Research Article

Essential Norm of Composition Operators on Banach Spaces of Hölder Functions

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Let (X, d) be a pointed compact metric space, let $0 < \alpha < 1$, and let $\varphi : X \rightarrow X$ be a base point preserving Lipschitz map. We prove that the essential norm of the composition operator C_φ induced by the symbol φ on the spaces $\text{lip}_0(X, d^\alpha)$ and $\text{Lip}_0(X, d^\alpha)$ is given by the formula $\|C_\varphi\|_e = \limsup_{t \rightarrow 0} \sup_{0 < d(x, y) < t} (d(\varphi(x), \varphi(y))^\alpha / d(x, y)^\alpha)$ whenever the dual space $\text{lip}_0(X, d^\alpha)^*$ has the approximation property. This happens in particular when X is an infinite compact subset of a finite-dimensional normed linear space.

1. Introduction

Let (X, d) be a compact metric space with a distinguished point $e \in X$ and $0 < \alpha < 1$. The formula $d^\alpha(x, y) = d(x, y)^\alpha$ defines a new metric on X , and the metric space (X, d^α) is said to be a *Hölder metric space of order α* . As usual, $\mathbb{K}$ denotes the field of real or complex numbers.

The *Lipschitz space* $\text{Lip}_0(X, d^\alpha)$ is the Banach space of all Lipschitz functions $f : X \rightarrow \mathbb{K}$ on the Hölder metric space (X, d^α) for which $f(e) = 0$ under the standard Lipschitz norm

$$L_\alpha(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y \right\}. \quad (1.1)$$

Notice that the Lipschitz functions on (X, d^α) are precisely the *Hölder functions of order α* on (X, d) .

The *little Lipschitz space* $\text{lip}_0(X, d^\alpha)$ is the closed subspace consisting of those functions $f \in \text{Lip}_0(X, d^\alpha)$ that satisfy the following *local flatness condition*:

$$\lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} = 0. \quad (1.2)$$

The Lipschitz space $\text{Lip}_0(X, d^\alpha)$ has a canonical predual $\mathcal{F}(X, d^\alpha)$, the *free Lipschitz space* on (X, d^α) , also known as the *Arens-Eells space* in [1], that can be defined as the closed linear span of the point evaluations

$$\delta_x(f) = f(x), \quad (x \in X, f \in \text{Lip}_0(X, d^\alpha)), \quad (1.3)$$

in the dual space $\text{Lip}_0(X, d^\alpha)^*$. As it turns out, $\mathcal{F}(X, d^\alpha)$ is itself the dual space of $\text{lip}_0(X, d^\alpha)$. The structure of spaces of Lipschitz and Hölder functions and their preduals on general metric spaces was studied by Kalton in [2]. We refer to the book [1] by Weaver for a complete study of the spaces of Lipschitz functions.

We denote by $\mathcal{L}(E)$ the algebra of all bounded linear operators on a Banach space E , and by $\mathcal{K}(E)$, the closed ideal of all compact operators on E . The *essential norm* $\|T\|_e$ of an operator $T \in \mathcal{L}(E)$ is just the distance from T to $\mathcal{K}(E)$, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(E)\}. \quad (1.4)$$

It is clear that an operator $T \in \mathcal{L}(E)$ is compact if and only if $\|T\|_e = 0$.

Recall that a Banach space E is said to have the *approximation property* if the identity operator on E can be approximated uniformly on every compact subset of E by operators of finite rank.

Let (X, d) be a pointed compact metric space with base point $e \in X$, let $0 < \alpha < 1$, and let $\varphi : X \rightarrow X$ be a Lipschitz mapping that preserves the base point, that is, $\varphi(e) = e$ and

$$L(\varphi) := \sup \left\{ \frac{d(\varphi(x), \varphi(y))}{d(x, y)} : x, y \in X, x \neq y \right\} < \infty. \quad (1.5)$$

The *composition operator* $C_\varphi : \text{lip}_0(X, d^\alpha) \rightarrow \text{lip}_0(X, d^\alpha)$ is defined by the expression

$$(C_\varphi f)(x) = f(\varphi(x)), \quad (x \in X, f \in \text{lip}_0(X, d^\alpha)). \quad (1.6)$$

