Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 679201, 9 pages
doi:10.1155/2011/679201

Research Article

Sharp Bounds for Power Mean in Terms of Generalized Heronian Mean

Hongya Gao, Jianling Guo, and Wanguo Yu

College of Mathematics and Computer Science, Hebei University, Baoding 071002, China

Correspondence should be addressed to Gao Hongya, hongya-gao@sohu.com

Received 2 March 2011; Accepted 7 April 2011

Academic Editor: Marcia Federson

For $1 < r < +\infty$, we find the least value α and the greatest value β such that the inequality $H_\alpha(a,b) < A_r(a,b) < H_\beta(a,b)$ holds for all $a,b > 0$ with $a \neq b$. Here, $H_\omega(a,b)$ and $A_r(a,b)$ are the generalized Heronian and the power means of two positive numbers a and b, respectively.

1. Introduction and Statement of Result

For $a,b > 0$ with $a \neq b$, the generalized Heronian mean of a and b is defined by Janous [1] as

$$H_\omega(a,b) = \begin{cases} \dfrac{a+\omega\sqrt{ab}+b}{\omega+2}, & 0 \le \omega < +\infty, \\ \sqrt{ab}, & \omega = +\infty. \end{cases} \tag{1.1}$$

If we take $\omega = 1$ in (1.1), then we arrive at the classical Heronian mean

$$H_e(a,b) = \frac{a+\sqrt{ab}+b}{3}. \tag{1.2}$$

The domain of definition for the function $\omega \mapsto H_\omega(a,b)$ can be extended to all ω with $\omega \in (-2,+\infty)$, that is,

$$H_\omega(a,b) = \begin{cases} \dfrac{a+\omega\sqrt{ab}+b}{\omega+2}, & -2 < \omega < +\infty, \\ \sqrt{ab}, & \omega = +\infty. \end{cases} \tag{1.3}$$

For all fixed $a, b > 0$, it is easy to derive that $\omega \mapsto H_\omega(a,b)$, $-2 < \omega < +\infty$ is monotonically decreasing, and

$$\lim_{\omega \to -2^+} H_\omega(a,b) = +\infty. \tag{1.4}$$

Let

$$A_r(a,b) = \begin{cases} \left(\dfrac{a^r + b^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0, \\ \max\{a,b\}, & r = +\infty, \\ \min\{a,b\}, & r = -\infty, \end{cases} \tag{1.5}$$

denote the power mean of order r. In particular, the harmonic, geometric, square-root, arithmetic, and root-square means of a and b are

$$\begin{aligned} H(a,b) &= A_{-1}(a,b) = \frac{2a}{a+b}, \\ G(a,b) &= A_0(a,b) = \sqrt{ab}, \\ N_1(a,b) &= A_{1/2}(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2, \\ A(a,b) &= A_1(a,b) = \frac{a+b}{2}, \\ S(a,b) &= A_2(a,b) = \sqrt{\frac{a^2+b^2}{2}}. \end{aligned} \tag{1.6}$$

It is well known that the power mean of order r given in (1.5) is monotonically increasing in r, then we can write

$$\min\{a,b\} < H(a,b) < G(a,b) < N_1(a,b) < A(a,b) < S(a,b) < \max\{a,b\}. \tag{1.7}$$

Recently, the inequalities for means have been the subject of intensive research [1–15]. In particular, many remarkable inequalities for the generalized Heronian and power means can be found in the literature [4–9].

In [4], the authors established two sharp inequalities

$$\begin{aligned} \frac{2}{3}G(a,b) + \frac{1}{3}H(a,b) &\geq A_{-1/3}(a,b), \\ \frac{1}{3}G(a,b) + \frac{2}{3}H(a,b) &\geq A_{-2/3}(a,b). \end{aligned} \tag{1.8}$$

In [5], Long and Chu found the greatest value p and the least value q such that the double inequality

$$A_p(a,b) \le A(a,b)^{\alpha} G(a,b)^{\beta} H(a,b)^{1-\alpha-\beta} \le A_q(a,b) \tag{1.9}$$

holds for all $a, b > 0$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$.

In [6], Shi et al. gave two optimal inequalities

$$\begin{aligned} A^{\alpha}(a,b)L^{1-\alpha}(a,b) &\le A_{(1+2\alpha)/3}(a,b), \\ G^{\alpha}(a,b)L^{1-\alpha}(a,b) &\le A_{(1-\alpha)/3}(a,b), \end{aligned} \tag{1.10}$$

for $0 < \alpha < 1$, where

$$L(a,b) = \frac{a-b}{\log a - \log b}, \quad a \ne b, \tag{1.11}$$

is the logarithmic mean for $a, b > 0$.

