Review Article

Some Identities on the q -Integral Representation of the Product of Several q -Bernstein-Type Polynomials

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The purpose of this paper is to give some properties of several q -Bernstein-type polynomials to express the q -integral on $[0, 1]$ in terms of q -beta and q -gamma functions. Finally, we derive some identities on the q -integral of the product of several q -Bernstein-type polynomials.

1. Introduction

Let $q \in \mathbb{R}$ with $0 \leq q < 1$. We assume that q -number is defined by $[x]_q = (1 - q^x)/(1 - q)$ and $[0]_q = 0$. Note that $\lim_{q \rightarrow 1} [x]_q = x$. The q -derivative of a map $f : \mathbb{R} \rightarrow \mathbb{R}$ at $x \in \mathbb{R} \setminus \{0\}$ is given by

$$D_q(f) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q - 1)x} \quad (1.1)$$

(see [1–6]). For $n \in \mathbb{N}$, by (1.1), we get $D_q^n(x^n) = [n]_q [n-1]_q \cdots [2]_q [1]_q = [n]_q!$. The q -binomial formula is given by

$$(a + b)_q^n = \prod_{i=0}^{n-1} (a + bq^i) = \sum_{l=0}^n \binom{n}{l}_q q^{\binom{l}{2}} a^{n-l} b^l \quad (1.2)$$

(see [2, 5, 7–11]), where $\binom{n}{k}_q = [n]_q! / [k]_q! [n-k]_q! = [n]_q [n-1]_q \cdots [n-k+1]_q / [k]_q!$.

For $a, b \in \mathbb{R}$, the Jackson q -integral of $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\int_a^b f(x) d_q x = (1-q) \sum_{n=0}^{\infty} q^n (bf(bq^n) - af(aq^n)) \quad (1.3)$$

(see [1, 2, 5, 6, 9, 12, 13]). From (1.2), we note that

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q. \quad (1.4)$$

By (1.2) and (1.4), we get

$$\begin{aligned} (1-b)_q^n &= (b : q)_n = \prod_{i=0}^{n-1} (1 - q^i b) = \sum_{i=0}^n \binom{n}{i}_q q^{\binom{i}{2}} (-b)^i, \\ \frac{1}{(1-b)_q^n} &= \frac{1}{(b : q)_n} = \frac{1}{\prod_{i=0}^{n-1} (1 - q^i b)} = \sum_{i=0}^{\infty} \binom{n+i-1}{i}_q b^i. \end{aligned} \quad (1.5)$$

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well-known linear operators (see [1, 4, 9, 11, 14]):

$$\mathbb{B}_n(f | x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x). \quad (1.6)$$

Here $\mathbb{B}_n(f | x)$ is called Bernstein operator of order n for f . For $k, n \in \mathbb{Z}_+ (= \mathbb{N} \cup \{0\})$, the Bernstein polynomials of degree n are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad (1.7)$$

(see [1, 3, 4, 11–14]). By the definition of Bernstein polynomials (see (1.6) and (1.7)), we can see that Bernstein basis is the probability mass function of binomial distribution. A Bernoulli trial involves performing an experiment once and noting whether a particular event A occurs. The outcome of Bernoulli trial is said to be “success” if A occurs and a “failure” otherwise. Let k be the number of successes in n independent Bernoulli trials, the probabilities of k are given by the binomial probability law:

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } k = 0, 1, \dots, n, \quad (1.8)$$

where $p_n(k)$ is the probability of k successes in n trials. For example, a communication system transmits binary information over channel that introduces random bit errors with probability $\xi = 10^{-3}$. The transmitter transmits each information bit three times, and a decoder takes a majority vote of the received bits to decide on what the transmitted bit was. The receiver can correct a single error, but it will make the wrong decision if the channel introduces two or more errors. If we view each transmission as a Bernoulli trial in which a "success" corresponds to the introduction of an error, then the probability of two or more errors in three Bernoulli trials is

$$p(k \geq 2) = \binom{3}{2} (0.001)^2 (0.999) + \binom{3}{3} (0.001)^3 \approx 3(10^{-6}), \quad (1.9)$$

see [9]. Based on the q -integers Phillips introduced the q -analogue of well-known Bernstein polynomials (see [4, 5, 9, 11, 15]). For $f \in C([0, 1])$, Phillips introduced the q -extension of (1.6) as follows:

$$\begin{aligned} \mathbb{B}_{n,q}(f | x) &= \sum_{k=0}^n f\left(\frac{[k]_q}{[n]_q}\right) \binom{n}{k}_q (1-x)_q^{n-k} \\ &= \sum_{k=0}^n f\left(\frac{[k]_q}{[n]_q}\right) B_{k,n}(x, q), \quad \text{for } k, n \in \mathbb{Z}_+ \end{aligned} \quad (1.10)$$

