Research Article

Approximate Best Proximity Pairs in Metric Space

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Let $A$ and $B$ be nonempty subsets of a metric space $X$ and also $T : A \cup B \rightarrow A \cup B$ and $T(A) \subseteq B$, $T(B) \subseteq A$. We are going to consider element $x \in A$ such that $d(x, Tx) \leq d(A, B) + \epsilon$ for some $\epsilon > 0$. We call pair $(A, B)$ an approximate best proximity pair. In this paper, definitions of approximate best proximity pair for a map and two maps, their diameters, $T$-minimizing a sequence are given in a metric space.

1. Introduction

Let $X$ be a metric space and $A$ and $B$ nonempty subsets of $X$, and $d(A, B)$ is distance of $A$ and $B$. If $d(x_0, y_0) = d(A, B)$, then the pair $(x_0, y_0)$ is called a best proximity pair for $A$ and $B$ and put

$$\text{prox}(A, B) := \{(x, y) \in A \times B : d(x, y) = d(A, B)\}$$

(1.1)

as the set of all best proximity pair $(A, B)$. Best proximity pair evolves as a generalization of the concept of best approximation. That reader can find some important result of it in [1–4].

Now, as in [5] (see also [4, 6–11]), we can find the best proximity points of the sets $A$ and $B$, by considering a map $T : A \cup B \rightarrow A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Best proximity pair also evolves as a generalization of the concept of fixed point of mappings. Because if $A \cap B \neq \emptyset$, every best proximity point is a fixed point of $T$.

We say that the point $x \in A \cup B$ is an approximate best proximity point of the pair $(A, B)$, if $d(x, Tx) \leq d(A, B) + \epsilon$, for some $\epsilon > 0$.

In the following, we introduce a concept of approximate proximity pair that is stronger than proximity pair.
Define 1.1. Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ a map such that $T(A) \subseteq B$, $T(B) \subseteq A$. Then

$$P^n_T(A, B) = \{x \in A \cup B : d(x, Tx) \leq d(A, B) + \varepsilon \text{ for some } \varepsilon > 0\}. \quad (1.2)$$

We say that the pair $(A, B)$ is an approximate best proximity pair if $P^n_T(A, B) \neq \emptyset$.

Example 1.2. Suppose that $X = \mathbb{R}^2$, $A = \{(x, y) \in X : (x - y)^2 + y^2 \leq 1\}$, and $B = \{(x, y) \in X : (x + y)^2 + y^2 \leq 1\}$ with $T(x, y) = (-x, y)$ for $(x, y) \in X$. Then $d((x, y), T(x, y)) \leq d(A, B) + \varepsilon$ for some $\varepsilon > 0$. Hence $P^n_T(A, B) \neq \emptyset$.

2. Approximate Best Proximity

In this section, we will consider the existence of approximate best proximity points for the map $T : A \cup B \to A \cup B$, such that $T(A) \subseteq B$, $T(B) \subseteq A$, and its diameter.

Theorem 2.1. Let $A$ and $B$ be nonempty subsets of a metric space $X$. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$, and

$$\lim_{n \to \infty} d\left(T^n x, T^{n+1} x\right) = d(A, B) \quad \text{for some } x \in A \cup B. \quad (2.1)$$

Then the pair $(A, B)$ is an approximate best proximity pair.

Proof. Let $\varepsilon > 0$ be given and $x \in A \cup B$ such that $\lim_{n \to \infty} d(T^n x, T^{n+1} x) = d(A, B)$; then there exists $N_0 > 0$ such that

$$\forall n \geq N_0 : d\left(T^n x, T^{n+1} x\right) < d(A, B) + \varepsilon. \quad (2.2)$$

If $n = N_0$, then $d(T^{N_0} (x), T(T^{N_0} (x))) < d(A, B) + \varepsilon$, and $T^{N_0} (x) \in P^n_T(A, B)$ and $P^n_T(A, B) \neq \emptyset$. \hfill \Box

Theorem 2.2. Let $A$ and $B$ be nonempty subsets of a metric space $X$. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma d(A, B) \quad (2.3)$$

for all $x, y \in A \cup B$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + \gamma < 1$. Then the pair $(A, B)$ is an approximate best proximity pair.