In [7], Guan and Zhu obtained sharp bounds for the generalized Heronian mean in terms of the power mean with $\omega > 0$. The optimal values α and β such that

$$A_{\alpha}(a,b) \le H_{\omega}(a,b) \le A_{\beta}(a,b) \tag{1.12}$$

holds in general are

(1) in case of $\omega \in (0,2]$, $\alpha_{\max} = \log 2/\log(\omega+2)$ and $\beta_{\min} = 2/(\omega+2)$,

(2) in case of $\omega \in [2,+\infty)$, $\alpha_{\max} = 2/(\omega+2)$ and $\beta_{\min} = \log 2/\log(\omega+2)$.

In this paper, we find the least value α and the greatest value β, such that for any fixed $1 < r < +\infty$, the inequality

$$H_{\alpha}(a,b) < A_r(a,b) < H_{\beta}(a,b) \tag{1.13}$$

holds for all $a, b > 0$ with $a \ne b$.

Theorem 1.1. *For $1 < r < +\infty$, the optimal numbers α and β such that*

$$H_{\alpha}(a,b) < A_r(a,b) < H_{\beta}(a,b) \tag{1.14}$$

is valid for all $a, b > 0$ with $a \ne b$, are $\alpha_{\min} = 2^{1/r} - 2$ and $\beta_{\max} = 2(1-r)/r$.

Notice that in our case $r > 1$; the two numbers $\alpha_{\min}$ and $\beta_{\max}$ are all negative see Corollary 2.2 below. Thus, the result in this paper is different from [7, Theorem A].

2. Preliminary Lemmas

The following lemma will be repeatedly used in the proof of Theorem 1.1.

Lemma 2.1. *For* $1 < r < +\infty$, *one has*

$$r2^{1/r-1} > 1. \tag{2.1}$$

Proof. We show that

$$m(r) = (1-r)\log 2 + r\log r > 0, \tag{2.2}$$

which is clearly equivalent to the claim. Equation (2.2) follows from the facts

$$\lim_{r\to 1^+} m(r) = 0, \qquad m'(r) = -\log 2 + \log r + 1 > 0. \tag{2.3}$$

□

Corollary 2.2. *If* $1 < r < +\infty$, *then*

$$-2 < \frac{2(1-r)}{r} < 2^{1/r} - 2 < 0. \tag{2.4}$$

Proof. Since for $1 < r < +\infty$, the two functions

$$\varphi_1(r) = \frac{2(1-r)}{r}, \qquad \varphi_2(r) = 2^{1/r} - 2 \tag{2.5}$$

are strictly decreasing, then one has

$$-2 = \lim_{r\to+\infty} \varphi_1(r) < \varphi_1(r), \qquad \varphi_2(r) < \lim_{r\to 1^+} \varphi_2(r) = 0. \tag{2.6}$$

It suffices to show that

$$2 - 2r < r2^{1/r} - 2r, \tag{2.7}$$

which is equivalent to (2.1). □

Lemma 2.3. *For* $x > 1$ *and* $r > 1$, *let*

$$\ell(x) = \left(x^{2r} + 1\right)^{1/r-2} x^{2(r-1)} \left(x^{2r} + 2r - 1\right). \tag{2.8}$$

Then, $\ell(x)$ *is strictly decreasing for* $x > 1$, *and*

$$\lim_{x\to 1^+} \ell(x) = r2^{1/r-1}, \qquad \lim_{x\to+\infty} \ell(x) = 1. \tag{2.9}$$

Proof. The fact $\ell(x) > 0$ for $x > 1$ and $r > 1$ is obvious, which allows us to take the logarithmic function of $\ell(x)$,

$$\log \ell(x) = \left(\frac{1}{r} - 2\right) \log\left(x^{2r} + 1\right) + 2(r-1)\log x + \log\left(x^{2r} + 2r - 1\right). \tag{2.10}$$

Some tedious, but not difficult calculations lead to

$$\begin{aligned}\left[\log \ell(x)\right]' &= \left(\frac{1}{r} - 2\right)\frac{2rx^{2r-1}}{x^{2r}+1} + \frac{2(r-1)}{x} + \frac{2rx^{2r-1}}{x^{2r}+2r-1} \\ &= \frac{m(x)}{x(x^{2r}+1)(x^{2r}+2r-1)},\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}m(x) &= 2(1-2r)x^{2r}\left(x^{2r}+2r-1\right) + (2r-1)\left(x^{2r}+1\right)\left(x^{2r}+2r-1\right) + 2rx^{2r}\left(x^{2r}+1\right) \\ &= 2(r-1)(2r-1)\left(1-x^{2r}\right).\end{aligned} \tag{2.12}$$