(see [4, 5, 9, 11, 15]). Here $\mathbb{B}_{n,q}(f | x)$ is called the q -Bernstein operator of order n for f . For $k, n \in \mathbb{Z}_+$, the q -Bernstein polynomial of degree n is defined by

$$B_{k,n}(x, q) = \binom{n}{k}_q x^k (1-x)_q^{n-k}, \quad \text{where } x \in [0, 1]. \quad (1.11)$$

Note that (1.11) is the q -extension of (1.7). That is, $\lim_{q \rightarrow 1} B_{k,n}(x, q) = B_{k,n}(x)$. For example, $B_{0,1}(x, q) = 1-x$, $B_{1,1}(x, q) = x$, and $B_{0,2}(x, q) = 1 - [2]_q x + qx^2, \dots$. Also $B_{k,n}(x, q) = 0$ for $k > n$, because $\binom{n}{k}_q = 0$. For $n, k \in \mathbb{Z}_+$, its probabilities are given by

$$p(x = k) = \binom{n}{k}_q x^k (1-x)_q^{n-k}, \quad \text{where } x \in [0, 1]. \quad (1.12)$$

This distributions are studied by several authors and they have applications in physics as well as in approximation theory due to the q -Bernstein polynomials and the q -Bernstein operators (see [1–16]). By the definition of the q -Bernstein polynomials, we easily see that

the q -Bernstein basis is the probability mass function of q -binomial distribution. In this paper we use the two q -analogues of exponential function as follows:

$$E_q(x) = ((1-q)x : q)_\infty = (1 + (1-q)x)_q^\infty = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!}, \quad (1.13)$$

$$e_q(x) = \frac{1}{((1-q)x : q)_\infty} = \frac{1}{(1 + (1-q)x)_q^\infty} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \quad (1.14)$$

(see [2–4, 6, 10]). From (1.3), the improper q -integral is given by

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n \in \mathbb{Z}} \frac{q^n}{A} f\left(\frac{q^n}{A}\right) \quad (1.15)$$

(see [6]), where the improper q -integral depends on A . The purpose of this paper is to give some properties of several q -Bernstein type polynomials to express the q -integral on $[0, 1]$ in terms of q -beta and q -gamma functions. Finally, we derive some identities on the q -integral of the product of several q -Bernstein type polynomials.

2. q -Integral Representation of q -Bernstein Polynomials

The gamma and beta functions are defined as the following definite integrals ($\alpha > 0, \beta > 0$):

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt, \quad (2.1)$$

(see [1–11, 14–16])

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \int_0^{\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}} dt. \quad (2.2)$$

From (2.1) and (2.2), we can derive the following equations:

$$\Gamma(\alpha+1) = \alpha \Gamma(\alpha), \quad B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}. \quad (2.3)$$

As the q -extensions of (2.1) and (2.2), the q -gamma and q -beta functions are defined as the following q -integrals ($\alpha > 0, \beta > 0$):

$$\Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q(-qx) d_q x \quad (2.4)$$

(see [2–6, 10]),

$$B_q(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-qx)_q^{\beta-1} d_q x \quad (2.5)$$

(see [2, 4, 6, 10]).

By (2.4) and (2.5), we obtain the following lemma.

