Research Article

A de Casteljau Algorithm for q -Bernstein-Stancu Polynomials

Grzegorz Nowak

*The Great Poland University of Social and Economics in Środa Wielkopolska, Paderewskiego 27,
63-000 Środa Wielkopolska, Poland*

Correspondence should be addressed to Grzegorz Nowak, grzegnow2@gmail.com

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This paper is concerned with a generalization of the q -Bernstein polynomials and Stancu operators, where the function is evaluated at intervals which are in geometric progression. It is shown that these polynomials can be generated by a de Casteljau algorithm, which is a generalization of that relating to the classical case and q -Bernstein case.

1. Introduction

Let $q > 0$. For any fixed real number $q > 0$ and for $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, the q -integers of the number $[n]$ are defined by

$$[n] = \frac{(1 - q^n)}{(1 - q)}, \quad \text{for } q \neq 1, \quad [n] = n, \quad \text{for } q = 1. \quad (1.1)$$

The q -factorial $[n]!$, for $n \in N_0 = \{0, 1, 2, \dots\}$, is defined by

$$[n]! = [1][2] \cdots [n] \quad (n = 1, 2, \dots), \quad [0]! = 1. \quad (1.2)$$

For the integers n, k , ($n \geq k \geq 0$), the q -binomial or the Gaussian coefficients are defined by (see [1, page 12])

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}. \quad (1.3)$$

For $f \in C[0;1]$, $q > 0$, $\alpha \geq 0$ and each positive integer n , we introduce (see [2]) the following generalized q -Bernstein operators:

$$B_n^{q,\alpha}(f; x) = \sum_{k=0}^n p_{n,k}^{q,\alpha}(x) f\left(\frac{[k]}{[n]}\right), \quad (1.4)$$

where

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{\prod_{i=0}^{k-1} (x + \alpha[i]) \prod_{s=0}^{n-1-k} (1 - q^s x + \alpha[s])}{\prod_{i=0}^{n-1} (1 + \alpha[i])}. \quad (1.5)$$

Note, that an empty product in (1.5) denotes 1. In the case where $\alpha = 0$, $B_n^{q,\alpha}(f; x)$ reduces to the well-known q -Bernstein polynomials introduced by Phillips [3, 4] in 1997

$$B_{n,q}(f; x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{i=0}^{n-k-1} (1 - q^i x) f\left(\frac{[k]}{[n]}\right). \quad (1.6)$$

In the case where $q = 1$, $B_n^{q,\alpha}(f; x)$ reduces to Bernstein-Stancu polynomials, introduced by Stancu [5] in 1968

$$S_n(f; x) = \sum_{k=0}^n \binom{n}{k} \frac{\prod_{i=0}^{k-1} (x + \alpha i) \prod_{s=0}^{n-k-1} (1 - x + s\alpha)}{\prod_{i=0}^{n-1} (1 + i\alpha)} f\left(\frac{k}{n}\right). \quad (1.7)$$

When $q = 1$ and $\alpha = 0$, we obtain the classical Bernstein polynomial defined by

$$B_n(f; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \quad (1.8)$$

Basic facts on Bernstein polynomials, their generalizations, and applications can be found for example in [6–8]. In recent years, the q -Bernstein polynomials have attracted much interest, and a great number of interesting results related to the $B_{n,q}(f)$ polynomials have been obtained (see [3, 4, 9–12]). Some approximation properties of the Stancu operators are presented in [5, 13–15].

Let $\Delta_q^0 f_j = f_j$, for $j = 0, 1, \dots, n$, and recursively,

$$\Delta_q^{k+1} f_j = \Delta_q^k f_{j+1} - q^k \Delta_q^k f_j, \quad (1.9)$$

for $k = 0, 1, \dots, n - j - 1$ and $f_j = f([j]/[n])$. It is easily established by induction that q -differences satisfy the relation

$$\Delta_q^k f_j = \sum_{i=0}^k (-1)^i q^{i(i-1)/2} \begin{bmatrix} k \\ i \end{bmatrix} f_{j+k-i}. \quad (1.10)$$

In [2], we prove that the operators $B_n^{q,\alpha}(f; x)$ defined by (1.4) can be expressed in terms of q -differences

$$B_n^{q,\alpha} f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \Delta_q^k f_0 \prod_{i=0}^{k-1} \frac{x + \alpha[i]}{1 + \alpha[i]}, \quad (1.11)$$

which generalized the well-known result [3, 4] for the q -Bernstein polynomial. In this paper, we show that polynomials defined by (1.4) can be generated by a de Casteljau algorithm, which is a generalization of that relating to the classical case [16] and q -Bernstein case [4, 11].

