Integro-differential Equations

Zdeněk Šmarda

*Department of Mathematics, Faculty of Electrical Engineering and Communication, Technická 8,
Brno University of Technology, 61600 Brno, Czech Republic*

Correspondence should be addressed to Zdeněk Šmarda, smarda@feec.vutbr.cz

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Some new integral inequalities are given, and bounds of solutions of the following integro-differential equation are determined: $x'(t) - \mathcal{F}(t, x(t), \int_0^t k(t, s, x(t), x(s))ds) = h(t)$, $x(0) = x_0$, where $h : R_+ \rightarrow R$, $k : R_+^2 \times R^2 \rightarrow R$, $\mathcal{F} : R_+ \times R^2 \rightarrow R$ are continuous functions, $R_+ = [0, \infty)$.

1. Introduction

Ou Yang [1] established and applied the following useful nonlinear integral inequality.

Theorem 1.1. *Let u and h be nonnegative and continuous functions defined on R_+ and let $c \geq 0$ be a constant. Then, the nonlinear integral inequality*

$$u^2(t) \leq c^2 + 2 \int_0^t h(s)u(s)ds, \quad t \in R_+ \quad (1.1)$$

implies

$$u(t) \leq c + \int_0^t h(s)ds, \quad t \in R_+. \quad (1.2)$$

This result has been frequently used by authors to obtain global existence, uniqueness, boundedness, and stability of solutions of various nonlinear integral, differential, and

integro-differential equations. On the other hand, Theorem 1.1 has also been extended and generalized by many authors; see, for example, [2–19]. Like Gronwall-type inequalities, Theorem 1.1 is also used to obtain *a priori* bounds to unknown functions. Therefore, integral inequalities of this type are usually known as Gronwall-Ou Yang type inequalities.

In the last few years there have been a number of papers written on the discrete inequalities of Gronwall inequality and its nonlinear version to the Bihari type, see [13, 16, 20]. Some applications discrete versions of integral inequalities are given in papers [21–23].

Pachpatte [11, 12, 14–16] and Salem [24] have given some new integral inequalities of the Gronwall-Ou Yang type involving functions and their derivatives. Lipovan [7] used the modified Gronwall-Ou Yang inequality with logarithmic factor in the integrand to the study of wave equation with logarithmic nonlinearity. Engler [5] used a slight variant of the Haraux's inequality for determination of global regular solutions of the dynamic antiplane shear problem in nonlinear viscoelasticity. Dragomir [3] applied his inequality to the stability, boundedness, and asymptotic behaviour of solutions of nonlinear Volterra integral equations.

In this paper, we present new integral inequalities which come out from above-mentioned inequalities and extend Pachpatte's results (see [11, 16]) especially. Obtained results are applied to certain classes of integro-differential equations.

2. Integral Inequalities

Lemma 2.1. *Let u , f , and g be nonnegative continuous functions defined on R_+ . If the inequality*

$$u(t) \leq u_0 + \int_0^t f(s) \left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau)) d\tau \right) ds \quad (2.1)$$

holds where u_0 is a nonnegative constant, $t \in R_+$, then

$$u(t) \leq u_0 \left[1 + \int_0^t f(s) \exp \left(\int_0^s \left(2g(\tau) + f(\tau) \left(1 + \int_0^\tau g(\sigma) d\sigma \right) \right) d\tau \right) ds \right] \quad (2.2)$$

for $t \in R_+$.

Proof. Define a function $v(t)$ by the right-hand side of (2.1)

$$v(t) = u_0 + \int_0^t f(s) \left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau)) d\tau \right) ds. \quad (2.3)$$

Then, $v(0) = u_0$, $u(t) \leq v(t)$ and

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t) \int_0^t g(s)(u(t) + u(s)) ds \\ &\leq f(t)v(t) + f(t) \int_0^t g(s)(v(t) + v(s)) ds. \end{aligned} \quad (2.4)$$