Proof. If $x \in A \cup B$, then

$$d\left(Tx, T^2x\right) \leq \alpha d(x, Tx) + \beta[d(x, Tx) + d(Tx, T^2x)] + \gamma d(A, B). \quad (2.4)$$
Therefore,
\[ d(Tx, T^2x) \leq \frac{\alpha + \beta}{1 - \beta} d(x, Tx) + \frac{\gamma}{1 - \beta} d(A, B). \] (2.5)

Now if \( k = (\alpha + \beta)/(1 - \beta) \), then
\[ d(Tx, T^2x) \leq kd(x, Tx) + (1 - k)d(A, B) \] (2.6)
also
\[ d(T^2x, T^3x) \leq k^2d(x, Tx) + (1 - k^2)d(A, B). \] (2.7)

Therefore,
\[ d(T^n x, T^{n+1} x) \leq k^n d(x, Tx) + (1 - k^n)d(A, B), \] (2.8)
and so
\[ d(T^n x, T^{n+1} x) \to d(A, B), \text{ as } n \to \infty. \] (2.9)

Therefore, by Theorem 2.1, \( P^n_T(A, B) \neq \emptyset \); then pair \((A, B)\) is an approximate best proximity pair.

**Definition 2.3.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \). Suppose that the mapping \( T : A \cup B \to A \cup B \) is satisfying \( T(A) \subseteq B, T(B) \subseteq A \). We say that the sequence \( \{z_n\} \subseteq A \cup B \) is \( T \)-minimizing if
\[ \lim_{n \to \infty} d(z_n, Tz_n) = d(A, B). \] (2.10)

**Theorem 2.4.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \), suppose that the mapping \( T : A \cup B \to A \cup B \) is satisfying \( T(A) \subseteq B, T(B) \subseteq A \). If \( \{T^n x\} \) is a \( T \)-minimizing for some \( x \in A \cup B \), then \((A, B)\) is an approximate best pair proximity.

**Proof.** Since
\[ \lim_{n \to \infty} d(T^n x, T^{n+1} x) = d(A, B) \quad \text{for some } x \in A \cup B, \] (2.11)
therefore, by Theorem 2.1, \( P^n_T(A, B) \neq \emptyset \); then pair \((A, B)\) is an approximate best proximity pair. \(\square\)
Theorem 2.5. Let $A$ and $B$ be nonempty subsets of a normed space $X$ such that $A \cup B$ is compact. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$, $T$ is continuous and

$$
\|Tx - Ty\| \leq \|x - y\|, 
$$

(2.12)

where $(x, y) \in A \times B$. Then $P_T^A(A, B)$ is nonempty and compact.

Proof. Since $A \cup B$ compact, there exists a $z_0 \in A \cup B$ such that

$$
\|z_0 - Tz_0\| = \inf_{z \in A \cup B} \|z - Tz\|.
$$

(\ast)

If $\|z_0 - Tz_0\| > d(A, B)$, then $\|Tz_0 - T^2z_0\| < \|z_0 - Tz_0\|$ which contradict to the definition of $z_0$, ($Tz_0 \in A \cup B$ and by (\ast) $\|Tz_0 - T(Tz_0)\| \geq \|z_0 - Tz_0\|$). Therefore, $\|z_0 - Tz_0\| = d(A, B) \leq d(A, B) + \epsilon$ for some $\epsilon > 0$ and $z_0 \in P_T^A(A, B)$. Therefore, $P_T^A(A, B)$ is nonempty.