The aim of this paper is to give lower and upper estimates for the essential norm of the composition operator C_φ on $\text{lip}_0(X, d^\alpha)$ in terms of φ . Results along these lines were obtained by Montes-Rodríguez [3, 4] and more recently by Galindo and Lindström [5], and also by Galindo et al. [6]. In Section 2, we compute the norm of C_φ . We show that

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \quad (1.7)$$

The prototype of a formula as above with $\alpha = 1$ was provided by Weaver in [1, Proposition 1.8.2] for the composition operator C_φ on the space $\text{Lip}_0(X, d)$. In Section 3, we give a lower bound for the essential norm of the operator C_φ , namely,

$$\lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \leq \|C_\varphi\|_e. \quad (1.8)$$

Section 4 contains our main result. When the dual space $\text{lip}_0(X, d^\alpha)^*$ has the approximation property, we show that

$$\|C_\varphi\|_e \leq \lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \quad (1.9)$$

The proof of this inequality depends on some results involving shrinking compact approximating sequences on $\text{lip}_0(X, d^\alpha)$. Using the fact that the space $\text{Lip}_0(X, d^\alpha)$ is isometrically isomorphic to the second dual of $\text{lip}_0(X, d^\alpha)$, and the relationship between the essential norm of an operator and its adjoint, we derive in Section 5 the same formula for the essential norm of the operator C_φ on $\text{Lip}_0(X, d^\alpha)$.

It is natural to ask for some examples where the dual space $\text{lip}_0(X, d^\alpha)^*$ has the approximation property. For instance, this happens if X is uniformly discrete, that is, $\inf_{x \neq y} d(x, y) > 0$ (see [2, Proposition 4.4]). Also, $\text{lip}_0(X, d^\alpha)^*$ has the approximation property whenever the space $\text{lip}_0(X, d^\alpha)$ is isomorphic to c_0 and hence $\text{Lip}_0(X, d^\alpha)$ is isomorphic to ℓ_∞ and $\mathcal{F}(X, d^\alpha)$ is isomorphic to ℓ_1 .

A classical result of Bonic et al. [7] ensures that $\text{lip}_0(X, d^\alpha)$ is isomorphic to c_0 whenever X is an infinite compact subset of a finite-dimensional normed linear space. This result was corrected by Weaver, who asked whether such an isomorphism could be extended to any compact metric space [1, page 98]. Kalton answered this question negatively by proving that a compact convex subset X of a Hilbert space containing the origin has the property that $\text{lip}_0(X, d^\alpha)$ is isomorphic to c_0 if and only if X is finite dimensional [2, Theorem 8.3]. In fact this statement is true for every general Banach space in place of a Hilbert space if $0 < \alpha \leq 1/2$ [2, Theorem 8.5] and for any Banach space that has nontrivial Rademacher type if $0 < \alpha < 1$ [2, Theorem 8.4]. Kalton conjectured that this holds in full generality for all Banach spaces.

Let us recall now that a metric space (X, d) satisfies the *doubling condition* (or has *finite Assouad dimension*) if there is an integer n such that for any $\delta > 0$, every closed ball of radius δ can be covered by at most n closed balls of radius $\delta/2$. A theorem of Assouad [8] asserts that whenever a metric space (X, d) satisfies the doubling condition, every Hölder metric space (X, d^α) Lipschitz embeds in the euclidean space $\mathbb{R}^n$. Using this result, Kalton observed that if a compact metric space (X, d) satisfies the doubling condition, then the space $\text{lip}_0(X, d^\alpha)$ is isomorphic to c_0 [2, Theorem 6.5]. Furthermore, he also showed that the converse is false by means of a counterexample [2, Proposition 6.8].