Define a function $m(t)$ by

$$m(t) = v(t) + \int_0^t g(s)v(s)ds + v(t) \int_0^t g(s)ds, \quad (2.5)$$

then $m(0) = v(0) = u_0$, $v(t) \leq m(t)$,

$$v'(t) \leq f(t)m(t), \quad (2.6)$$

$$\begin{aligned} m'(t) &= 2g(t)v(t) + v'(t) \left(1 + \int_0^t g(s)ds \right) \\ &\leq m(t) \left[2g(t) + f(t) \left(1 + \int_0^t g(s)ds \right) \right]. \end{aligned} \quad (2.7)$$

Integrating (2.7) from 0 to t , we have

$$m(t) \leq u_0 \exp \left(\int_0^t \left(2g(s) + f(s) \left(1 + \int_0^s g(\sigma)d\sigma \right) \right) ds \right). \quad (2.8)$$

Using (2.8) in (2.6), we obtain

$$v'(t) \leq u_0 f(t) \exp \left(\int_0^t \left(2g(s) + f(s) \left(1 + \int_0^s g(\sigma)d\sigma \right) \right) ds \right). \quad (2.9)$$

Integrating from 0 to t and using $u(t) \leq v(t)$, we get inequality (2.2). The proof is complete. $\square$

Lemma 2.2. Let u , f , and g be nonnegative continuous functions defined on R_+ , $w(t)$ be a positive nondecreasing continuous function defined on R_+ . If the inequality

$$u(t) \leq w(t) + \int_0^t f(s) \left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau))d\tau \right) ds, \quad (2.10)$$

holds, where u_0 is a nonnegative constant, $t \in R_+$, then

$$u(t) \leq w(t) \left[1 + \int_0^t f(s) \exp \left(\int_0^s \left(2g(\tau) + f(\tau) \left(1 + \int_0^\tau g(\sigma)d\sigma \right) \right) d\tau \right) ds \right], \quad (2.11)$$

where $t \in R_+$.

Proof. Since the function $w(t)$ is positive and nondecreasing, we obtain from (2.10)

$$\frac{u(t)}{w(t)} \leq 1 + \int_0^t f(s) \left(\frac{u(s)}{w(s)} + \int_0^s g(\tau) \left(\frac{u(s)}{w(s)} + \frac{u(\tau)}{w(\tau)} \right) d\tau \right) ds. \quad (2.12)$$

Applying Lemma 2.1 to inequality (2.12), we obtain desired inequality (2.11). $\square$

Lemma 2.3. *Let u , f , g , and h be nonnegative continuous functions defined on R_+ , and let c be a nonnegative constant.*

If the inequality

$$u^2(t) \leq c^2 + 2 \left[\int_0^t f(s) u(s) \left(u(s) + \int_0^s g(\tau) (u(\tau) + u(s)) d\tau \right) + h(s) u(s) \right] ds \quad (2.13)$$

holds for $t \in R_+$, then

$$u(t) \leq p(t) \left[1 + \int_0^t f(s) \exp \left(\int_0^s \left(2g(\tau) + f(\tau) \left(1 + \int_0^\tau g(\sigma) d\sigma \right) \right) d\tau \right) ds \right], \quad (2.14)$$

where

$$p(t) = c + \int_0^t h(s) ds. \quad (2.15)$$

Proof. Define a function $z(t)$ by the right-hand side of (2.13)

$$z(t) = c^2 + 2 \left[\int_0^t f(s) u(s) \left(u(s) + \int_0^s g(\tau) (u(\tau) + u(s)) d\tau \right) + h(s) u(s) \right] ds. \quad (2.16)$$

Then $z(0) = c^2$, $u(t) \leq \sqrt{z(t)}$ and

$$\begin{aligned} z'(t) &= 2 \left[f(t) u(t) \left(u(t) + \int_0^t g(s) (u(t) + u(s)) ds \right) + h(t) u(t) \right] \\ &\leq 2 \sqrt{z(t)} \left[f(t) \left(\sqrt{z(t)} + \int_0^t g(s) \left(\sqrt{z(t)} + \sqrt{z(s)} \right) ds \right) + h(t) \right]. \end{aligned} \quad (2.17)$$

