Research Article

Monotonicity, Convexity, and Inequalities Involving the Agard Distortion Function

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We present some monotonicity, convexity, and inequalities for the Agard distortion function \( \eta_K(t) \) and improve some well-known results.

1. Introduction

For \( r \in [0, 1] \), Legendre’s complete elliptic integrals of the first and second kind [1] are defined by

\[
\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \left( 1 - r^2 \sin^2 \theta \right)^{-1/2} d\theta, \\
\mathcal{K}'(r) = \mathcal{K}(r'), \quad \mathcal{K}(0) = \frac{\pi}{2}, \quad \mathcal{K}(1) = \infty,
\]

\[
\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \left( 1 - r^2 \sin^2 \theta \right)^{1/2} d\theta, \\
\mathcal{E}'(r) = \mathcal{E}(r'), \quad \mathcal{E}(0) = \frac{\pi}{2}, \quad \mathcal{E}(1) = 1,
\]

respectively. Here and in what follows, we set \( r' = \sqrt{1-r^2} \).
Let $\mu(r)$ be the modulus of the plan Grötzsch ring $B^2 \setminus [0, r]$ for $r \in (0, 1)$, where $B^2$ is the unit disk. Then, it follows from [2] that

$$\mu(r) = \frac{\pi}{2} \mathcal{K}'(r).$$

(1.3)

For $K \in (0, \infty)$, the Hersch-Pfluger distortion function $\varphi_K(r)$ is defined as

$$\varphi_K(r) = \mu^{-1}\left(\frac{\mu(r)}{K}\right) \quad \text{for} \quad r \in (0, 1), \quad \varphi_K(0) = \varphi_K(1) - 1 = 0,$$

(1.4)

while the Agard distortion function $\eta_K(t)$ and the linear distortion function $\lambda(K)$ are defined by

$$\eta_K(t) = \left[ \frac{\varphi_K(r)}{\varphi_{1/K}(r')} \right], \quad \lambda(K) = \eta_K(1), \quad r = \sqrt{\frac{t}{1+t}} \quad t \in (0, \infty),$$

(1.5)

respectively.

It is well known that the functions $\eta_K(t)$ and $\lambda(K)$ play a very important role in quasiconformal theory, quasiregular theory, and some other related fields [3–8]. For example, Martin [8] found that the sharp upper bound in Schottky’s theorem can be expressed by $\eta_K(t)$, and in [9–15] the authors established a number of remarkable properties for the Agard distortion function $\eta_K(t)$.

In [14], the authors proved that

$$e^{\pi(K^{-1})} < \lambda(K) < e^{a(K^{-1})},$$

(1.6)

$$e^{b(K^{-1})} < \lambda(K) < e^{\pi(K^{-1})},$$

(1.7)

for all $K \in (1, \infty)$, where $a = (4/\pi)\mathcal{K}(1/\sqrt{2})^2 = 4.3768 \ldots$, $b = a/2$. Recently, Anderson et al. [15] established that

$$\lambda(K) < e^{(\pi+b/K)(K^{-1})} \quad (1.8)$$

$$e^{[\log_2(a-b/2)/K](K^{-1})} < \lambda(K) < e^{[\pi+(a-log_2)/K](K^{-1})}$$

(1.9)

for all $K \in (1, \infty)$, where $a$ and $b$ are defined as in inequalities (1.6) and (1.7), respectively.

The purpose of this paper is to present the new monotonicity, convexity, and inequalities for the Agard distortion function $\eta_K(t)$ and improve inequalities (1.6)–(1.9).

Our main results are Theorems 1.1 and 1.2 as follows.

**Theorem 1.1.** Let $K \in (1, \infty)$, $a = (4/\pi)\mathcal{K}(1/\sqrt{2})^2 = 4.3768 \ldots$, $b = a/2$, and $c \in \mathbb{R}$. Then, the following statements are true.

