Research Article

Sharp Bounds for Power Mean in Terms of Generalized Heronian Mean

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For $1 < r < +\infty$, we find the least value $\alpha$ and the greatest value $\beta$ such that the inequality $H_{\omega}(a,b) < A_r(a,b) < H_\beta(a,b)$ holds for all $a, b > 0$ with $a \neq b$. Here, $H_{\omega}(a,b)$ and $A_r(a,b)$ are the generalized Heronian and the power means of two positive numbers $a$ and $b$, respectively.

1. Introduction and Statement of Result

For $a, b > 0$ with $a \neq b$, the generalized Heronian mean of $a$ and $b$ is defined by Janous [1] as

$$H_\omega(a,b) = \begin{cases} \frac{a + \omega \sqrt{ab} + b}{\omega + 2}, & 0 \leq \omega < +\infty, \\ \sqrt{ab}, & \omega = +\infty. \end{cases}$$

If we take $\omega = 1$ in (1.1), then we arrive at the classical Heronian mean

$$H_e(a,b) = \frac{a + \sqrt{ab} + b}{3}.$$ 

The domain of definition for the function $\omega \mapsto H_\omega(a,b)$ can be extended to all $\omega$ with $\omega \in (-2, +\infty)$, that is,

$$H_\omega(a,b) = \begin{cases} \frac{a + \omega \sqrt{ab} + b}{\omega + 2}, & -2 < \omega < +\infty, \\ \sqrt{ab}, & \omega = +\infty. \end{cases}$$

$$H_e(a,b) = \frac{a + \sqrt{ab} + b}{3}.$$
For all fixed \(a, b > 0\), it is easy to derive that \(\omega \mapsto H_{\omega}(a, b), -2 < \omega < +\infty\) is monotonically decreasing, and

\[
\lim_{\omega \to -2^+} H_{\omega}(a, b) = +\infty.
\] (1.4)

Let

\[
A_r(a, b) = \begin{cases} \left( \frac{a^r + b^r}{2} \right)^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0, \\ \max\{a, b\}, & r = +\infty, \\ \min\{a, b\}, & r = -\infty, \end{cases}
\] (1.5)

denote the power mean of order \(r\). In particular, the harmonic, geometric, square-root, arithmetic, and root-square means of \(a\) and \(b\) are

\[
H(a, b) = A_{-1}(a, b) = \frac{2a}{a + b},
\]

\[
G(a, b) = A_0(a, b) = \sqrt{ab},
\]

\[
N_1(a, b) = A_{1/2}(a, b) = \left( \frac{\sqrt{a} + \sqrt{b}}{2} \right)^2,
\]

\[
A(a, b) = A_1(a, b) = \frac{a + b}{2},
\]

\[
S(a, b) = A_2(a, b) = \sqrt{\frac{a^2 + b^2}{2}}.
\]

It is well known that the power mean of order \(r\) given in (1.5) is monotonically increasing in \(r\), then we can write

\[
\min\{a, b\} < H(a, b) < G(a, b) < N_1(a, b) < A(a, b) < S(a, b) < \max\{a, b\}.
\] (1.7)

Recently, the inequalities for means have been the subject of intensive research [1–15]. In particular, many remarkable inequalities for the generalized Heronian and power means can be found in the literature [4–9].

In [4], the authors established two sharp inequalities

\[
\frac{2}{3} G(a, b) + \frac{1}{3} H(a, b) \geq A_{-1/3}(a, b),
\]

\[
\frac{1}{3} G(a, b) + \frac{2}{3} H(a, b) \geq A_{-2/3}(a, b).
\] (1.8)
In [5], Long and Chu found the greatest value $p$ and the least value $q$ such that the double inequality

$$A_p(a,b) \leq A(a,b)^a G(a,b)^\beta H(a,b)^{1-a-\beta} \leq A_q(a,b)$$ (1.9)

holds for all $a, b > 0$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$.

In [6], Shi et al. gave two optimal inequalities

$$A^a(a,b) L^{1-a}(a,b) \leq A_{(1+2\alpha)/3}(a,b),$$
$$G^a(a,b) L^{1-a}(a,b) \leq A_{(1-\alpha)/3}(a,b),$$ (1.10)

for $0 < a < 1$, where

$$L(a,b) = \frac{a-b}{\log a - \log b}, \quad a \neq b,$$ (1.11)

is the logarithmic mean for $a,b > 0$.

In [7], Guan and Zhu obtained sharp bounds for the generalized Heronian mean in terms of the power mean with $\omega > 0$. The optimal values $\alpha$ and $\beta$ such that

$$A_\alpha(a,b) \leq H_\omega(a,b) \leq A_\beta(a,b)$$ (1.12)

holds in general are

(1) in case of $\omega \in (0, 2]$, $\alpha_{\text{max}} = \log 2 / \log (\omega + 2)$ and $\beta_{\text{min}} = 2 / (\omega + 2)$,

(2) in case of $\omega \in [2, +\infty)$, $\alpha_{\text{max}} = 2 / (\omega + 2)$ and $\beta_{\text{min}} = \log 2 / \log (\omega + 2)$.