Differentiating $\sqrt{z(t)}$ and using (2.17), we get

$$\begin{aligned} \frac{d}{dt} \left(\sqrt{z(t)} \right) &= \frac{z'(t)}{2\sqrt{z(t)}} \\ &\leq f(t) \left(\sqrt{z(t)} + \int_0^t g(s) \left(\sqrt{z(t)} + \sqrt{z(s)} \right) ds \right) + h(t). \end{aligned} \quad (2.18)$$

Integrating inequality (2.18) from 0 to t , we have

$$\sqrt{z(t)} \leq p(t) + \int_0^t f(s) \left(\sqrt{z(s)} + \int_0^s g(\tau) \left(\sqrt{z(s)} + \sqrt{z(\tau)} \right) d\tau \right) ds, \quad (2.19)$$

where $p(t)$ is defined by (2.15), $p(t)$ is positive and nondecreasing for $t \in R_+$. Now, applying Lemma 2.2 to inequality (2.19), we get

$$\sqrt{z(t)} \leq p(t) \left[1 + \int_0^t f(s) \exp \left(\int_0^s \left(2g(\tau) + f(\tau) \left(1 + \int_0^\tau g(\sigma) d\sigma \right) \right) d\tau \right) ds \right]. \quad (2.20)$$

Using (2.20) and the fact that $u(t) \leq \sqrt{z(t)}$, we obtain desired inequality (2.14). $\square$

3. Application of Integral Inequalities

Consider the following initial value problem

$$x'(t) - \mathcal{F} \left(t, x(t), \int_0^t k(t, s, x(t), x(s)) ds \right) = h(t), \quad x(0) = x_0, \quad (3.1)$$

where $h : R_+ \rightarrow R$, $k : R_+^2 \times R^2 \rightarrow R$, $\mathcal{F} : R_+ \times R^2 \rightarrow R$ are continuous functions. We assume that a solution $x(t)$ of (3.1) exists on R_+ .

Theorem 3.1. *Suppose that*

$$\begin{aligned} |k(t, s, u_1, u_2)| &\leq f(t)g(s)(|u_1| + |u_2|) \quad \text{for } (t, s, u_1, u_2) \in R_+^2 \times R^2, \\ |\mathcal{F}(t, u_1, v_1)| &\leq f(t)|u_1| + |v_1| \quad \text{for } (t, u_1, v_1) \in R_+ \times R^2, \end{aligned} \quad (3.2)$$

where f, g are nonnegative continuous functions defined on R_+ . Then, for the solution $x(t)$ of (3.1) the inequality

$$\begin{aligned} |x(t)| &\leq r(t) \left[1 + \int_0^t f(s) \exp \left(\int_0^s \left(2g(\tau) + f(\tau) \left(1 + \int_0^\tau g(\sigma) d\sigma \right) \right) d\tau \right) ds \right], \\ r(t) &= |x_0| + \int_0^t |h(t)| dt \end{aligned} \quad (3.3)$$

holds on R_+ .

Proof. Multiplying both sides of (3.1) by $x(t)$ and integrating from 0 to t we obtain

$$x^2(t) = x_0^2 + 2 \int_0^t \left[x(s) \mathcal{F} \left(s, x(s), \int_0^s k(s, \tau, x(s), x(\tau)) d\tau \right) + x(s)h(s) \right] ds. \quad (3.4)$$

From (3.2) and (3.4), we get

$$|x(t)|^2 \leq |x_0|^2 + 2 \int_0^t \left[f(s)|x(s)| \times \left(|x(s)| + \int_0^s g(\tau)(|x(s)| + |x(\tau)|)d\tau \right) + |h(s)||x(s)| \right] ds. \quad (3.5)$$

Using inequality (2.14) in Lemma 2.3, we have

$$|x(t)| \leq r(t) \left[1 + \int_0^t f(s) \exp \left(\int_0^s \left(2g(\tau) + f(\tau) \left(1 + \int_0^\tau g(\sigma)d\sigma \right) \right) d\tau \right) ds \right], \quad (3.6)$$

where

$$r(t) = |x_0| + \int_0^t |h(t)|dt, \quad (3.7)$$

which is the desired inequality (3.3). $\square$

Remark 3.2. It is obvious that inequality (3.3) gives the bound of the solution $x(t)$ of (3.1) in terms of the known functions.

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