2. The Norm of C_φ on $\text{Lip}_0(X, d^\alpha)$

The aim of this section is to derive a formula for the norm of the composition operator C_φ on $\text{lip}_0(X, d^\alpha)$ in terms of the Lipschitz constant of φ . A similar expression was already provided by Weaver for the composition operator C_φ on the space $\text{Lip}_0(X, d)$, obtaining in [1, Proposition 1.8.2] the following identity:

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))}{d(x, y)}. \quad (2.1)$$

Theorem 2.1. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \rightarrow X$ a base point preserving Lipschitz mapping. Then the norm of the composition operator $C_\varphi : \text{lip}_0(X, d^\alpha) \rightarrow \text{lip}_0(X, d^\alpha)$ is given by the expression*

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \quad (2.2)$$

Proof. We follow the steps of the proof of Weaver's formula. One inequality is formally identical, while the other inequality needs an adjustment of the suitable attaining functions. For any $f \in \text{lip}_0(X, d^\alpha)$ with $L_\alpha(f) \leq 1$, we have

$$\begin{aligned} L_\alpha(C_\varphi f) &= \sup_{x \neq y} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)^\alpha} \\ &\leq \sup_{\varphi(x) \neq \varphi(y)} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(\varphi(x), \varphi(y))^\alpha} \cdot \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \\ &\leq L_\alpha(f) \cdot \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \\ &\leq \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}, \end{aligned} \quad (2.3)$$

and so

$$\|C_\varphi\| = \sup_{L_\alpha(f) \leq 1} L_\alpha(C_\varphi f) \leq \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \quad (2.4)$$

For the converse inequality, fix two points $x, y \in X$ such that $\varphi(x) \neq \varphi(y)$ and choose β strictly between α and 1. Define $h : X \rightarrow \mathbb{R}$ by

$$h(z) = \frac{d(z, \varphi(y))^\beta - d(z, \varphi(x))^\beta}{2d(\varphi(x), \varphi(y))^{\beta-\alpha}}, \quad (2.5)$$

and $f : X \rightarrow \mathbb{R}$ by

$$f(z) = h(z) - h(e). \quad (2.6)$$

It is not hard to show that $f \in \text{lip}_0(X, d^\alpha)$ with $L_\alpha(f) = 1$ (see, for instance, [9]), so

$$\|C_\varphi\| \geq L_\alpha(C_\varphi f) \geq \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)^\alpha} = \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \quad (2.7)$$

Taking supremum over x and y , we conclude that

$$\|C_\varphi\| \geq \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \quad (2.8) \quad \square$$

3. The Lower Estimate of the Essential Norm of C_φ on $\text{Lip}_0(X, d^\alpha)$

Next we bound from below the essential norm of C_φ on $\text{lip}_0(X, d^\alpha)$ by means of an asymptotic quantity that measures the local flatness of φ .

Theorem 3.1. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \rightarrow X$ a base point preserving Lipschitz mapping. Then the essential norm of the operator $C_\varphi : \text{lip}_0(X, d^\alpha) \rightarrow \text{lip}_0(X, d^\alpha)$ satisfies the lower estimate*

$$\lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \leq \|C_\varphi\|_e. \quad (3.1)$$

We will need the following description of the weak convergence in $\text{lip}_0(X, d^\alpha)$. This result is part of the folklore and it is immediate from the Banach-Steinhaus theorem since $\text{lip}_0(X, d^\alpha)^* = \overline{\text{span}}\{\delta_x : x \in X\}$ (see Weaver [1, Theorem 3.3.3]).

Lemma 3.2. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\{f_n\}$ a sequence in $\text{lip}_0(X, d^\alpha)$. Then $\{f_n\}$ converges to 0 weakly in $\text{lip}_0(X, d^\alpha)$ if and only if $\{f_n\}$ is bounded in $\text{lip}_0(X, d^\alpha)$ and converges to 0 pointwise on X .*

Proof of Theorem 3.1. Since the mapping $\varphi : X \rightarrow X$ is Lipschitz, the function

$$t \mapsto \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}, \quad (t > 0) \quad (3.2)$$

is well defined. It is easy to check that

$$\lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} = \inf_{t > 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \quad (3.3)$$

Now, for every natural number n we can take a real number t_n such that $0 < t_n < [2/n(1 + L(\varphi)^\alpha)]^{1/\alpha}$ and two points $x_n, y_n \in X$ such that $0 < d(x_n, y_n) < t_n$, satisfying

$$\begin{aligned} \sup_{0 < d(x, y) < t_n} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} &< \inf_{t > 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} + \frac{1}{n}, \\ \sup_{0 < d(x, y) < t_n} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} - \frac{1}{n} &< \frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha}. \end{aligned} \quad (3.4)$$