2. Auxiliary Results

We note that $B_n^{q,\alpha}(f; x)$ defined by (1.4), is a monotone linear operator for any $0 < q \leq 1$ and $\alpha \geq 0$. These operators reproduces linear functions [2], that is,

$$B_n^{q,\alpha}(ax + b; x) = ax + b, \quad a, b \in R. \quad (2.1)$$

They also satisfy the end point interpolation conditions $B_n^{q,\alpha}(f; 0) = f(0)$ and $B_n^{q,\alpha}(f; 1) = f(1)$. These properties are significant in designing curves and surfaces.

Moreover, the following holds.

Lemma 2.1. *Let $0 < q \leq 1$, $\alpha \geq 0$. Then,*

$$\prod_{u=0}^{m-1} (q^r - q^u x + \alpha([u] - [r])) = \sum_{s=0}^m (-1)^s q^{s(s-1)/2 + (m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]), \quad (2.2)$$

for all $m \in N$, $r \in N_0 = N \cup \{0\}$ and $x \in [0; 1]$.

Proof. We use induction on m . First, we see from equality $[-r] = -q^{-r}[r]$, ($r \in N$), that (2.2) is evident for $m = 1$. Let us assume that (2.2) holds for a given $m \in N$. Then, using (2.2), we obtain

$$\begin{aligned}
& \prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r])) \\
&= (q^r - q^m x + \alpha([m] - [r])) \sum_{s=0}^m (-1)^s q^{s(s-1)/2 + (m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \\
&\quad \cdot \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]) \\
&= \sum_{s=0}^m (-1)^s q^{s(s-1)/2 + (m-s)r} (q^r + \alpha[m] - \alpha[r] + \alpha q^m[s]) \begin{bmatrix} m \\ s \end{bmatrix} \\
&\quad \cdot \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]) \\
&\quad + \sum_{s=1}^{m+1} (-1)^s q^{s(s-1)(s-2)/2 + (m-s+1)r+m} (1 + \alpha[s-r-1]) \begin{bmatrix} m \\ s-1 \end{bmatrix} \\
&\quad \cdot \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]) \\
&= q^{mr} (q^r + \alpha[m] - \alpha[r]) \prod_{j=-r}^{m-r-1} (1 + \alpha[j]) \\
&\quad + (-1)^{m+1} q^{m(m-1)/2+m} (1 + \alpha[m-r]) \prod_{i=0}^m (x + \alpha[i]) \\
&\quad + \sum_{s=1}^m (-1)^s q^{s(s-1)/2 + (m+1-s)r} U_s \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]),
\end{aligned} \tag{2.3}$$

where

$$U_s = \begin{bmatrix} m \\ s \end{bmatrix} (q^r + \alpha[m] - \alpha[r] + \alpha q^m[s]) q^{-r} + q^{m-s+1} \begin{bmatrix} m \\ s-1 \end{bmatrix} (1 + \alpha[s-r-1]). \tag{2.4}$$

Using the obvious equalities

$$\begin{aligned} (q^r + \alpha[m] - \alpha[r])q^{-r} &= 1 + \alpha[m - r], \\ \begin{bmatrix} m \\ s \end{bmatrix} [s] &= \begin{bmatrix} m \\ s-1 \end{bmatrix} [m - s + 1], \end{aligned} \quad (2.5)$$

we have

$$\begin{aligned} U_s &= \begin{bmatrix} m \\ s \end{bmatrix} (1 + \alpha[m - r]) \\ &+ \begin{bmatrix} m \\ s-1 \end{bmatrix} q^{m-s+1} \left(1 + \alpha \left([m - s + 1] q^{s-r-1} + [s - r - 1] \right) \right). \end{aligned} \quad (2.6)$$