Lemma 2.1 (see [2, 6]). (a) Γ_q can be equivalently expressed as

$$\Gamma_q(\alpha) = \frac{(1-q)_q^{\alpha-1}}{(1-q)^{\alpha-1}}, \quad \text{where } \alpha > 0. \quad (2.6)$$

In particular, one has

$$\Gamma_q(\alpha + 1) = [\alpha]_q \Gamma_q(\alpha), \quad \text{for } \alpha > 0, \quad \Gamma_q(1) = 1. \quad (2.7)$$

(b) The q -gamma and q -beta functions are related to each other by the following two equations:

$$\Gamma_q(\alpha) = \frac{B_q(\alpha, \infty)}{(1-q)^\alpha}, \quad B_q(\alpha, \beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}, \quad \text{where } \alpha > 0, \beta > 0. \quad (2.8)$$

Now one takes the q -integral for one q -Bernstein polynomial as follows: for $n, k \in \mathbb{Z}_+$,

$$\begin{aligned} q^{-k} \int_0^1 B_{k,n}(qx, q) d_q x &= \binom{n}{k}_q \int_0^1 x^k (1-qx)_q^{n-k} d_q x \\ &= \binom{n}{k}_q \sum_{l=0}^{n-k} \binom{n-k}{l}_q (-1)^l q^{\binom{l+1}{2}} \int_0^1 x^{l+k} d_q x \\ &= \binom{n}{k}_q \sum_{l=0}^{n-k} \binom{n-k}{l}_q (-1)^{n-k-l} q^{\binom{n-k-l+1}{2}} \frac{1}{[n-l+1]_q}. \end{aligned} \quad (2.9)$$

Therefore, by (2.9), one obtains the following proposition.

Proposition 2.2. For $n, k \in \mathbb{Z}_+$, one has

$$\int_0^1 B_{k,n}(qx, q) d_q x = q^k \binom{n}{k}_q \sum_{l=0}^{n-k} \binom{n-k}{l}_q (-1)^{n-k-l} q^{\binom{n-k-l+1}{2}} \frac{1}{[n-l+1]_q}. \quad (2.10)$$

The Proposition 2.2 is closely related to the q -beta function which is given by

$$B_q(n, m) = \int_0^1 x^{n-1} (1 - qx)_q^{m-1} d_q x, \quad (2.11)$$

$$\Gamma_q(m) = \int_0^{1/(1-q)} x^{m-1} E_q(-qx) d_q x, \quad (2.12)$$

(see (2.5)). From Lemma 2.1, one has

$$B_q(n, m) = \frac{\Gamma_q(m) \Gamma_q(n)}{\Gamma_q(n+m)}, \quad \text{where } m, n \in \mathbb{N}. \quad (2.13)$$

By (2.9) and (2.13), one gets

$$\begin{aligned} q^{-k} \int_0^1 B_{k,n}(qx, q) d_q x &= \binom{n}{k}_q B_q(k+1, n-k+1) \\ &= \binom{n}{k}_q \frac{\Gamma_q(k+1) \Gamma_q(n-k+1)}{\Gamma_q(n+2)}, \quad \text{where } k > -1, n > k-1. \end{aligned} \quad (2.14)$$

Therefore, by (2.14), one obtains the following theorem.

Theorem 2.3. For $n, k \in \mathbb{Z}_+$ with $k > -1$ and $n > k-1$, one has

$$\int_0^1 B_{k,n}(qx, q) d_q x = \binom{n}{k}_q [k]_q [n-k]_q ((q-1)[k]_q + 1) \frac{\Gamma_q(k) \Gamma_q(n-k)}{\Gamma_q(n+2)}. \quad (2.15)$$

By comparing the coefficients on the both sides of Proposition 2.2 and Theorem 2.3, one obtains the following corollary.

Corollary 2.4. For $n, k \in \mathbb{Z}_+$ with $k > -1$ and $n > k-1$, one has

$$\sum_{l=0}^{n-k} \binom{n-k}{l}_q (-1)^{n-k-l} \frac{q^{\binom{n-k-l+1}{2}}}{[n-l+1]_q} = \frac{\Gamma_q(k+1) \Gamma_q(n-k+1)}{\Gamma_q(n+2)}. \quad (2.16)$$

According to this result one can say that the q -integral of q -Bernstein polynomials from 0 to 1 is symmetric. Now one considers the q -integral for the multiplication of two q -Bernstein polynomials which is given by the following relation:

$$\begin{aligned} \frac{\int_0^1 B_{k,n}(qx, q) B_{k,m}(q^{n-k+1}x, q) d_q x}{q^{nk-k^2+2k}} &= \binom{n}{k}_q \binom{m}{k}_q \int_0^1 x^2 k (1 - qx)_q^{n+m-2k} d_q x \\ &= \binom{n}{k}_q \binom{m}{k}_q \int_0^1 u^{n+m-2k} (1 - qu)_q^{2k} d_q u. \end{aligned} \quad (2.17)$$