In this way we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$0 < d(x_n, y_n) < \left[\frac{2}{n(1 + L(\varphi)^\alpha)} \right]^{1/\alpha}, \quad (3.5)$$

$$\left| \frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha} - \inf_{t > 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \right| < \frac{1}{n},$$

for all $n \in \mathbb{N}$, and this last inequality implies that

$$\inf_{t > 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} = \lim_{n \rightarrow \infty} \frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha}. \quad (3.6)$$

Now take $x_n, y_n \in X$ for which $\varphi(x_n) \neq \varphi(y_n)$ and choose $\beta \in]\alpha, 1[$. Define $h_n, f_n : X \rightarrow \mathbb{R}$ by

$$\begin{aligned} h_n(x) &= \frac{d(x, \varphi(y_n))^\beta - d(x, \varphi(x_n))^\beta}{2d(\varphi(x_n), \varphi(y_n))^{\beta-\alpha}}, \\ f_n(x) &= h_n(x) - h_n(e). \end{aligned} \quad (3.7)$$

Using (3.5), we have $\|h_n\|_\infty \leq 1/n$. Clearly, $f_n \in \text{lip}_0(X, d^\alpha)$ with $L_\alpha(f_n) = L_\alpha(h_n) = 1$ and $\|f_n\|_\infty \leq 2/n$ for all $n \in \mathbb{N}$. Moreover, an easy calculation shows that $|f_n(\varphi(x_n)) - f_n(\varphi(y_n))| = d(\varphi(x_n), \varphi(y_n))^\alpha$. Since

$$\frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha} = \frac{|f_n(\varphi(x_n)) - f_n(\varphi(y_n))|}{d(x_n, y_n)^\alpha} \leq L_\alpha(C_\varphi f_n), \quad (3.8)$$

for all $n \in \mathbb{N}$, we have

$$\lim_{n \rightarrow \infty} \frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha} \leq \limsup_{n \rightarrow \infty} L_\alpha(C_\varphi f_n). \quad (3.9)$$

By Lemma 3.2, $\{f_n\} \rightarrow 0$ weakly in $\text{lip}_0(X, d^\alpha)$. Thus, if K is any compact operator from $\text{lip}_0(X, d^\alpha)$ into $\text{lip}_0(X, d^\alpha)$, then $\lim_{n \rightarrow \infty} L_\alpha(Kf_n) = 0$ because compact operators map weakly convergent sequences into norm convergent sequences. It follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} L_\alpha(C_\varphi f_n) &= \limsup_{n \rightarrow \infty} (L_\alpha(C_\varphi f_n) - L_\alpha(Kf_n)) \\ &\leq \limsup_{n \rightarrow \infty} L_\alpha((C_\varphi - K)f_n) \\ &\leq \|C_\varphi - K\|. \end{aligned} \quad (3.10)$$

Combining (3.3), (3.6), (3.9), and (3.10), we conclude that

$$\lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \leq \|C_\varphi - K\|. \quad (3.11)$$

By taking the infimum on both sides of this inequality over all compact operators K on $\text{lip}_0(X, d^\alpha)$, we obtain the lower estimate

$$\lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \leq \|C_\varphi\|_e. \quad (3.12) \quad \square$$

4. The Upper Estimate of the Essential Norm of C_φ on $\text{Lip}_0(X, d^\alpha)$

Now we prove that the lower bound of the essential norm of C_φ on $\text{lip}_0(X, d^\alpha)$ obtained in Section 3 is also an upper bound whenever the dual space $\text{lip}_0(X, d^\alpha)^*$ has the approximation property.