It is easy to see that

$$\lim_{x\to 1^+} m(x) = 0, \tag{2.13}$$

$$m'(x) = -4r(r-1)(2r-1)x^{2r-1} < 0. \tag{2.14}$$

Equation (2.14) implies that $m(x)$ is strictly decreasing for $x > 1$, which together with (2.13) implies $m(x) < 0$ for $x > 1$. Thus, by (2.11),

$$\left[\log \ell(x)\right]' < 0, \tag{2.15}$$

which implies

$$\ell'(x) = \left[\log \ell(x)\right]' \ell(x) < 0. \tag{2.16}$$

Hence, $\ell(x)$ is strictly decreasing.

It remains to show (2.9). The first equality in (2.9) is obvious. The second one follows from

$$\begin{aligned}\lim_{x\to+\infty} \ell(x) &= \lim_{x\to+\infty} \left(x^{2r}+1\right)^{1/r-2} x^{2(r-1)}\left(x^{2r}+2r-1\right) \\ &= \lim_{t\to 0^+} \frac{(2r-1)t^{2r}+1}{(1+t^{2r})^{(2r-1)/r}} \\ &= 1.\end{aligned} \tag{2.17}$$

This ends the proof of Lemma 2.3. □

Lemma 2.4. *For $x > 1$, $r > 1$, and $\omega = 2^{1/r} - 2$, let*

$$f_r(x) = 2^{1/r}\left(x^2 + \omega x + 1\right) - (\omega + 2)\left(x^{2r} + 1\right)^{1/r}. \tag{2.18}$$

Then,

$$\begin{aligned} \lim_{x \to +\infty} f_r(x) &= -\infty, \\ \lim_{x \to +\infty} f_r'(x) &= 2^{1/r}\left(2^{1/r} - 2\right). \end{aligned} \tag{2.19}$$

Proof. Simple calculations lead to

$$\begin{aligned} \lim_{x \to +\infty} f_r(x) &= \lim_{x \to +\infty} 2^{1/r}\left(x^2 + \omega x + 1\right) - (\omega + 2)\left(x^{2r} + 1\right)^{1/r} \\ &= \lim_{t \to 0^+} \frac{2^{1/r}\left(t^2 + \omega t + 1\right) - (\omega + 2)\left(t^{2r} + 1\right)^{1/r}}{t^2} \\ &= -\infty, \\ \lim_{x \to +\infty} f_r'(x) &= \lim_{x \to +\infty} 2^{1/r}(2x + \omega) - 2(\omega + 2)\left(x^{2r} + 1\right)^{1/r} \\ &= \lim_{t \to 0^+} \frac{2^{1/r}(2 + \omega t) - 2(\omega + 2)\left(1 + t^{2r}\right)^{(1-r)/r}}{t} \\ &= \lim_{t \to 0^+} \frac{2^{1/r}(2 + \omega t)\left(1 + t^{2r}\right)^{(r-1)/r} - 2(\omega + 2)}{t\left(1 + t^{2r}\right)^{(r-1)/r}} \\ &= \lim_{t \to 0^+} \frac{2^{1/r}\omega\left(1 + t^{2r}\right)^{(r-1)/r} + 2^{(1/r)+1}(r-1)(2 + \omega t)\left(1 + t^{2r}\right)^{-1/r} t^{2r-1}}{\left(1 + t^{2r}\right)^{(r-1)/r} + 2(r-1)\left(1 + t^{2r}\right)^{-1/r} t^{2r}} \\ &= 2^{1/r}\omega = 2^{1/r}\left(\omega^{1/r} - 2\right) < 0, \end{aligned} \tag{2.20}$$

where we have used L'Hospital's law. This ends the proof of Lemma 2.4. □

3. Proof of Theorem 1.1

Proof. Firstly, we prove that for $1 < r < +\infty$,

$$H_{2(1-r)/r}(a, b) > A_r(a, b), \tag{3.1}$$

$$H_{2^{1/r}-2}(a, b) < A_r(a, b) \tag{3.2}$$

hold true for all $a, b > 0$ with $a \neq b$. It is no loss of generality to assume that $a > b > 0$. Let $x = \sqrt{b/a} > 1$ and $\omega \in \{2(1-r)/r, 2^{1/r} - 2\}$. In view of Corollary 2.2, $-2 < \omega < 0$. Equations (1.3) and (1.5) lead to