For $n, k, m \in \mathbb{Z}_+$, one can derive the following equation (2.20) from (2.17):

$$\begin{aligned} \frac{\int_0^1 B_{k,n}(qx, q) B_{k,m}(q^{n-k+1}x, q) d_q x}{q^{nk-k^2+2k}} &= \binom{n}{k}_q \binom{m}{k}_q \sum_{l=0}^{2k} \frac{\binom{2k}{l}_q (-1)^l q^{\binom{l+1}{2}}}{[n+m+l-2k+1]_q} \\ &= \binom{n}{k}_q \binom{m}{k}_q \sum_{l=0}^{2k} \frac{\binom{2k}{l}_q (-1)^{2k-l} q^{\binom{2k-l+1}{2}}}{[n+m-l+1]_q}. \end{aligned} \quad (2.18)$$

Therefore, one obtains the following theorem.

Theorem 2.5. For $m, n, k \in \mathbb{Z}_+$, one has

$$\int_0^1 B_{k,n}(qx, q) B_{k,m}(q^{n-k+1}x, q) d_q x = q^{nk-k^2+2k} \binom{n}{k}_q \binom{m}{k}_q \sum_{l=0}^{2k} \frac{\binom{2k}{l}_q (-1)^{2k-l} q^{\binom{2k-l+1}{2}}}{[n+m-l+1]_q}. \quad (2.19)$$

For $m, n, k \in \mathbb{Z}_+$, by (2.5) and (2.9), one gets

$$\frac{\int_0^1 B_{k,n}(qx, q) B_{k,m}(q^{n-k+1}x, q) d_q x}{q^{nk-k^2+2k}} = \binom{n}{k}_q \binom{m}{k}_q B_q(n+m-2k+1, 2k+1). \quad (2.20)$$

Therefore, by Theorem 2.5 and (2.20), one obtains the following corollary.

Corollary 2.6. For $k > -1$ and $n+m-2k > -1$, one has

$$\sum_{l=0}^{2k} \frac{\binom{2k}{l}_q (-1)^{2k-l} q^{\binom{2k-l+1}{2}}}{[n+m-l+1]_q} = \frac{\Gamma_q(n+m-2k+1) \Gamma_q(2k+1)}{\Gamma_q(n+m+2)}. \quad (2.21)$$

By the same method, the multiplication of three q -Bernstein polynomials is given by the following relation: for $k, n, m, s \in \mathbb{Z}_+$,

$$\begin{aligned}
 & \frac{\int_0^1 B_{k,n}(qx, q) B_{k,m}(q^{n-k+1}x, q) B_{k,s}(q^{n+m-2k+1}x, q) d_q x}{q^{3k+2nk-3k^2+mk}} \\
 &= \binom{n}{k}_q \binom{m}{k}_q \binom{s}{k}_q \int_0^1 x^{3k} (1-qx)_q^{n+m+s-3k} d_q x \\
 &= \binom{n}{k}_q \binom{m}{k}_q \binom{s}{k}_q \int_0^1 u^{n+m+s-3k} (1-qu)_q^{3k} d_q u \\
 &= \binom{n}{k}_q \binom{m}{k}_q \binom{s}{k}_q \sum_{l=0}^{3k} \binom{3k}{l}_q q^{\binom{l+1}{2}} (-1)^l \int_0^1 u^{n+m+s-3k+l} d_q u \\
 &= \binom{n}{k}_q \binom{m}{k}_q \binom{s}{k}_q \sum_{l=0}^{3k} \binom{3k}{l}_q q^{\binom{3k-l+1}{2}} (-1)^{l+3k} \frac{1}{[n+m+s-l+1]_q}.
 \end{aligned} \tag{2.22}$$

Therefore, by (2.22), one obtains the following theorem.