Theorem 4.1. *Let X be a pointed compact metric space and $0 < \alpha < 1$. Suppose that the dual space $\text{lip}_0(X, d^\alpha)^*$ has the approximation property. Let $\varphi : X \rightarrow X$ be a base point preserving Lipschitz mapping. Then the essential norm of the composition operator $C_\varphi : \text{lip}_0(X, d^\alpha) \rightarrow \text{lip}_0(X, d^\alpha)$ satisfies*

$$\|C_\varphi\|_e \leq \lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \quad (4.1)$$

The strategy for the proof of Theorem 4.1 is to work with a sequence $\{K_n\}$ of compact operators on $\text{lip}_0(X, d^\alpha)$ that satisfies some prescribed conditions that are stated in Lemma 4.3 below. We borrow this technique from the work of Montes-Rodríguez [3].

First we recall some notions and results. A sequence $\{K_n\}$ is called a *compact approximating sequence* for a separable Banach space E if each $K_n : E \rightarrow E$ is a compact operator and $\lim_{n \rightarrow \infty} \|(I - K_n)f\| = 0$ for every $f \in E$, where I denotes the identity operator on E . Also, we say that $\{K_n\}$ is *shrinking* if $\lim_{n \rightarrow \infty} \|(I - K_n)^* f^*\| = 0$ for every $f^* \in E^*$.

Johnson [10, Theorem 2] showed that both the Banach space $\text{lip}_0(X, d^\alpha)$ and its dual space $\text{lip}_0(X, d^\alpha)^*$ are separable.

On the other hand, the *Banach-Mazur distance* between isomorphic Banach spaces E, F is defined by

$$d(E, F) = \inf \left\{ \|T\| \cdot \|T^{-1}\| : T \text{ is an isomorphism of } E \text{ onto } F \right\}. \quad (4.2)$$

We say that E *embeds almost isometrically into* F provided that for every $\varepsilon > 0$, there is a subspace $F_\varepsilon \subset F$ such that $d(E, F_\varepsilon) < 1 + \varepsilon$.

The next proposition is immediate from a result of Kalton [11].

Proposition 4.2 (see [11, Corollary 3]). *Let $\{K_n\}$ be a sequence of compact operators between Banach spaces E and F and let us suppose that $\lim_{n \rightarrow \infty} \langle K_n^* f^*, f^{**} \rangle = 0$ for all $f^* \in F^*$ and $f^{**} \in E^{**}$. Then there exists a sequence $\{K_n^c\}$ of compact operators such that $K_n^c \in \text{conv}\{K_m : m \geq n\}$ and $\lim_{n \rightarrow \infty} \|K_n^c\| = 0$.*

Lemma 4.3. *Let X be a pointed compact metric space and $0 < \alpha < 1$. Suppose that the dual space $\text{lip}_0(X, d^\alpha)^*$ has the approximation property. Then there is a shrinking compact approximating sequence $\{K_n\}$ on $\text{lip}_0(X, d^\alpha)$ such that $\limsup_{n \rightarrow \infty} \|I - K_n\| \leq 1$.*

Proof. Since the dual space $\text{lip}_0(X, d^\alpha)^*$ is separable and has the approximation property, it has the metric approximation property and therefore there is a shrinking compact approximating sequence $\{S_n\}$ on $\text{lip}_0(X, d^\alpha)$. We claim that for every $j \in \mathbb{N}$, there exist a natural $n_j \geq j$ and a compact operator K_j on $\text{lip}_0(X, d^\alpha)$ in the convex hull of the set $\{S_m : m \geq n_j\}$ such that $\|I - K_j\| < (1 + 1/j)^2$.

Fix $j \in \mathbb{N}$. Now, for the proof of our claim, we use a result that ensures that $\text{lip}_0(X, d^\alpha)$ embeds almost isometrically into c_0 . We refer to Kalton [2, Theorem 6.6] for a simple proof of this result due to Yoav Benyamini. Thus, there is a closed subspace $F_j \subset c_0$ and an isomorphism $T_j : \text{lip}_0(X, d^\alpha) \rightarrow F_j$ such that $\|T_j\| \cdot \|T_j^{-1}\| < 1 + (1/j)$. Now consider $M_n := T_j S_n T_j^{-1}$, and notice that $\{M_n\}$ is a shrinking compact approximating sequence on F_j . Next, let $\{P_n\}$ be the sequence of projections on c_0 defined by

$$(P_n x)(k) = x(k), \quad (1 \leq k \leq n), \quad (P_n x)(k) = 0, \quad (k > n), \quad (4.3)$$