1. $f(K) = \lambda(K)/K^c$ is strictly increasing from $(1, \infty)$ onto $(1, \infty)$ for $c \leq a$; if $c > a$, then there exists $K_0 \in (1, \infty)$, such that $f$ is strictly decreasing in $(1, K_0)$ and strictly increasing
in \((K_0, \infty)\). In particular, the inequality \(\lambda(K) \geq K^c\) holds for all \(K \in (1, \infty)\) with the best possible constant \(c = a\).

(2) \(g(K) = [\log \eta_K(t) - \log t]/(K - 1)\) is convex in \((1, \infty)\) for fixed \(t \in (0, \infty)\).

(3) If \(t \geq 1\) and \(r = \sqrt{t/(1 + t)}\), then \(h(K) = [\log \eta_K(t) - \log t]/(K - 1/K)\) is strictly increasing from \(1, \infty)\) onto \((2K(r)K'(r)/\pi, \pi K(r)/K'(r))\).

**Theorem 1.2.** Let \(t \in (0, \infty), r = \sqrt{t/(1 + t)}, a = (4/\pi)K(1/\sqrt{2})^2, b = a/2, A(r) = \pi^2/(2\mu(r)), B(r) = 8K(r)K'(r)^2[\xi(r) - r^2K(r)]/\pi^2\), and \(F_c(K) = K[(\log \eta_K(t) - \log t)/(K - 1 - c)]\). Then, the following statements are true.

(1) \(F_c(K)\) is strictly decreasing from \((1, \infty)\) onto \((-\infty, 4K(r)K'(r)/\pi - c)\) for \(c > A(r)\). If \(c = A(r)\), then \(F_c(K)\) is strictly decreasing from \((1, \infty)\) onto \((A(r) - 4\log 2 - \log t, 4K(r)K'(r)/\pi - A(r))\). Moreover,

\[
t e^{(K-1)(A(r)+((A(r)-4\log 2-\log t)/K))} < \eta_K(t) < t e^{(K-1)(A(r)+((4K(r)K'(r)/\pi-A(r))/K))}
\]  

(1.10)  

for all \(t \in (0, \infty)\) and \(K \in (1, \infty)\). In particular, if \(t = 1\), then (1.10) becomes

\[
e^{(K-1)(\pi+((\pi-4\log 2)/K))} < \lambda(K) < e^{(K-1)(\pi+((\pi-A)/K))}.
\]  

(1.11)  

(2) If \(c \leq B(r)\), then \(F_c(K)\) is strictly increasing from \((1, \infty)\) onto \((4K(r)K'(r)/\pi - c, \infty)\). Moreover,

\[
\eta_K(t) > t e^{(K-1)(B(r)+((4K(r)K'(r)/\pi-B(r))/K))}
\]  

(1.12)  

for all \(t \in (0, \infty)\) and \(K \in (1, \infty)\). In particular, if \(t = 1\), then (1.12) becomes

\[
\lambda(K) > e^{(K-1)(b+((\pi-b)/K))} = e^{b(K-1/K)}.
\]  

(1.13)  

(3) If \(B(r) < c < A(r)\), then there exists \(K_1 \in (1, \infty)\) such that \(F_c(K)\) is strictly decreasing on \((1, K_1)\) and strictly increasing on \((K_1, \infty)\).

(4) \(F_c(K)\) is convex in \((1, \infty)\).

**2. Lemmas**

In order to prove our main results, we need several formulas and lemmas, which we present in this section.
The following formulas were presented in [14, Appendix E, pp. 474-475]. Let $t \in (0, \infty), K \in (0, \infty), r = \sqrt{t/(1+t)} \in (0, 1)$, and $s = \varphi_K(r)$. Then,

\[
\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r) - r^2\mathcal{K}(r)}{rr^2}, \quad \frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r},
\]

\[
\mathcal{K}(r)\mathcal{E}'(r) + \mathcal{K}'(r)\mathcal{E}(r) - \mathcal{K}(r)\mathcal{K}'(r) = \frac{\pi}{2},
\]

\[
\frac{d\mu(r)}{dr} = -\frac{\pi^2}{4rr^2\mathcal{K}(r)^2}.
\]

\[
\frac{\partial s}{\partial r} = \frac{s^2}{rr^2\mathcal{K}(r)\mathcal{K}'(r)};
\]