In this paper, we find the least value $\alpha$ and the greatest value $\beta$, such that for any fixed $1 < r < +\infty$, the inequality

$$H_\alpha(a,b) < A_r(a,b) < H_\beta(a,b)$$ (1.13)

holds for all $a, b > 0$ with $a \neq b$.

**Theorem 1.1.** For $1 < r < +\infty$, the optimal numbers $\alpha$ and $\beta$ such that

$$H_\alpha(a,b) < A_r(a,b) < H_\beta(a,b)$$ (1.14)

is valid for all $a, b > 0$ with $a \neq b$, are $\alpha_{\text{min}} = 2^{1/r} - 2$ and $\beta_{\text{max}} = 2(1 - r)/r$.

Notice that in our case $r > 1$, the two numbers $\alpha_{\text{min}}$ and $\beta_{\text{max}}$ are all negative see Corollary 2.2 below. Thus, the result in this paper is different from [7, Theorem A].
2. Preliminary Lemmas

The following lemma will be repeatedly used in the proof of Theorem 1.1.

**Lemma 2.1.** For $1 < r < +\infty$, one has

\[ r^{2^{1/r-1}} > 1. \]  \tag{2.1}

*Proof.* We show that

\[ m(r) = (1 - r) \log 2 + r \log r > 0, \]  \tag{2.2}

which is clearly equivalent to the claim. Equation (2.2) follows from the facts

\[ \lim_{r \to 1^-} m(r) = 0, \quad m'(r) = -\log 2 + \log r + 1 > 0. \]  \tag{2.3}

**Corollary 2.2.** If $1 < r < +\infty$, then

\[ -2 < \frac{2(1 - r)}{r} < 2^{1/r} - 2 < 0. \]  \tag{2.4}

*Proof.* Since for $1 < r < +\infty$, the two functions

\[ \varphi_1(r) = \frac{2(1 - r)}{r}, \quad \varphi_2(r) = 2^{1/r} - 2 \]  \tag{2.5}

are strictly decreasing, then one has

\[ -2 = \lim_{r \to +\infty} \varphi_1(r) < \varphi_1(r), \quad \varphi_2(r) < \lim_{r \to 1^+} \varphi_2(r) = 0. \]  \tag{2.6}

It suffices to show that

\[ 2 - 2r < r^{2^{1/r} - 2}, \]  \tag{2.7}

which is equivalent to (2.1). \qed

**Lemma 2.3.** For $x > 1$ and $r > 1$, let

\[ \ell(x) = \left( x^{2r} + 1 \right)^{1/r-2} x^{2(r-1)} \left( x^{2r} + 2r - 1 \right). \]  \tag{2.8}

Then, $\ell(x)$ is strictly decreasing for $x > 1$, and

\[ \lim_{x \to 1^+} \ell(x) = r^{2^{1/r-1}}, \quad \lim_{x \to +\infty} \ell(x) = 1. \]  \tag{2.9}
Proof. The fact $\ell(x) > 0$ for $x > 1$ and $r > 1$ is obvious, which allows us to take the logarithmic function of $\ell(x)$,

$$
\log \ell(x) = \left(\frac{1}{r} - 2\right) \log(x^{2r} + 1) + 2(r - 1) \log x + \log(x^{2r} + 2r - 1).
$$

(2.10)

Some tedious, but not difficult calculations lead to

$$
[\log \ell(x)]' = \left(\frac{1}{r} - 2\right) \frac{2rx^{2r-1}}{x^{2r} + 1} + \frac{2(r - 1)}{x} + \frac{2rx^{2r-1}}{x^{2r} + 2r - 1} = m(x) \frac{m(x)}{x(x^{2r} + 1)(x^{2r} + 2r - 1)}.
$$

(2.11)

where

$$
m(x) = 2(1 - 2r)x^{2r}(x^{2r} + 2r - 1) + (2r - 1)(x^{2r} + 1)(x^{2r} + 2r - 1) + 2rx^{2r}(x^{2r} + 1)
= 2(r - 1)(2r - 1)(1 - x^{2r}).
$$

(2.12)

It is easy to see that

$$
\lim_{x \to 1^+} m(x) = 0, \quad (2.13)
$$

$$
m'(x) = -4r(r - 1)(2r - 1)x^{2r-1} < 0. \quad (2.14)
$$

Equation (2.14) implies that $m(x)$ is strictly decreasing for $x > 1$, which together with (2.13) implies $m(x) < 0$ for $x > 1$. Thus, by (2.11),

$$
[\log \ell(x)]' < 0, \quad (2.15)
$$

which implies

$$
\ell'(x) = [\log \ell(x)]' \ell(x) < 0. \quad (2.16)
$$

Hence, $\ell(x)$ is strictly decreasing.