Theorem 2.7. For $n, m, s, k \in \mathbb{Z}_+$, one has

$$\begin{aligned}
 & \int_0^1 B_{k,n}(qx, q) B_{k,m}(q^{n-k+1}x, q) B_{k,s}(q^{n+m-2k+1}x, q) d_q x \\
 &= q^{3k+2nk-3k^2+mk} \binom{n}{k}_q \binom{m}{k}_q \binom{s}{k}_q \sum_{l=0}^{3k} \binom{3k}{l}_q q^{\binom{3k-l+1}{2}} \frac{(-1)^{l+3k}}{[n+m+s-l+1]_q}.
 \end{aligned} \tag{2.23}$$

From (2.5) and (2.22), one has

$$\begin{aligned}
 & \frac{\int_0^1 B_{k,n}(qx, q) B_{k,m}(q^{n-k+1}x, q) B_{k,s}(q^{n+m-2k+1}x, q) d_q x}{q^{3k+2nk-3k^2+mk}} \\
 &= \binom{n}{k}_q \binom{m}{k}_q \binom{s}{k}_q B_q(n+m+s-3k+1, 3k+1).
 \end{aligned} \tag{2.24}$$

Therefore, by Theorem 2.7 and (2.24), one obtains the following corollary.

Corollary 2.8. For $k > -1/3$ and $n+m+s-3k > -1$, one has

$$\sum_{k=0}^{3k} \binom{3k}{l}_q \frac{(-1)^{l+3k} q^{\binom{3k-l+1}{2}}}{[n+m+s-l+1]_q} = \frac{\Gamma_q(n+m+s-3k+1) \Gamma_q(3k+1)}{\Gamma_q(n+m+s+2)}. \tag{2.25}$$

For $s \in \mathbb{N}$, let $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$. Then one has

$$\begin{aligned}
 & \frac{\int_0^1 B_{k,n_1}(qx, q) \left(\prod_{i=1}^{s-1} B_{k,n_{i+1}} \left(q^{\sum_{l=1}^i n_l - ik + 1} x, q \right) d_q x \right)}{q^{sk+k \sum_{i=1}^{s-1} n_{s-i} - k^2 \binom{s}{2}}} \\
 &= \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q \int_0^1 x^{sk} (1 - qx)_q^{n_1 + \cdots + n_s - sk} d_q x \\
 &= \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q \sum_{l=0}^{sk} \binom{sk}{l}_q (-1)^l q^{\binom{l+1}{2}} \int_0^1 x^{n_1 + \cdots + n_s - sk + l} d_q x \\
 &= \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q \sum_{l=0}^{sk} \binom{sk}{l}_q \frac{(-1)^{l+sk} q^{\binom{sk-l+1}{2}}}{[n_1 + \cdots + n_s - l + 1]_q}.
 \end{aligned} \tag{2.26}$$

Therefore, by (2.26), one obtains the following theorem.

Theorem 2.9. For $s \in \mathbb{N}$, let $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$. Then one has

$$\begin{aligned}
 & \int_0^1 B_{k,n_1}(qx, q) \left(\prod_{i=1}^{s-1} B_{k,n_{i+1}} \left(q^{\sum_{l=1}^i n_l - ik + 1} x, q \right) d_q x \right) \\
 &= q^{sk+k \sum_{i=1}^{s-1} n_{s-i} - k^2 \binom{s}{2}} \binom{n_1}{k}_q \cdots \binom{n_s}{k}_q \sum_{l=0}^{sk} \frac{\binom{sk}{l}_q (-1)^{l+sk} q^{\binom{sk-l+1}{2}}}{[n_1 + \cdots + n_s - l + 1]_q}.
 \end{aligned} \tag{2.27}$$

By (2.5) and (2.26), we get

$$\begin{aligned}
 & \frac{\int_0^1 B_{k,n_1}(qx, q) \left(\prod_{i=1}^{s-1} B_{k,n_{i+1}} \left(q^{\sum_{l=1}^i n_l - ik + 1} x, q \right) d_q x \right)}{q^{sk+k \sum_{i=1}^{s-1} n_{s-i} - k^2 \binom{s}{2}}} \\
 &= \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q B_q(sk + 1, n_1 + \cdots + n_s - sk + 1) \\
 &= \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q \frac{\Gamma_q(sk + 1) \Gamma_q(n_1 + \cdots + n_s - sk + 1)}{\Gamma_q(n_1 + \cdots + n_s + 2)}.
 \end{aligned} \tag{2.28}$$

By comparing the coefficients on the both sides of Theorem 2.9 and (2.28), one obtains the following corollary.