\[
\frac{\partial s}{\partial K} = \frac{2}{\pi K} s^2 \mathcal{K}(s) \mathcal{K}'(s),
\]

\[
\varphi_K(r)^2 + \varphi_{1/K}(r)^2 = 1,
\]

\[
\eta_K(t) = \left(\frac{s}{s'}\right)^2,
\]

\[
\frac{\partial \eta_K(t)}{\partial K} = \frac{4}{\pi K} \eta_K(t) \mathcal{K}(s) \mathcal{K}'(s) = \frac{2}{\mu(r)} \mathcal{K}'(s)^2 \eta_K(t).
\]

**Lemma 2.1** (see [14, Theorem 1.25]). For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$, and let $g'(x) \neq 0$ on $(a, b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a, b)$, then so are

\[
\frac{f(x) - f(a)}{g(x) - g(a)}, \quad \frac{f(x) - f(b)}{g(x) - g(b)}.
\]

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

The following lemma can be found in [14, Theorem 3.21(1) and (7), Lemma 3.32(1) and Theorem 5.13(2)].

**Lemma 2.2.** (1) $[\mathcal{E}(r) - r^2\mathcal{K}(r)]/r^2$ is strictly increasing from $(0, 1)$ onto $(\pi/4, 1)$;

(2) $r^c\mathcal{K}(r)$ is strictly decreasing from $(0, 1)$ onto $(0, \pi/2)$ if and only if $c \geq 1/2$;

(3) $\mathcal{K}(r)\mathcal{K}'(r)$ is strictly decreasing in $(0, \sqrt{2}/2)$ and strictly increasing in $(\sqrt{2}/2, 1)$;

(4) $\mu(r) + \log r$ is strictly decreasing from $(0, 1)$ onto $(0, \log 4)$.

**Lemma 2.3.** Let $r \in [1/\sqrt{2}, 1)$, $K \in (1, \infty)$, and $s = \varphi_K(r)$. Then, $G(K) = [\pi/[2\mathcal{K}(s)]^2 + [\mu(r)/\mathcal{K}'(s)]^2$ is strictly decreasing from $(1, \infty)$ onto $(\mathcal{K}'(r)^2/\mathcal{K}(r)^2, \pi^2/[2\mathcal{K}(r)^2])$.

**Proof.** Clearly $G(1^+) = \pi^2/(2\mathcal{K}(r)^2), G(\infty) = \mathcal{K}'(r)^2/\mathcal{K}(r)^2$. Differentiating $G(K)$, one has

\[
G'(K) = \frac{4}{\pi K} \mu(r)^2 \mathcal{K}(s) \mathcal{K}'(s)^2 \left[\mathcal{E}'(s) - s^2 \mathcal{K}'(s)\right]
\]

\[
- \frac{\pi}{K} \mathcal{K}(s)^2 \mathcal{K}'(s) \left[\mathcal{E}(s) - s^2 \mathcal{K}(s)\right]
\]

\[
= \frac{4}{\pi K} \mathcal{K}(s)^2 \mathcal{K}'(s)^2 \eta_K(t),
\]

where $G_1(K) = [\mathcal{E}'(s) - s^2 \mathcal{K}'(s)] \mathcal{K}(s)^3 \mu(r)^2 - \pi^2 [\mathcal{E}(s) - s^2 \mathcal{K}(s)] \mathcal{K}'(s)^3/4.$
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From Lemma 2.2(1) and (2), we clearly see that $G_1(K)$ is strictly decreasing in $(1, \infty)$. Moreover,

$$
\lim_{K \to 1^+} G_1(K) = \frac{\pi^2}{4}K^2 G_2(r),
$$

(2.4)

where $G_2(r) = \mathcal{K}(r) [\mathcal{E}'(r) - r^2 \mathcal{K}'(r)] - \mathcal{K}'(r) [\mathcal{E}(r) - r^2 \mathcal{K}(r)]$ is also strictly decreasing in $(0,1)$. Thus, $G_2(r) \leq G_2(\sqrt{2}/2) = 0$ for $r \in [1/\sqrt{2}, 1)$, and $G_1(K) < G_1(1^+) \leq 0$ for $K \in (1, \infty)$.