Also, if $\{z_n\} \subseteq P_T^A(A, B)$, then $\|z_n - Tz_n\| < d(A, B) + \epsilon$, for some $\epsilon > 0$, and by compactness of $A \cup B$, there exists a subsequence $z_{n_k}$ and a $z_0 \in A \cup B$ such that $z_{n_k} \to z_0$ and so

$$
\|z_0 - Tz_0\| = \lim_{k \to \infty} \|z_{n_k} - Tz_{n_k}\| < d(A, B) + \epsilon
$$

(2.13)

for some $\epsilon > 0$, hence $P_T^A(A, B)$ is compact. \hfill \Box

Example 2.6. If $A = [-3, -1], B = [1, 3]$, and $T : A \cup B \to A \cup B$ such that

$$
T(x) = \begin{cases} 
1 - x \over 2, & x \in A, \\
-1 - x \over 2, & x \in B,
\end{cases}
$$

(2.14)

then $P_T^A(A, B)$ is compact, and we have

$$
P_T^A(A, B) = \{ x \in A \cup B : d(x, Tx) < d(A, B) + \epsilon \text{ for some } \epsilon > 0 \}
$$

(2.15)

$$
= \{ x \in A \cup B : d(x, Tx) < 2 + \epsilon \text{ for some } \epsilon > 0 \}
$$

$$
= \{ 1, -1 \}.
$$

That is compact.

In the following, by diam($P_T^A(A, B)$) for a set $P_T^A(A, B) \neq \emptyset$, we will understand the diameter of the set $P_T^A(A, B)$. 


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Definition 2.7. Let \( T : A \cup B \rightarrow A \cup B \) be a continuous map such that \( T(A) \subseteq B, T(B) \subseteq A \) and \( \epsilon > 0 \). We define diameter \( P_T^\alpha(A,B) \) by

\[
\text{diam}(P_T^\alpha(A,B)) = \sup \{ d(x,y) : x,y \in P_T^\alpha(A,B) \}. \tag{2.16}
\]

Theorem 2.8. Let \( T : A \cup B \rightarrow A \cup B \), such that \( T(A) \subseteq B, T(B) \subseteq A \) and \( \epsilon > 0 \). If there exists an \( \alpha \in [0,1] \) such that for all \( (x,y) \in A \times B \)

\[
d(Tx,Ty) \leq ad(x,y), \tag{2.17}
\]

then

\[
\text{diam}(P_T^\alpha(A,B)) \leq \frac{2\epsilon}{1-\alpha} + \frac{2d(A,B)}{1-\alpha}. \tag{2.18}
\]

Proof. If \( x,y \in P_T^\alpha(A,B) \), then

\[
d(x,y) \leq d(x,Tx) + d(Tx,Ty) + d(Ty,y) \leq \epsilon_1 + ad(x,y) + 2d(A,B) + \epsilon_2.
\]

Put \( \epsilon = \text{Max}\{\epsilon_1, \epsilon_2\} \), therefore, \( d(x,y) \leq 2\epsilon/(1-\alpha) + (2d(A,B))/(1-\alpha) \). Hence \( \text{diam}(P_T^\alpha(A,B)) \leq 2\epsilon/(1-\alpha) + (2d(A,B))/(1-\alpha) \).

3. Approximate Best Proximity for Two Maps

In this section, we will consider the existence of approximate best proximity points for two maps \( T : A \cup B \rightarrow A \cup B \) and \( S : A \cup B \rightarrow A \cup B \), and its diameter.

Definition 3.1. Let \( A \) and \( B \) be nonempty subsets of a metric space \( (X,d) \) and let \( T : A \cup B \rightarrow A \cup B \) and \( S : A \cup B \rightarrow A \cup B \) two maps such that \( T(A) \subseteq B, S(B) \subseteq A \). A point \( (x,y) \in A \times B \) is said to be an approximate-pair fixed point for \( (T,S) \) in \( X \) if there exists \( \epsilon > 0 \)

\[
d(Tx,Sy) \leq d(A,B) + \epsilon. \tag{3.1}
\]

We say that the pair \( (T,S) \) has the approximate-pair fixed property in \( X \) if \( P_{(T,S)}^\alpha(A,B) \neq \emptyset \), where

\[
P_{(T,S)}^\alpha(A,B) = \{(x,y) \in A \times B : d(Tx,Sy) \leq d(A,B) + \epsilon \text{ for some } \epsilon > 0\}. \tag{3.2}
\]