for all $x \in c_0$, and let $J : F_j \hookrightarrow c_0$ be the inclusion map. Then consider the sequence of compact operators $D_n := P_n J - J M_n$ defined from F_j into c_0 . Notice that $\lim_{n \rightarrow \infty} \langle D_n^* f^*, f^{**} \rangle = 0$ for all $f^* \in c_0^*$ and $f^{**} \in F_j^*$. It follows from Proposition 4.2 that there is a sequence of operators $\{D_n^c\}$ such that $D_n^c \in \text{conv}\{D_m : m \geq n\}$ and $\lim_{n \rightarrow \infty} \|D_n^c\| = 0$. This gives rise to a pair of shrinking compact approximating sequences $\{P_n^c\}$ and $\{M_n^c\}$ such that, for each $n \in \mathbb{N}$, $P_n^c \in \text{conv}\{P_m : m \geq n\}$, $M_n^c \in \text{conv}\{M_m : m \geq n\}$, and $\lim_{n \rightarrow \infty} \|P_n^c J - J M_n^c\| = 0$. Now, consider $L_n := T_j^{-1} M_n^c T_j$. We have

$$\begin{aligned} \|I - L_n\| &= \|I - T_j^{-1} M_n^c T_j\| \leq \|T_j^{-1}\| \cdot \|T_j\| \cdot \|I - M_n^c\| \\ &\leq \left(1 + \frac{1}{j}\right) \cdot \|I - M_n^c\| = \left(1 + \frac{1}{j}\right) \cdot \|J(I - M_n^c)\| \end{aligned}$$

$$\begin{aligned}
&\leq \left(1 + \frac{1}{j}\right) \cdot (\|(I - P_n^c)J\| + \|P_n^c J - JM_n^c\|) \\
&\leq \left(1 + \frac{1}{j}\right) \cdot (1 + \|P_n^c J - JM_n^c\|),
\end{aligned} \tag{4.4}$$

for all $n \in \mathbb{N}$. Finally, choose $n_j \geq j$ large enough so that $\|P_{n_j}^c J - JM_{n_j}^c\| < 1/j$ and conclude that $\|I - L_{n_j}\| < (1 + (1/j))^2$. The claim is proved if we take $K_j := L_{n_j}$.

The proof of the lemma will be finished if we show that $\{K_j\}$ is a shrinking compact approximating sequence on $\text{lip}_0(X, d^\alpha)$. Let $f \in \text{lip}_0(X, d^\alpha)$ and $\varepsilon > 0$. Since $\lim_{n \rightarrow \infty} L_\alpha(f - S_n f) = 0$, there exists $m_0 \in \mathbb{N}$ such that $L_\alpha(f - S_n f) < \varepsilon$ for $n \geq m_0$. If $j \geq m_0$, using that $K_j \in \text{conv}\{S_m : m \geq n_j\}$ and $n_j \geq j$, we conclude that $L_\alpha(f - K_j f) < \varepsilon$. Hence $\lim_{j \rightarrow \infty} L_\alpha(f - K_j f) = 0$. This shows that $\{K_j\}$ is approximating on $\text{lip}_0(X, d^\alpha)$ and similarly it is seen that $\{K_j\}$ is shrinking. $\square$

There is another preliminary result that is needed for the proof of Theorem 4.1 and that can be stated as follows.

Lemma 4.4. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \rightarrow X$ a base point preserving Lipschitz mapping. Let $\{K_n\}$ be a shrinking compact approximating sequence on $\text{lip}_0(X, d^\alpha)$.*

Then, for each $t > 0$,

$$\lim_{n \rightarrow \infty} \sup_{L_\alpha(f) \leq 1} \sup_{d(x, y) \geq t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} = 0. \tag{4.5}$$

Proof. Fix $t > 0$. Since the inequality $\|f\|_\infty \leq \text{diam}(X)^\alpha \cdot L_\alpha(f)$ is satisfied for all $f \in \text{lip}_0(X, d^\alpha)$, there is a continuous injection $J : (\text{lip}_0(X, d^\alpha), L_\alpha(\cdot)) \hookrightarrow (\text{lip}_0(X, d^\alpha), \|\cdot\|_\infty)$. Moreover, it follows from the Arzelà-Ascoli Theorem that J is a compact operator, and by Schauder's theorem, its adjoint J^* is a compact operator, too.