It remains to show (2.9). The first equality in (2.9) is obvious. The second one follows from

$$
\lim_{x \to +\infty} \ell(x) = \lim_{x \to +\infty} \left(x^{2r} + 1\right)^{r-2} x^{2r(1 - r)} \left(x^{2r} + 2r - 1\right)
= \lim_{t \to +\infty} \frac{(2r - 1)t^{2r} + 1}{(1 + t^{2r})^{2r-1/r}}
= 1.
$$

(2.17)

This ends the proof of Lemma 2.3. $\Box$
Lemma 2.4. For $x > 1$, $r > 1$, and $\omega = 2^{1/r} - 2$, let

$$f_r(x) = 2^{1/r}(x^2 + \omega x + 1) - (\omega + 2)(x^{2r} + 1)^{1/r}.$$  \hfill (2.18)

Then,

$$\lim_{x \to +\infty} f_r(x) = -\infty,$$

$$\lim_{x \to +\infty} f_r'(x) = 2^{1/r}(2^{1/r} - 2).$$  \hfill (2.19)

Proof. Simple calculations lead to

$$\lim_{x \to +\infty} f_r(x) = \lim_{x \to +\infty} 2^{1/r}(x^2 + \omega x + 1) - (\omega + 2)(x^{2r} + 1)^{1/r}$$

$$= \lim_{t \to 0^+} \frac{2^{1/r}(t^2 + t\omega + 1) - (\omega + 2)(t^{2r} + 1)^{1/r}}{t^2}$$

$$= -\infty,$$

$$\lim_{x \to +\infty} f_r'(x) = \lim_{x \to +\infty} 2^{1/r}(2x + \omega) - 2(\omega + 2)(x^{2r} + 1)^{1/r}$$

$$= \lim_{t \to 0^+} \frac{2^{1/r}(2 + \omega t) - 2(\omega + 2)(1 + t^{2r})^{(1/r)}}{t}$$

$$= \lim_{t \to 0^+} \frac{2^{1/r}(2 + \omega t)(1 + t^{2r})^{(r-1)/r} - 2(\omega + 2)}{t(1 + t^{2r})^{(r-1)/r}}$$

$$= \lim_{t \to 0^+} \frac{2^{1/r}\omega(1 + t^{2r})^{(r-1)/r} + 2(1/r+1)r(2 + \omega t)(1 + t^{2r})^{1/r}t^{2r-1}}{(1 + t^{2r})^{(r-1)/r} + 2(r - 1)(1 + t^{2r})^{-1/r}t^{2r}}$$

$$= 2^{1/r}\omega = 2^{1/r}(\omega^{1/r} - 2) < 0,$$

where we have used L’Hospital’s law. This ends the proof of Lemma 2.4. \hfill $\square$

3. Proof of Theorem 1.1

Proof. Firstly, we prove that for $1 < r < +\infty$,

$$H_{2(1-r)/r}(a,b) > A_r(a,b),$$  \hfill (3.1)

$$H_{2/r-2}(a,b) < A_r(a,b).$$  \hfill (3.2)
hold true for all \(a, b > 0\) with \(a \neq b\). It is no loss of generality to assume that \(a > b > 0\). Let \(x = \sqrt{b/a} > 1\) and \(\omega \in [2(1-r)/r, 2^{1/r} - 2]\). In view of Corollary 2.2, \(-2 < \omega < 0\). Equations (1.3) and (1.5) lead to

\[
\frac{1}{a} [H_\omega(a, b) - A_r(a, b)] = H_\omega(x^2, 1) - A_r(x^2, 1)
\]

\[
= \frac{x^2 + \omega x + 1}{\omega + 2} - \left(\frac{x^{2r} + 1}{2}\right)^{1/r}
\]

\[
= \frac{2^{1/r} (x^2 + \omega x + 1) - (\omega + 2) (x^{2r} + 1)^{1/r}}{2^{1/r} (\omega + 2)}
\]

\[
= \frac{f_r(x)}{2^{1/r} (\omega + 2)}
\]

where \(f_r(x)\) is defined by (2.18). It is easy to see that

\[
\lim_{x \to 1^+} f_r(x) = 0,
\]

\[
f_r'(x) = 2^{1/r} (2x + \omega) - 2(\omega + 2) \left(x^{2r} + 1\right)^{1/r-1} x^{2r-1},
\]

\[
\lim_{x \to 1^+} f_r'(x) = 0.
\]