Therefore, the monotonicity of $G(K)$ follows from (2.3) and (2.4) together with the fact that $G_1(K) < 0$ for $K \in (1, \infty)$.

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. For part (1), clearly $f(1^+) = 1$. Let $r = \mu^{-1} [\pi/(2K)]$ for $K \in (1, \infty)$, then

$$
\lambda(K) = \left(\frac{r}{r}\right)^{1/2},
$$

$\lambda(K)$ is strictly increasing in $(1, \infty)$ onto $(a - c, \infty)$. Then, from (3.2) and (3.3), we know that $f$ is strictly increasing from $(1, \infty)$ onto $(1, \infty)$ for $c \leq a$. If $c > a$, then there exists $K_0 \in (1, \infty)$ such that $f$ is strictly decreasing in $(1, K_0)$ and strictly increasing in $(K_0, \infty)$. Moreover, the inequality $\lambda(K) \geq K^c$ holds for all $K \in (1, \infty)$ with the best possible constant $c = a$.

For part (2), denote $r = \sqrt{t/(1 + t)}$. Differentiating $g(K)$, we get

$$
g'(K) = \frac{2\mathcal{K}'(s)^2 (K - 1)/\mu(r) - (\log \eta_K(t) - \log t)}{(K - 1)^2}.
$$

(3.4)

Let $g_1(K) = 2\mathcal{K}'(s)^2 (K - 1)/\mu(r) - (\log \eta_K(t) - \log t)$ and $g_2(K) = (K - 1)^2$, then $g_1(1) = g_2(1) = 0$, $g'(K) = g_1(K)/g_2(K)$ and

$$
\frac{g_1(K)}{g_2(K)} = g_3(K) \equiv -\frac{2}{\mu(r)^2} \left[\mathcal{E}(s) - s^2 \mathcal{K}'(s)\right] \mathcal{K}'(s)^3.
$$

(3.5)
Clearly, \( g_3(K) \) is strictly increasing in \((1, \infty)\). Then, (3.5) and Lemma 2.1 lead to the conclusion that \( g'(K) \) is strictly increasing in \((1, \infty)\). Therefore, \( g(K) \) is convex in \((1, \infty)\).

For part (3), if \( t \geq 1 \), then \( r \geq \sqrt{2}/2 \). Let \( h_1(K) = \log \eta_K(t) - \log t \) and \( h_2(K) = K - 1/K \), then \( h_1(1) = h_2(1) = 0 \), \( h(K) = h_1(K)/h_2(K) \), and

\[
\frac{h_1'(K)}{h_2'(K)} = \frac{2K'(s)^2/\mu(r)}{1 + K^{-2}} = \frac{2\mu(r)}{G(K)},
\]

where \( G(K) \) is defined as in Lemma 2.2.

Therefore, \( h(K) \) is strictly increasing in \((1, \infty)\) for \( t \geq 1 \) follows from Lemmas 2.1 and 2.2 together with (3.6). Moreover, making use of l'Hôpital's rule, we have \( h(1^+) = 2K(K)K'(r)/\pi, h(\infty) = \pi K(r)/K'(r) \).

**Proof of Theorem 1.2.** Differentiating \( F_c(K) \) gives

\[
F_c'(K) = \frac{\log \eta_K(t) - \log t}{K - 1} - c + K \left[ \frac{2K'(s)^2(K - 1)/\mu(r) - (\log \eta_K(t) - \log t)}{(K - 1)^2} \right] - \frac{2K'(s)^2K(K - 1)/\mu(r) - (\log \eta_K(t) - \log t)}{(K - 1)^2} - c.
\]

Let

\[
H(K) = \frac{2K'(s)^2K(K - 1)/\mu(r) - [\log \eta_K(t) - \log t]}{(K - 1)^2}.
\]