Let B be the unit ball of $(\text{lip}_0(X, d^\alpha), \|\cdot\|_\infty)^*$. Since the bounded sequence of operators $\{(I - K_n)^*\}$ converges to zero pointwise on $\text{lip}_0(X, d^\alpha)^*$ and $J^*(B)$ is a relatively compact set in $(\text{lip}_0(X, d^\alpha), L_\alpha(\cdot))^*$, it follows that $\lim_{n \rightarrow \infty} \|(I - K_n)^* J^*\| = 0$. Thus, $\lim_{n \rightarrow \infty} \|J(I - K_n)\| = 0$. Let $\varepsilon > 0$ be given and choose $m \in \mathbb{N}$ such that if $n \geq m$, then $\|(I - K_n)f\|_\infty < \varepsilon t^\alpha / 4$ for all $f \in \text{lip}_0(X, d^\alpha)$ with $L_\alpha(f) \leq 1$. For $n \geq m$, we have

$$\frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \leq \frac{2\|(I - K_n)f\|_\infty}{t^\alpha} < \frac{\varepsilon}{2}, \tag{4.6}$$

whenever $f \in \text{lip}_0(X, d^\alpha)$ with $L_\alpha(f) \leq 1$ and $x, y \in X$ such that $d(x, y) \geq t$. Finally, we get

$$\sup_{L_\alpha(f) \leq 1} \sup_{d(x, y) \geq t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} < \varepsilon, \tag{4.7}$$

as required. $\square$

We now are ready to prove our main result.

Proof of Theorem 4.1. Let $\{K_n\}$ be the sequence of operators on $\text{lip}_0(X, d^\alpha)$ provided by Lemma 4.3. Since each K_n is a compact operator, so is the product $C_\varphi K_n : \text{lip}_0(X, d^\alpha) \rightarrow \text{lip}_0(X, d^\alpha)$ and therefore

$$\|C_\varphi\|_e \leq \|C_\varphi(I - K_n)\|. \quad (4.8)$$

Next, fix $t > 0$ and notice that

$$\begin{aligned} \|C_\varphi(I - K_n)\| &= \sup_{L_\alpha(f) \leq 1} L_\alpha(C_\varphi(I - K_n)f) \\ &= \sup_{L_\alpha(f) \leq 1} \sup_{x \neq y} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \\ &\leq \sup_{L_\alpha(f) \leq 1} \sup_{0 < d(x, y) < t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \\ &\quad + \sup_{L_\alpha(f) \leq 1} \sup_{d(x, y) \geq t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha}. \end{aligned} \quad (4.9)$$

Then, for every $f \in \text{lip}_0(X, d^\alpha)$, we have

$$\begin{aligned} &\sup_{0 < d(x, y) < t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \\ &\leq \sup_{\varphi(x) \neq \varphi(y)} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(\varphi(x), \varphi(y))^\alpha} \cdot \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \\ &\leq L_\alpha[(I - K_n)f] \cdot \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}, \end{aligned} \quad (4.10)$$

so that

$$\sup_{L_\alpha(f) \leq 1} \sup_{0 < d(x, y) < t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \leq \|I - K_n\| \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}. \quad (4.11)$$

Now, combining the above inequalities, we obtain

$$\begin{aligned} \|C_\varphi\|_e &\leq \|I - K_n\| \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} \\ &\quad + \sup_{L_\alpha(f) \leq 1} \sup_{d(x,y) \geq t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x,y)^\alpha}. \end{aligned} \quad (4.12)$$

Letting $n \rightarrow \infty$ and using Lemmas 4.3 and 4.4, we conclude that

$$\|C_\varphi\|_e \leq \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}. \quad (4.13)$$

Finally, taking limits as $t \rightarrow 0$ yields the desired inequality. $\square$