Theorem 3.2. Let \( A \) and \( B \) be nonempty subsets of a metric space \( (X,d) \) and let \( T : A \cup B \rightarrow A \cup B \) and \( S : A \cup B \rightarrow A \cup B \) be two maps such that \( T(A) \subseteq B, S(B) \subseteq A \). If, for every \( (x,y) \in A \times B \),

\[
d(T^n(x),S^n(y)) \rightarrow d(A,B), \tag{3.3}
\]

then \( (T,S) \) has the approximate-pair fixed property.
Proof. For \( \varepsilon > 0 \), suppose \((x, y) \in A \times B\). Since

\[
d(T^n(x), S^n(y)) \to d(A, B),
\]

there exists a \( \varepsilon > 0 \) such that for every \( n \geq n_0 \),

\[
d(T^n(x), S^n(y)) < d(A, B) + \varepsilon,
\]

then \( d(T(T^{n-1}(x), S(S^{n-1}(y)) < d(A, B) + \varepsilon \) for every \( n \geq n_0 \). Put \( x_0 = T^{n_0-1}(x) \) and \( y_0 = S^{n_0-1}(y) \). Hence \( d(T(x_0), S(y_0)) \leq d(A, B) + \varepsilon \) and \( P^a_{(T, S)}(A, B) \neq \emptyset \).

**Theorem 3.3.** Let \( A \) and \( B \) be nonempty subsets of a metric space \((X, d)\) and let \( T : A \cup B \to A \cup B \) and \( S : A \cup B \to A \cup B \) be two maps such that \( T(A) \subseteq B \), \( S(B) \subseteq A \) and, for every \((x, y) \in A \times B\),

\[
d(Tx, Sy) \leq \alpha d(x, y) + \beta [d(x, Tx) + d(y, Sy)] + \gamma d(A, B),
\]

where \( \alpha, \beta, \gamma \geq 0 \) and \( \alpha + 2\beta + \gamma < 1 \). Then if \( x \) is an approximate fixed point for \( T \), or \( y \) is an approximate fixed point for \( S \), then \( P^a_{(T, S)}(A, B) \neq \emptyset \).

Proof. If \((x, y) \in A \times B\), then

\[
d(Tx, S(Tx)) \leq \alpha d(x, Tx) + \beta [d(x, Tx) + d(Tx, S(Tx))] + \gamma d(A, B).
\]

Therefore,

\[
d(Tx, S(Tx)) \leq \frac{\alpha + \beta}{1 - \beta} d(x, Tx) + \frac{\gamma}{1 - \beta} d(A, B).
\]

Now if \( k = (\alpha + \beta)/(1 - \beta) \), then

\[
d(Tx, S(Tx)) \leq kd(x, Tx) + (1 - k)d(A, B) \tag{\star}
\]

also

\[
d(Sy, T(Sy)) \leq kd(y, Sy) + (1 - k)d(A, B) \tag{\star\star}
\]

If \( x \) is an approximate fixed point for \( T \), then there exists a \( \varepsilon > 0 \) and by \((\star)\)

\[
d(Tx, S(Tx)) \leq kd(x, Tx) + (1 - k)d(A, B)
\]

\[
\leq k(d(A, B) + \varepsilon) + (1 - k)d(A, B)
\]

\[
= d(A, B) + k\varepsilon
\]

\[
< d(A, B) + \varepsilon.
\]
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And \((x, Tx) \in P_{(T,S)}^a(A, B)\); also if \(y\) is an approximate fixed point for \(S\), then there exists a \(\epsilon > 0\) and by (**)

\[
d(Sy, T(Sy)) \leq kd(y, Sy) + (1 - k)d(A, B)
\]

\[
\leq k(d(A, B) + \epsilon) + (1 - k)d(A, B)
\]

\[
= d(A, B) + k\epsilon
\]

\[
< d(A, B) + \epsilon.
\]

And \((y, Sy) \in P_{(T,S)}^a(A, B)\). Therefore, \(P_{(T,S)}^a(A, B) \neq \emptyset\).