Then, from (3.8) and (3.9) together with Lemma 2.1, we know that \( H(K) \) is strictly increasing in \((1, \infty)\). Moreover, l'Hôpital's rule leads to

\[
\lim_{K \to 1} H(K) = B(r), \quad \lim_{K \to \infty} H(K) = A(r).
\]

For part (1), if \( c > A(r) \), then from (3.7) and (3.8), we know that \( F_c'(K) < 0 \) for \( K \in (1, \infty) \) and \( F_c(K) \) is strictly decreasing in \((1, \infty)\). Moreover,

\[
\lim_{K \to 1} F_c(K) = [4K(r)K'(r)/\pi] - c, \quad \lim_{K \to \infty} F_c(K) = -\infty.
\]
If \( c = A(r) \), then \( F_c(K) \) is also strictly decreasing in \((1, \infty)\) and \( F_c(1^+) = [4\mathcal{K}(r)\mathcal{K}'(r)/\pi] - A(r) \), and from Lemma 2.2(4) we get

\[
\lim_{k \to \infty} F_c(K) = \lim_{k \to \infty} \frac{K}{K-1} \left[-2\log(s') - 2\mu(s') + A(r) - \log t\right] = A(r) - 4\log 2 - \log t. \tag{3.12}
\]

Therefore, inequalities (1.10) and (1.11) follows from (3.12) and the monotonicity of \( F_c(K) \) when \( c = A(r) \).

For part (2), if \( c \leq B(r) \), then that \( F_c(K) \) is strictly increasing in \((1, \infty)\) follows from (3.7) and (3.8). Note that

\[
\lim_{k \to 1} F_c(K) = [4\mathcal{K}(r)\mathcal{K}'(r)/\pi] - c, \quad \lim_{k \to \infty} F_c(K) = +\infty. \tag{3.13}
\]

Therefore, inequalities (1.12) and (1.13) follow from (3.13) and the monotonicity of \( F_c(K) \) when \( c = B(r) \).

For part (3), if \( B(r) < c < A(r) \), then from (3.7) and (3.8) together with the monotonicity of \( H(K) \) we clearly see that there exists \( K_1 \in (1, \infty) \), such that \( F_c'(K) < 0 \) for \( K \in (1, K_1) \) and \( F_c'(K) > 0 \) for \( K \in (K_1, \infty) \). Hence, \( F_c(K) \) is strictly decreasing in \((1, K_1)\) and strictly increasing in \((K_1, \infty)\).

Part (4) follows from (3.7) and (3.8) together with the monotonicity of \( H(K) \). \( \square \)

Taking \( t = 1 \) in Theorem 1.2, we get the following corollary.

**Corollary 3.1.** Let \( a \) and \( b \) be defined as in Theorem 1.2, \( c \in \mathbb{R} \), and \( f_c(K) = K[\log \lambda(K)]/(K - 1) - c \). Then,

1. if \( c > \pi \), then \( f_c(K) \) is strictly decreasing from \((1, \infty)\) onto \((\infty, a - c)\); if \( c = \pi \), then \( f_c(K) \) is strictly decreasing from \((1, \infty)\) onto \((\pi - 4\log 2, a - \pi)\);
2. if \( c \leq b \), then \( f_c(K) \) is strictly increasing from \((1, \infty)\) onto \((a - c, \infty)\);
3. if \( b < c < \pi \), then there exists \( K_2 \in (1, \infty) \), such that \( f_c(K) \) is strictly decreasing in \((1, K_2)\) and strictly increasing in \((K_2, \infty)\);
4. \( f_c(K) \) is convex in \((1, \infty)\).

Inequalities (1.11) and (1.13) lead to the following corollary, which improve inequalities (1.6)–(1.9).

**Corollary 3.2.** Let \( a \) and \( b \) be defined as in Theorem 1.2, then the following inequality

\[
\max \left\{ e^{(K-1)(\pi + (\pi - 4\log 2)/K)}, e^{b(K - (1/K))} \right\} < \lambda(K) < e^{(K-1)(\pi + (a-\pi)/K)}. \tag{3.14}
\]

holds for all \( K \in (1, \infty) \).
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