$$\begin{aligned}
\frac{1}{a}[H_\omega(a,b) - A_r(a,b)] &= H_\omega\left(x^2, 1\right) - A_r\left(x^2, 1\right) \\
&= \frac{x^2 + \omega x + 1}{\omega + 2} - \left(\frac{x^{2r} + 1}{2}\right)^{1/r} \\
&= \frac{2^{1/r}\left(x^2 + \omega x + 1\right) - (\omega + 2)\left(x^{2r} + 1\right)^{1/r}}{2^{1/r}(\omega + 2)} \\
&= \frac{f_r(x)}{2^{1/r}(\omega + 2)},
\end{aligned} \tag{3.3}$$

where $f_r(x)$ is defined by (2.18). It is easy to see that

$$\lim_{x \to 1^+} f_r(x) = 0, \tag{3.4}$$

$$f_r'(x) = 2^{1/r}(2x + \omega) - 2(\omega + 2)\left(x^{2r} + 1\right)^{1/r-1} x^{2r-1}, \tag{3.5}$$

$$\lim_{x \to 1^+} f_r'(x) = 0. \tag{3.6}$$

By Lemma 2.3,

$$\begin{aligned}
f_r''(x) &= 2\left\{2^{1/r} - (\omega + 2)\left[2(1-r)\left(x^{2r} + 1\right)^{1/r-2} x^{4r-2} + (2r-1)\left(x^{2r} + 1\right)^{1/r-1} x^{2(r-1)}\right]\right\} \\
&= 2\left[2^{1/r} - (\omega + 2)\ell(x)\right] > 2\left[2^{1/r} - (\omega + 2)r2^{1/r-1}\right] = 2^{1/r}[2 - (\omega + 2)r],
\end{aligned} \tag{3.7}$$

$$\lim_{x \to 1^+} f_r''(x) = 2\left\{2^{1/r} - (\omega + 2)\left[2(1-r)2^{1/r-2} + (2r-1)2^{1/r-1}\right]\right\} = 2^{1/r}[2 - (\omega + 2)r]. \tag{3.8}$$

We now distinguish between two cases.

Case 1 ($\omega = 2(1-r)/r$). Since $2 - (\omega + 2)r = 0$, then by (3.7), $f_r''(x) > 0$. Thus, $f_r'(x)$ is strictly increasing for $x > 1$, which together with (3.6) implies $f_r'(x) > 0$. Hence, $f_r(x)$ is strictly increasing for $x > 1$. Since (3.4), then $f_r(x) > 0$. Equation (3.1) follows from (3.3).

Case 2 ($\omega = 2^{1/r} - 2$). By (3.5) and (2.11),

$$\begin{aligned} f_r'''(x) &= -2(\omega+2)\ell'(x) = -2(\omega+2)\left[\log \ell(x)\right]'\ell(x) \\ &= -2(\omega+2)\frac{m(x)\ell(x)}{x(x^{2r}+1)(x^r+2r-1)} \\ &> 0. \end{aligned} \tag{3.9}$$

Thus, $f_r''(x)$ is strictly increasing. Equations (3.8) and (2.1) imply

$$\lim_{x\to 1^+} f_r''(x) = 2^{1/r}[2-(\omega+2)r] = 2^{1/r+1}\left(1 - r2^{1/r-1}\right) < 0. \tag{3.10}$$

Equations (3.7) and (2.9) imply

$$\lim_{x\to+\infty} f_r''(x) = \lim_{x\to+\infty} 2\left[2^{1/r} - (\omega+2)\ell(x)\right] = 2\left(2^{1/r}-1\right) > 0. \tag{3.11}$$

Combining (3.10) with (3.11), we obviously know that there exists $\lambda_1 > 1$ such that $f_r''(x) < 0$ for $x \in (1,\lambda_1)$ and $f_r''(x) > 0$ for $x \in (\lambda_1,+\infty)$. This implies that $f_r'(x)$ is strictly decreasing for $x \in (1,\lambda_1)$ and strictly increasing for $x \in (\lambda_1,+\infty)$. By (3.6) and Lemma 2.4, we know that $f_r'(x) < 0$ for $x > 1$. Therefore, $f_r(x)$ is strictly decreasing. By (3.4) and Lemma 2.4 again, we derive that $f_r(x) < 0$ for $x > 1$. Equation (3.2) follows from (3.3).