5. The Essential Norm of C_φ on $\text{Lip}_0(X, d^\alpha)$

Now we extend the estimates on the essential norm of a composition operator to the spaces $\text{Lip}_0(X, d^\alpha)$. Recall that $\text{lip}_0(X, d^\alpha)^{**}$ is isometrically isomorphic to $\text{Lip}_0(X, d^\alpha)$ whenever (X, d) is a pointed compact metric space and $\alpha \in (0, 1)$ (see [1, Theorem 3.3.3 and Proposition 3.2.2]). As a matter of fact, the mapping $\Delta : \text{lip}_0(X, d^\alpha)^{**} \rightarrow \text{Lip}_0(X, d^\alpha)$, defined by

$$\Delta(F)(x) = F(\delta_x), \quad (F \in \text{lip}_0(X, d^\alpha)^{**}, x \in X), \quad (5.1)$$

is an isometric isomorphism.

If T is a bounded linear operator on a Banach space, then $\|T^*\| = \|T\|$. However, this identity is no longer true for the essential norm. Since the adjoint of a compact operator is again a compact operator, we always have $\|T^*\|_e \leq \|T\|_e$ and therefore $\|T^{**}\|_e \leq \|T^*\|_e \leq \|T\|_e$. Axler et al. [12] showed that in fact $\|T^{**}\|_e = \|T^*\|_e$, but they gave a counterexample where $\|T^*\|_e < \|T\|_e$.

Theorem 5.1. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi : X \rightarrow X$ a point preserving Lipschitz mapping. Then the essential norm of the operator $C_\varphi : \text{Lip}_0(X, d^\alpha) \rightarrow \text{Lip}_0(X, d^\alpha)$ satisfies the lower estimate*

$$\|C_\varphi\|_e \geq \lim_{t \rightarrow 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}. \quad (5.2)$$

If, in addition, the space $\text{lip}_0(X, d^\alpha)^*$ has the approximation property, then one has the upper estimate

$$\|C_\varphi\|_e \leq \lim_{t \rightarrow 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}. \quad (5.3)$$

Proof. Let us start with the lower estimate. Let $\{f_n\}$ be the weakly null sequence in $\text{lip}_0(X, d^\alpha)$ that we constructed for the proof of Theorem 3.1. Then the sequence $\{f_n\}$ is weakly null in $\text{Lip}_0(X, d^\alpha)$. Thus, if K is any compact operator on $\text{Lip}_0(X, d^\alpha)$, we have $\lim_{n \rightarrow \infty} L_\alpha(Kf_n) = 0$. Hence, the same computation we performed in Theorem 3.1 yields the lower estimate

$$\lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \leq \|C_\varphi\|_e. \quad (5.4)$$

Now, for the upper estimate, given $F \in \text{lip}_0(X, d^\alpha)^{**}$ and $x \in X$, notice that

$$\begin{aligned} (\Delta^{-1}C_\varphi\Delta)(F)(\delta_x) &= ((C_\varphi\Delta)(F))(x) = C_\varphi(\Delta(F))(x) = \Delta(F)(\varphi(x)) \\ &= F(\delta_{\varphi(x)}) = F(\delta_x \circ C_\varphi|_{\text{lip}_0(X, d^\alpha)}) = (F \circ (C_\varphi|_{\text{lip}_0(X, d^\alpha)}))^*(\delta_x) \\ &= (C_\varphi|_{\text{lip}_0(X, d^\alpha)})^{**}(F)(\delta_x). \end{aligned} \quad (5.5)$$

Since $\text{lip}_0(X, d^\alpha)^* = \overline{\text{span}}\{\delta_x : x \in X\}$, we conclude that $\Delta^{-1}C_\varphi\Delta = (C_\varphi|_{\text{lip}_0(X, d^\alpha)})^{**}$. Finally, using the relationship between the essential norm of an operator and that of its second adjoint, and applying Theorem 4.1, we get

$$\|C_\varphi\|_e = \|\Delta^{-1}C_\varphi\Delta\|_e = \|(C_\varphi|_{\text{lip}_0(X, d^\alpha)})^{**}\|_e \leq \|C_\varphi|_{\text{lip}_0(X, d^\alpha)}\|_e \leq \lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}, \quad (5.6)$$

as we wanted. $\square$

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