It is easy to see that

$$\begin{aligned} [m - s + 1] q^{s-r-1} + [s - r - 1] &= [m - r], \\ \begin{bmatrix} m \\ s \end{bmatrix} + \begin{bmatrix} m \\ s-1 \end{bmatrix} q^{m-s+1} &= \begin{bmatrix} m+1 \\ s \end{bmatrix}. \end{aligned} \quad (2.7)$$

Therefore,

$$U_s = (1 + \alpha[m - r]) \begin{bmatrix} m+1 \\ s \end{bmatrix}. \quad (2.8)$$

From last equality and (2.3), we obtain

$$\begin{aligned} &\prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r])) \\ &= q^{mr} (q^r + \alpha[m] - \alpha[r]) \prod_{j=-r}^{m-r-1} (1 + \alpha[j]) \\ &+ (-1)^{m+1} q^{m(m-1)/2+m} (1 + \alpha[m - r]) \prod_{i=0}^m (x + \alpha[i]) \\ &+ \sum_{s=1}^m (-1)^s q^{s(s-1)/2+(m+1-s)r} \begin{bmatrix} m+1 \\ s \end{bmatrix} (1 + \alpha[m - r]) \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]) \\ &= \sum_{s=0}^{m+1} (-1)^s q^{s(s-1)/2+(m+1-s)r} \begin{bmatrix} m+1 \\ s \end{bmatrix} \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r} (1 + \alpha[j]). \end{aligned} \quad (2.9)$$

This completes the proof of the lemma. $\square$

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input:  $q; f([0]/[n]), f([1]/[n]), \dots, f([n]/[n])$ 
for  $r = 0$  to  $n$ 
     $f_r^{[0]} := f\left(\frac{[r]}{[n]}\right)$ 
next  $r$ 
for  $m = 1$  to  $n$ 
    for  $r = 0$  to  $n - m$ 
         $f_r^{[m]} := \frac{\{(q^r - q^{m-1}x + \alpha([m-1] - [r]))f_r^{[m-1]} + (x + \alpha[r])f_{r+1}^{[m-1]}\}}{1 + \alpha[m-1]}$ 
    next  $r$ 
next  $m$ 

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Algorithm 1: De Casteljaeu type algorithm.

3. Main Result

The generalized q -Bernstein polynomials, defined by (1.4), may be evaluated by Algorithm 1.

In the case, where $\alpha = 0$, this is the de Casteljaeu algorithm for evaluating the q -Bernstein polynomial [3, 4]. Note that with $q = 1$ and $\alpha = 0$, we recover the original classical de Casteljaeu algorithm (see Hoschek and Lasser [16]). The algorithm is justified by the following theorem.

Theorem 3.1. *Each intermediate point $f_r^{[m]}$ of the algorithm can be expressed as*

$$\begin{aligned}
 f_r^{[m]} = & \left(\prod_{i=0}^{m-1} (1 + \alpha[i]) \right)^{-1} \\
 & \cdot \sum_{t=0}^m f_{r+t} \binom{m}{t} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])),
 \end{aligned} \tag{3.1}$$

and, in particular

$$f_0^{[n]} = B_n^{q,\alpha}(f; x). \tag{3.2}$$

Proof. We use induction on m . From the initial conditions in the algorithm, $f_r^{[0]} = f([r]/[n]) = f_r$, $0 \leq r \leq n$, it is clear that (3.1) holds for $m = 0$ and $0 \leq r \leq n$. Let us assume that (3.1) holds for some m such that $0 \leq m < n$, and for all r such that $0 \leq r \leq n - m$. Then, for $0 \leq r \leq n - m - 1$, it follows from the algorithm that

$$f_r^{[m+1]} := \left\{ (q^r - q^m x + \alpha([m] - [r]))f_r^{[m]} + (x + \alpha[r])f_{r+1}^{[m]} \right\} \frac{1}{1 + \alpha[m]}, \tag{3.3}$$