**Theorem 3.4.** Let \(A\) and \(B\) be nonempty subsets of a metric space \((X, d)\) and let \(T : A \cup B \to A \cup B\) and \(S : A \cup B \to A \cup B\) be two continuous maps such that \(T(A) \subseteq B\), \(S(B) \subseteq A\). If, for every \((x, y) \in A \times B\),

\[
d(Tx, Sy) \leq \alpha d(x, y) + \gamma d(A, B),
\]

where \(\alpha, \gamma \geq 0\) and \(\alpha + \gamma = 1\), also let \(\{x_n\}\) and \(\{y_n\}\) be as follows:

\[
x_{n+1} = Sy_n, \quad y_{n+1} = Tx_n \quad \text{for some} \quad (x_1, y_1) \in A \times B, \quad n \in \mathbb{N}.
\]

If \(\{x_n\}\) has a convergent subsequence in \(A\), then there exists a \(x_0 \in A\) such that \(d(x_0, Tx_0) = d(A, B)\).

**Proof.** We have

\[
d(x_{n+1}, y_{n+1}) = d(Tx_n, Sy_n)
\]

\[
\leq \alpha d(x_n, y_n) + \gamma d(A, B)
\]

\[
\leq \cdots
\]

\[
\leq \alpha^{n+1} d(x_0, y_0) + (1 + \alpha + \cdots + \alpha^n)\gamma d(A, B).
\]

If \(\{x_{n_k}\}_{k \geq 1}\) converges to \(x_1 \in A\), that is, \(x_{n_k} \to x_1\), then

\[
d(x_{n_{k+1}}, y_{n_{k+1}}) \leq \alpha^{n_{k+1}} d(x_0, y_0) + (1 + \alpha + \cdots + \alpha^{n_k})\gamma d(A, B).
\]

Since \(T\) is continuous, then

\[
d(x_{n_{k+1}}, Tx_{n_k}) \to \frac{\gamma}{1 - \alpha} d(A, B) = d(A, B).
\]

Therefore, \(d(x_1, Tx_1) = d(A, B)\). \(\square\)
Definition 3.5. Let \( T : A \cup B \to A \cup B \) and \( S : A \cup B \to A \cup B \) be continuous maps such that \( T(A) \subseteq B \) and \( S(B) \subseteq A \). We define diameter \( \text{diam} \left( P_{(T,S)}^a(A,B) \right) \) by

\[
\text{diam} \left( P_{(T,S)}^a(A,B) \right) = \sup \left\{ d(x,y) : d(Tx, Ty) \leq \epsilon + d(A,B) \text{ for some } \epsilon > 0 \right\}.
\] (3.15)

Example 3.6. Suppose \( A = \{(x,0) : 0 \leq x \leq 1\} \), \( B = \{(x,1) : 0 \leq x \leq 1\} \), \( T(x,0) = T(x,1) = (1/2, 1) \), and \( S(x,1) = S(x,0) = (1/2, 0) \). Then \( d(T(x,0), S(y,1)) = 1 \) and \( \text{diam} \left( P_{(T,S)}^a(A,B) \right) = \text{diam}(A \times B) = \sqrt{2} \).

Theorem 3.7. Let \( T : A \cup B \to A \cup B \) and \( S : A \cup B \to A \cup B \) be continuous maps such that \( T(A) \subseteq B \), \( S(B) \subseteq A \). If there exists a \( k \in [0,1] \),

\[
d(x,Tx) + d(Sy,y) \leq kd(x,y),
\] (3.16)

then

\[
\text{diam} \left( P_{(T,S)}^a(A,B) \right) \leq \frac{\epsilon}{1-k} + \frac{d(A,B)}{1-k} \text{ for some } \epsilon > 0.
\] (3.17)

Proof. If \( (x,y) \in P_{(T,S)}^a(A,B) \), then

\[
d(x,y) \leq d(x,Tx) + d(Tx, Sy) + d(Sy,y)
\leq \epsilon + kd(x,y) + d(A,B).
\] (3.18)

Therefore, \( d(x,y) \leq \epsilon/(1-k) + (d(A,B))/(1-k) \). Then \( \text{diam}(P_{(T,S)}^a(A,B)) \leq \epsilon/(1-k) + (d(A,B))/(1-k) \).

\[ \square \]

References


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