Secondly, we prove that $H_{2^{1/r}-2}(a,b)$ is the best lower bound for the power mean $A_r(a,b)$ for $1 < r < +\infty$. For any $\alpha < 2^{1/r} - 2$,

$$\lim_{x\to+\infty}\frac{H_\alpha(x,1)}{A_r(x,1)} = \lim_{x\to+\infty}\frac{2^{1/r}\left(x+\alpha\sqrt{x}+1\right)}{(\alpha+2)(x^r+1)^r} = \frac{2^{1/r}}{\alpha+2} > 1. \tag{3.12}$$

Hence, there exists $X = X(\alpha) > 1$ such that $H_\alpha(x,1) > A_r(x,1)$ for $x \in (X,+\infty)$.

Finally, we prove that $H_{2(1-r)/r}(a,b)$ is the best upper bound for the power mean $A_r(a,b)$ for $1 < r < +\infty$. For any $\beta > 2(1-r)/r$, by (3.7) (with β in place of ω), we have

$$\lim_{x\to 1^+} f_r''(x) = 2^{1/r}\left[2-(\beta+2)r\right] < 0. \tag{3.13}$$

Hence, by the continuity of $f_r''(x)$, there exists $\delta = \delta(\beta) > 0$ such that $f_r''(x) < 0$ for $x \in (1,1+\delta)$. Thus $f_r(x)$ is strictly decreasing for $x \in (1,1+\delta)$. From (3.6), $f_r'(x) < 0$ for $x \in (1,1+\delta)$. This result together with (3.4) implies that $f_r(x) < 0$ for $x \in (1,1+\delta)$. Hence, by (3.3),

$$H_\beta\left(x^2,1\right) < A_r\left(x^2,1\right), \tag{3.14}$$

for $x \in (1,1+\delta)$. □

Acknowledgments

This work was supported by NSFC (10971224) and NSF of Hebei Province (A2011201011).

References

[1] W. Janous, "A note on generalized Heronian means," *Mathematical Inequalities & Applications*, vol. 4, no. 3, pp. 369–375, 2001.
[2] H. J. Seiffert, "Problem 887," *Nieuw Archief voor Wiskunde*, vol. 11, no. 2, pp. 176–176, 1993.
[3] M. qiji, "Dual mean, logatithmic and Heronian dual mean of two positive numbers," *Journal of Suzhou College of Education*, vol. 16, pp. 82–85, 1999.
[4] Y. M. Chu and W. F. Xia, "Two sharp inequalities for power mean, geometric mean, and harmonic mean," *Journal of Inequalities and Applications*, Article ID 741923, 6 pages, 2009.
[5] B. Y. Long and Y. M. Chu, "Optimal power mean bounds for the weighted geometric mean of classical means," *Journal of Inequalities and Applications*, Article ID 905679, 6 pages, 2010.
[6] M. Y. Shi, Y. M. Chu, and Y. P. Jiang, "Optimal inequalities among various means of two arguments," *Abstract and Applied Analysis*, Article ID 694394, 10 pages, 2009.
[7] K. Z. Guan and H. T. Zhu, "The generalized Heronian mean and its inequalities," *Univerzitet u Beogradu. Publikacije Elektrotehničkog Fakulteta. Serija Matematika*, vol. 17, pp. 60–75, 2006.
[8] P. A. Hästö, "A monotonicity property of ratios of symmetric homogeneous means," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 5, article 71, pp. 1–54, 2002.
[9] J. J. Wen and W. L. Wang, "The optimization for the inequalities of power means," *Journal of Inequalities and Applications*, vol. 2006, Article ID 46782, 25 pages, 2006.
[10] P. A. Hästö, "Optimal inequalities between Seiffert's mean and power means," *Mathematical Inequalities & Applications*, vol. 7, no. 1, pp. 47–53, 2004.
[11] Y. M. Chu and B. Y. Long, "Best possible inequalities between generalized logarithmic mean and classical means," *Abstract and Applied Analysis*, vol. 2010, Article ID 303286, 13 pages, 2010.
[12] Y. M. Chu and W. F. Xia, "Inequalities for generalized logarithmic means," *Journal of Inequalities and Applications*, vol. 2009, Article ID 763252, 7 pages, 2009.
[13] W. F. Xia, Y. M. Chu, and G. D. Wang, "The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means," *Abstract and Applied Analysis*, vol. 2010, Article ID 604804, 9 pages, 2010.
[14] T. Hara, M. Uchiyama, and S. E. Takahasi, "A refinement of various mean inequalities," *Journal of Inequalities and Applications*, vol. 2, no. 4, pp. 387–395, 1998.
[15] E. Neuman and J. Sándor, "On certain means of two arguments and their extensions," *International Journal of Mathematics and Mathematical Sciences*, no. 16, pp. 981–993, 2003.