and using (3.1), we obtain

$$\begin{aligned}
f_r^{[m+1]} \left(\prod_{i=0}^m (1 + \alpha[i]) \right) &:= (q^r - q^m x + \alpha([m] - [r])) \\
&\cdot \sum_{t=0}^m f_{r+t} \begin{bmatrix} m \\ t \end{bmatrix} \prod_{s=0}^{t-1} (x + \alpha[r + s]) \cdot \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) \\
&+ (x + \alpha[r]) \cdot \sum_{t=0}^m f_{r+t+1} \begin{bmatrix} m \\ t \end{bmatrix} \prod_{s=0}^{t-1} (x + \alpha[r + s + 1]) \\
&\cdot \prod_{u=0}^{m-t-1} (q^{r+1} - q^u x + \alpha([u] - [r + 1])) \\
&= (q^r - q^m x + \alpha([m] - [r])) f_r \prod_{u=0}^{m-1} (q^r - q^u x + \alpha([u] - [r])) \\
&+ (q^r - q^m x + \alpha([m] - [r])) \cdot \sum_{t=1}^m f_{r+t} \begin{bmatrix} m \\ t \end{bmatrix} \\
&\cdot \prod_{s=0}^{t-1} (x + \alpha[r + s]) \cdot \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) \\
&+ (x + \alpha[r]) \cdot \sum_{t=1}^m f_{r+t} \begin{bmatrix} m \\ t-1 \end{bmatrix} \prod_{s=0}^{t-2} (x + \alpha[r + s + 1]) \\
&\cdot \prod_{u=0}^{m-t} (q^{r+1} - q^u x + \alpha([u] - [r + 1])) \\
&+ (x + \alpha[r]) f_{r+m+1} \prod_{s=0}^{m-1} (x + \alpha[r + s + 1]) \\
&= f_r \prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r])) \\
&+ \sum_{t=1}^m \left\{ (q^r - q^m x + \alpha([m] - [r])) \begin{bmatrix} m \\ t \end{bmatrix} \right. \\
&\quad \cdot \prod_{s=0}^{t-1} (x + \alpha[r + s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) \\
&\quad + (x + \alpha[r]) \begin{bmatrix} m \\ t-1 \end{bmatrix} \prod_{s=0}^{t-2} (x + \alpha[r + s + 1]) \\
&\quad \left. \prod_{u=0}^{m-t} (q^{r+1} - q^u x + \alpha([u] - [r + 1])) \right\} f_{r+t} \\
&+ f_{r+m+1} \prod_{s=0}^m (x + \alpha[r + s]).
\end{aligned} \tag{3.4}$$

We see that

$$\begin{aligned}
& \prod_{u=0}^{m-t} (q^{r+1} - q^u x + \alpha([u] - [r+1])) \\
&= (q^{r+1} - x - \alpha[r+1]) \prod_{u=0}^{m-t-1} (q^{r+1} - q^{u+1} x + \alpha([u+1] - [r+1])) \\
&= (q^{r+1} - x - \alpha[r+1]) \prod_{u=0}^{m-t-1} (q^{r+1} - q^{u+1} x + \alpha q([u] - [r])) \\
&= (q^{r+1} - x - \alpha[r+1]) q^{m-t} \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])),
\end{aligned} \tag{3.5}$$

and hence,

$$\begin{aligned}
& f_r^{[m+1]} \left(\prod_{i=0}^m (1 + \alpha[i]) \right) \\
&:= f_r \prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r])) \\
&+ \sum_{t=1}^m \left\{ \begin{bmatrix} m \\ t \end{bmatrix} (q^r - q^m x + \alpha([m] - [r])) + \begin{bmatrix} m \\ t-1 \end{bmatrix} (q^{r+1} - x - \alpha[r+1]) q^{m-t} \right\} \\
&\cdot f_{r+t} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) + f_{r+m+1} \prod_{s=0}^m (x + \alpha[r+s]).
\end{aligned} \tag{3.6}$$