By Lemma 2.3,

\[
f_r''(x) = 2 \left\{2^{1/r} - (\omega + 2) \left[2(1-r) \left(x^{2r} + 1\right)^{1/r-2} x^{4r-2} + (2r-1) \left(x^{2r} + 1\right)^{1/r-1} x^{2(r-1)}\right]\right\}
\]

\[
= 2 \left[2^{1/r} - (\omega + 2) \ell'(x)\right] > 2 \left[2^{1/r} - (\omega + 2)r2^{1/r-1}\right] = 2^{1/r} [2 - (\omega + 2)r],
\]

\[
\lim_{x \to 1^+} f_r''(x) = 2 \left\{2^{1/r} - (\omega + 2) \left[2(1-r)2^{1/r-2} + (2r-1)2^{1/r-1}\right]\right\}
\]

\[
= 2^{1/r} [2 - (\omega + 2)r].
\]

We now distinguish between two cases.

**Case 1** \((\omega = 2(1-r)/r)\). Since \(2 - (\omega + 2)r = 0\), then by (3.7), \(f''_r(x) > 0\). Thus, \(f'_r(x)\) is strictly increasing for \(x > 1\), which together with (3.6) implies \(f'_r(x) > 0\). Hence, \(f_r(x)\) is strictly increasing for \(x > 1\). Since (3.4), then \(f_r(x) > 0\). Equation (3.1) follows from (3.3).
Case 2 ($\omega = 2^{1/r} - 2$). By (3.5) and (2.11),

$$f''_r(x) = -2(\omega + 2)e'(x) = -2(\omega + 2) [\log e(x)]' \ell'(x)$$
$$= -2(\omega + 2) \frac{m(x)e(x)}{x(x^{2r} + 1)(x^r + 2r - 1)}$$

$$> 0.$$  \hfill (3.9)

Thus, $f''_r(x)$ is strictly increasing. Equations (3.8) and (2.1) imply

$$\lim_{x \to 1^+} f''_r(x) = 2^{1/r} [2 - (\omega + 2)r] = 2^{1/r+1} \left(1 - r 2^{1/r-1}\right) < 0.$$  \hfill (3.10)

Equations (3.7) and (2.9) imply

$$\lim_{x \to +\infty} f''_r(x) = \lim_{x \to +\infty} 2^{1/r} \left[2^{1/r} - (\omega + 2)\ell'(x)\right] = 2^{1/r - 1} > 0.$$  \hfill (3.11)

Combining (3.10) with (3.11), we obviously know that there exists $\lambda_1 > 1$ such that $f''_r(x) < 0$ for $x \in (1, \lambda_1)$ and $f''_r(x) > 0$ for $x \in (\lambda_1, +\infty)$. This implies that $f'_r(x)$ is strictly decreasing for $x \in (1, \lambda_1)$ and strictly increasing for $x \in (\lambda_1, +\infty)$. By (3.6) and Lemma 2.4, we know that $f'_r(x) < 0$ for $x > 1$. Therefore, $f'_r(x)$ is strictly decreasing. By (3.4) and Lemma 2.4 again, we derive that $f'_r(x) < 0$ for $x > 1$. Equation (3.2) follows from (3.3).

Secondly, we prove that $H_{2^{1/r}-2}(a, b)$ is the best lower bound for the power mean $A_r(a, b)$ for $1 < r < +\infty$. For any $a < 2^{1/r} - 2$,

$$\lim_{x \to +\infty} \frac{H_a(x, 1)}{A_r(x, 1)} = \lim_{x \to +\infty} \frac{2^{1/r} (x + a \sqrt{x} + 1)}{(x^r + 1)^{1/r}} = \frac{2^{1/r}}{a + 2} > 1.$$  \hfill (3.12)

Hence, there exists $X = X(a) > 1$ such that $H_a(x, 1) > A_r(x, 1)$ for $x \in (X, +\infty)$.

Finally, we prove that $H_{2(1-\beta)/r}(a, b)$ is the best upper bound for the power mean $A_r(a, b)$ for $1 < r < +\infty$. For any $\beta > 2(1-r)/r$, by (3.7) (with $\beta$ in place of $\omega$), we have

$$\lim_{x \to 1^r} f''_r(x) = 2^{1/r} [2 - (\beta + 2)r] < 0.$$  \hfill (3.13)

Hence, by the continuity of $f''_r(x)$, there exists $\delta = \delta(\beta) > 0$ such that $f''_r(x) < 0$ for $x \in (1, 1+\delta)$. Thus $f'_r(x)$ is strictly decreasing for $x \in (1, 1+\delta)$. From (3.6), $f'_r(x) > 0$ for $x \in (1, 1+\delta)$. This result together with (3.4) implies that $f'_r(x) < 0$ for $x \in (1, 1+\delta)$. Hence, by (3.3),

$$H_\beta(x^2, 1) < A_r(x^2, 1),$$  \hfill (3.14)

for $x \in (1, 1+\delta)$. \hfill $\Box$
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References