Corollary 2.10. For $s \in \mathbb{N}$, let $k > -1/s$ and $n_1 + \cdots + n_s - sk > -1$. Then one has

$$\sum_{l=0}^{sk} \frac{\binom{sk}{l}_q (-1)^{l+sk} q^{\binom{sk-l+1}{2}}}{[n_1 + \cdots + n_s - l + 1]_q} = \frac{\Gamma_q(sk + 1) \Gamma_q(n_1 + \cdots + n_s - sk + 1)}{\Gamma_q(n_1 + \cdots + n_s + 2)}. \tag{2.29}$$

For $n \in \mathbb{Z}_+$, one gets

$$\begin{aligned}
 & \frac{\int_0^1 B_{0,n}(qx, q) \left(\prod_{l=1}^n B_{l,n} \left(q^{nl - \binom{l}{2} + 1} x, q \right) \right) d_q x}{q^{\sum_{l=1}^n (nl - \binom{l}{2} + 1)l}} \\
 &= \left(\prod_{i=0}^n \binom{n}{i}_q \right) \int_0^1 x^{\binom{n+1}{2}} (1 - qx)_q^{\binom{n+1}{2}} d_q x \\
 &= \left(\prod_{i=0}^n \binom{n}{i}_q \right) B_q \left(\binom{n+1}{2} + 1, \binom{n+1}{2} + 1 \right) \\
 &= \left(\frac{(\Gamma_q(n+1))^{n+1}}{(\prod_{i=1}^n \Gamma_q(i+1))^2} \right) \left(\frac{(\Gamma_q(n(n+1)/2 + 1))^2}{\Gamma_q(n(n+1) + 2)} \right).
 \end{aligned} \tag{2.30}$$

Therefore, by (2.30), one obtains the following theorem.

Theorem 2.11. For $n \in \mathbb{Z}_+$, one has

$$\begin{aligned}
 & \int_0^1 B_{0,n}(qx, q) \left(\prod_{l=1}^n B_{l,n} \left(q^{nl - \binom{l}{2} + 1} x, q \right) \right) d_q x \\
 &= q^{\sum_{l=1}^n (nl - \binom{l}{2} + 1)l} \left(\frac{(\Gamma_q(n+1))^{n+1}}{(\prod_{i=1}^n \Gamma_q(i+1))^2} \right) \left(\frac{(\Gamma_q(n(n+1)/2 + 1))^2}{\Gamma_q(n(n+1) + 2)} \right).
 \end{aligned} \tag{2.31}$$

From (2.30), one can also derive the following equation:

$$\begin{aligned}
 & \frac{\int_0^1 B_{0,n}(qx, q) \left(\prod_{l=1}^n B_{l,n} \left(q^{nl - \binom{l}{2} + 1} x, q \right) \right) d_q x}{q^{\sum_{l=1}^n (nl - \binom{l}{2} + 1)l}} \\
 &= \left(\prod_{i=0}^n \binom{n}{i}_q \right) \sum_{l=0}^{\binom{n+1}{2}} \left(\binom{n+1}{2} \right)_q (-1)^l q^{\binom{l+1}{2}} \int_0^1 x^{\binom{n+1}{2} + l} d_q x \\
 &= \left(\prod_{i=0}^n \binom{n}{i}_q \right) \sum_{l=0}^{\binom{n+1}{2}} \left(\binom{n+1}{2} \right)_q (-1)^l q^{\binom{l+1}{2}} \frac{1}{[n(n+1)/2 + l + 1]_q}.
 \end{aligned} \tag{2.32}$$

By comparing the coefficients on the both sides of Theorem 2.11 and (2.30), one can see that

$$\sum_{l=0}^{n(n+1)/2} \frac{\binom{n(n+1)/2}{l}_q (-1)^l q^{\binom{l+1}{2}}}{[n(n+1)/2 + l + 1]_q} = B_q \left(\frac{n(n+1)}{2} + 1, \frac{n(n+1)}{2} + 1 \right). \quad (2.33)$$

Therefore, by (2.33), one obtains the following corollary.

Corollary 2.12. For $n \in \mathbb{Z}_+$, one has

$$\sum_{l=0}^{n(n+1)/2} \frac{\binom{n(n+1)/2}{l}_q (-1)^l q^{\binom{l+1}{2}}}{[n(n+1)/2 + l + 1]_q} = \left(\frac{(\Gamma_q(n(n+1)/2 + 1))^2}{\Gamma_q(n(n+1) + 2)} \right). \quad (2.34)$$

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