It is easy to verify that

$$\begin{aligned}
\begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1} \begin{bmatrix} m \\ t-1 \end{bmatrix} &= \begin{bmatrix} m+1 \\ t \end{bmatrix}, \\
\begin{bmatrix} m \\ t-1 \end{bmatrix} + q^t \begin{bmatrix} m \\ t \end{bmatrix} &= \begin{bmatrix} m+1 \\ t \end{bmatrix}.
\end{aligned} \tag{3.7}$$

Therefore,

$$\begin{aligned}
& \begin{bmatrix} m \\ t \end{bmatrix} (q^r - q^m x + \alpha([m] - [r])) + \begin{bmatrix} m \\ t-1 \end{bmatrix} (q^{r+1} - x - \alpha[r+1]) q^{m-t} \\
&= q^r \left(\begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1} \begin{bmatrix} m \\ t-1 \end{bmatrix} \right) - x q^{m-t} \left(\begin{bmatrix} m \\ t-1 \end{bmatrix} + q^t \begin{bmatrix} m \\ t \end{bmatrix} \right) \\
&+ \frac{\alpha}{1-q} \left\{ q^r \left(\begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1} \begin{bmatrix} m \\ t-1 \end{bmatrix} \right) - q^{m-t} \left(\begin{bmatrix} m \\ t-1 \end{bmatrix} + q^t \begin{bmatrix} m \\ t \end{bmatrix} \right) \right\} \\
&= \begin{bmatrix} m+1 \\ t \end{bmatrix} \{ (q^r - x q^{m-t}) + \alpha([m-t] - [r]) \}.
\end{aligned} \tag{3.8}$$

Consequently,

$$\begin{aligned}
f_r^{[m+1]} \left(\prod_{i=0}^m (1 + \alpha[i]) \right) &:= f_r \prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r])) \\
&+ \sum_{t=1}^m \begin{bmatrix} m+1 \\ t \end{bmatrix} \{ (q^r - x q^{m-t}) + \alpha([m-t] - [r]) \} \\
&\cdot f_{r+t} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) \\
&+ f_{r+m+1} \prod_{s=0}^m (x + \alpha[r+s]) \\
&= \sum_{t=0}^{m+1} \begin{bmatrix} m+1 \\ t \end{bmatrix} \cdot f_{r+t} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t} (q^r - q^u x + \alpha([u] - [r])).
\end{aligned} \tag{3.9}$$

Thus, one has the desired result. $\square$

Theorem 3.2. For $0 \leq m \leq n$ and $0 \leq r \leq n - m$, we have

$$f_r^{[m]} = \sum_{s=0}^m q^{(m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \Delta_q^s f_r \frac{\prod_{i=r}^{s+r-1} (x + \alpha[i])}{\prod_{j=0}^{s-1} (1 + \alpha[j])}, \tag{3.10}$$

for all $x \in [0; 1]$.

Proof. Using (2.2) and (3.1), we have

$$f_r^{[m]} \prod_{i=0}^{m-1} (1 + \alpha[i]) = \sum_{t=0}^m \begin{bmatrix} m \\ t \end{bmatrix} f_{r+t} S_t(m), \tag{3.11}$$

where

$$\begin{aligned}
 S_t(m) &= \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \begin{bmatrix} m-t \\ u \end{bmatrix} \\
 &\times \prod_{s=r}^{t+r-1} (x + \alpha[s]) \prod_{i=0}^{u-1} (x + \alpha[i]) \prod_{j=u-r}^{m-t-r-1} (1 + \alpha[j]) \quad (0 \leq t \leq m).
 \end{aligned} \tag{3.12}$$

First, we prove that

$$S_t(m) = \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \begin{bmatrix} m-t \\ u \end{bmatrix} \cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]) \tag{3.13}$$

for all $m \in N_0 = \{0, 1, 2, \dots\}$, $t \in N_0$, and $x \in [0; 1]$. Note that an empty sum denotes 0.

We use the induction on m . First, we see that (3.13) holds for $m = 0$ and all $t \in N_0$. Let us assume that (3.13) holds for a given m , and for all $t \in N_0$. Then, from (3.12) and (3.13), we obtain

$$\begin{aligned}
 S_t(m+1) &= (x + \alpha[t+r-1]) \sum_{u=0}^{m+1-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
 &\cdot \prod_{i=r}^{t+u-2+r} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]) \\
 &= \sum_{u=0}^{m+1-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
 &\cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]) \\
 &+ \alpha \sum_{u=1}^{m-t+1} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
 &\cdot ([t+r+1] - [t+u+r+1]) \prod_{i=r}^{t+u-2+r} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]).
 \end{aligned} \tag{3.14}$$

We see that

$$\begin{bmatrix} m-t+1 \\ u \end{bmatrix} ([t+r-1] - [t+u+r-1]) = -q^{t+r-1} [m-t-u+2] \begin{bmatrix} m-t+1 \\ u-1 \end{bmatrix}, \tag{3.15}$$

and hence,

$$\begin{aligned}
S_t(m+1) &= \sum_{u=0}^{m-t+1} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
&\quad \cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]) \\
&\quad + \alpha \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} q^{u+t-1} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
&\quad \cdot [m-t-u+1] \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]) \\
&= (-1)^{m-t+1} q^{(m-t+1)(m-t)/2} \prod_{i=r}^{m+r} (x + \alpha[i]) \\
&\quad + \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \\
&\quad \cdot \left(1 + \alpha[u+t-1] + \alpha q^{u+t-1} [m-t-u+1] \right) \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
&\quad \cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]).
\end{aligned} \tag{3.16}$$

Next, in view of the equality

$$\left(1 + \alpha[u+t-1] + \alpha q^{u+t-1} [m-t-u+1] \right) = 1 + \alpha[m], \tag{3.17}$$

we obtain (3.13). Consequently, in view of (3.11) and (3.13), we get

$$\begin{aligned}
f_r^{[m]} \prod_{i=0}^{m-1} (1 + \alpha[i]) &= \sum_{t=0}^m \begin{bmatrix} m \\ t \end{bmatrix} f_{r+t} \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \\
&\quad \cdot \begin{bmatrix} m-t \\ u \end{bmatrix} \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]) \\
&= \sum_{t=0}^m \sum_{u=t}^m \begin{bmatrix} m \\ t \end{bmatrix} f_{r+t} (-1)^{u-t} q^{(u-t)(u-t-1)/2+(m-u)r} \\
&\quad \cdot \begin{bmatrix} m-t \\ u-t \end{bmatrix} \prod_{i=r}^{u+r-1} (x + \alpha[i]) \prod_{j=u}^{m-1} (1 + \alpha[j]).
\end{aligned} \tag{3.18}$$

Next, in view of the equality

$$\begin{bmatrix} m \\ t \end{bmatrix} \begin{bmatrix} m-t \\ u-t \end{bmatrix} = \begin{bmatrix} m \\ u \end{bmatrix} \begin{bmatrix} u \\ t \end{bmatrix}, \quad (3.19)$$

we obtain

$$\begin{aligned} f_r^{[m]} \prod_{i=0}^{m-1} (1 + \alpha[i]) &= \sum_{u=0}^m \sum_{t=0}^u \begin{bmatrix} m \\ u \end{bmatrix} f_{r+t} (-1)^{u-t} q^{(u-t)(u-t-1)/2 + (m-u)r} \\ &\quad \cdot \begin{bmatrix} u \\ t \end{bmatrix} \prod_{i=r}^{u+r-1} (x + \alpha[i]) \prod_{j=u}^{m-1} (1 + \alpha[j]) \\ &= \sum_{u=0}^m \begin{bmatrix} m \\ u \end{bmatrix} q^{(m-u)r} \prod_{i=r}^{u+r-1} (x + \alpha[i]) \prod_{j=u}^{m-1} (1 + \alpha[j]) \\ &\quad \cdot \sum_{t=0}^u \begin{bmatrix} u \\ t \end{bmatrix} (-1)^{u-t} q^{(u-t)(u-t-1)/2} f_{r+t}. \end{aligned} \quad (3.20)$$

The condition (1.10) completes the proof. $\square$

Theorems 3.1 and 3.2 are generalizations of Theorems 2.1 and 2.3 in [11].

Note that when $m = n$ and $r = 0$, (3.10) does indeed reduce to (1.11)

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