Research Article

Arens Regularity of Certain Class of Banach Algebras

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Received 6 February 2011; Accepted 2 May 2011

Academic Editor: Marcia Federson

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We study Arens regularity of the left and right module actions of $A$ on $A^{(n)}$, where $A^{(n)}$ is the $n$th dual space of a Banach algebra $A$, and then investigate Arens regularity of $U = A \oplus A^{(n)}$ as a module extension of Banach algebras.

1. Introduction and Preliminaries

In 1951, Arens showed that every bounded bilinear map $m : X \times Y \rightarrow Z$ on normed spaces has two natural but different extensions $m^{''}$ and $m^{''t}$ from $X'' \times Y''$ to $Z''$ [1]. The first extension $m^{''}$ of $m$ is constructed by forming in turn the following bilinear maps:

\[ m' : Z' \times X \rightarrow Y', \quad \langle m'(z', x), y \rangle = \langle z', m(x, y) \rangle, \]
\[ m'' : Y'' \times Z' \rightarrow X', \quad \langle m''(y'', z'), x \rangle = \langle y'', m'(z', x) \rangle, \]
\[ m''' : X'' \times Y'' \rightarrow Z'', \quad \langle m'''(x'', y''), z'' \rangle = \langle x'', m''(y'', z'') \rangle. \] (1.1)

The bilinear map $m'''$ is the unique extension of $m$ which is $\omega^*$-separately continuous on $X \times Y''$. The second extension $m''t$ of $m$ can be made in the same way if we start by transpose map $m' : Y \times X \rightarrow Z$ instead of $m$, which is defined by $m'(y, x) = m(x, y)$. Similarly, it is the unique extension of $m$ that is $\omega^*$-separately continuous on $X'' \times Y$. It is easy to check that

\[ m'''(x'', y'') = \omega^* - \lim_{i} \lim_{j} m(x_i, y_j), \quad m''t(x'', y'') = \omega^* - \lim_{j} \lim_{i} m(x_i, y_j). \] (1.2)
where \((x_i)\) and \((y_j)\) are nets in \(X\) and \(Y\) that converge, in \(w^*-\)topologies, to \(x''\) and \(y''\), respectively. According to [1], \(m\) is said to be Arens regular if \(m'' = m'''\).

For the product map \(\pi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}\) of a Banach algebra \(\mathcal{A}\), we denote \(\pi''(\Phi, \Psi)\) and \(\pi'''(\Phi, \Psi)\) by the symbols \(\Phi \square \Psi\) and \(\Phi \circ \Psi\), respectively. These are called the first and second Arens products on \(\mathcal{A}''\). The Banach algebra \(\mathcal{A}\) is said to be Arens regular if \(\Phi \square \Psi = \Phi \circ \Psi\) on the whole of \(\mathcal{A}''\). The higher extensions \(\pi^{(3n)}\) and \(\pi^{(3n)\ast}\) of \(\pi\) and Arens products on \(\mathcal{A}^{(2n)}\) can be defined similarly. For any fixed \(\Phi \in \mathcal{A}''\), the maps \(\Psi \mapsto \Psi \Phi \Phi\) and \(\Psi \mapsto \Phi \Phi \Psi\) are \(w^*-w^*\) continuous on \(\mathcal{A}''\). Thus with the \(w^*\)-topology, \((\mathcal{A}''', \square)\) is a right topological semigroup and \((\mathcal{A}'', \circ)\) is a left topological semigroup. The following sets

\[
Z_1^{\square}(\mathcal{A}'') = \{ \Phi \in \mathcal{A}'': \Psi \mapsto \Phi \Psi \Phi \text{ is } w^*-w^* \text{ continuous on } \mathcal{A}'' \},
\]

\[
Z_2^{\circ}(\mathcal{A}'') = \{ \Phi \in \mathcal{A}'': \Psi \mapsto \Phi \Psi \Phi \text{ is } w^*-w^* \text{ continuous on } \mathcal{A}'' \}
\]

are called the first and the second topological centres of \(\mathcal{A}''\), respectively. One can verify that \(\mathcal{A}\) is Arens regular if and only if \(Z_1^{\square}(\mathcal{A}'') = Z_2^{\circ}(\mathcal{A}'') = \mathcal{A}''\). For example, the group algebra \(L^1(G)\) for locally compact group \(G\) is Arens regular if and only if \(G\) is finite [2]. The reader is referred to [3, 4] for more information on Arens products and topological centres.

Throughout the paper we identify an element of a Banach space \(X\) with its canonical image in \(X''\). Also for closed linear subspace \(E\) of \(X\) we write \(E^1 = \{ f \in X' : \|f\|_E = 0 \}\).

In [5], Eshaghi Gordji and Filali obtained significant results related to the topological centres of Banach module actions and regularity of bilinear maps. They showed that if \(\mathcal{A}\) enjoys a bounded approximate identity, then the left (right) module action of \(\mathcal{A}\) on \(\mathcal{A}''\) is regular if and only if \(\mathcal{A}\) is reflexive; see also [6].

In this paper, under certain conditions we prove that the left and right module actions of \(\mathcal{A}\) on \(\mathcal{A}^{(n)}\) are regular, where \(\mathcal{A}\) has not bounded approximate identity. Then we apply this fact to determine Arens regularity and quotient Arens regularity of certain class of Banach algebras.

### 2. Arens Regularity of Module Extension Banach Algebras

Suppose that \(X\) is a Banach \(\mathcal{A}\)-bimodule with the left and right module actions \(\pi_1 : \mathcal{A} \times X \to X\) and \(\pi_2 : X \times \mathcal{A} \to X\), respectively. According to [7], \(X''\) is a Banach \(\mathcal{A}''\)-bimodule, where \(\mathcal{A}''\) is equipped with the first Arens product. The module actions are defined by

\[
\Phi \cdot v = w^* - \lim_{i \to j} \hat{a}_i \cdot x_j, \quad v \cdot \Phi = w^* - \lim_{j \to i} x_j \cdot \hat{a}_i,
\]

where \((a_i)\) and \((x_j)\) are nets in \(\mathcal{A}\) and \(X\) that converge, in \(w^*\)-topologies, to \(\Phi\) and \(v\), respectively.

Now suppose that \(\mathcal{H} = \mathcal{A} \oplus X\). Then \(\mathcal{H}\) with norm \(\|(a, x)\| = \|a\| + \|x\|\) and product

\[
(a, x)(b, y) = (ab, a \cdot y + x \cdot b) \quad (a, b \in \mathcal{A}, x, y \in X)
\]
is a Banach algebra which is known as a module extension Banach algebra. The second dual $\mathcal{H}''$ of $\mathcal{H}$ is identified with $\mathcal{A}'' \oplus X''$, as a Banach space. Also the first Arens product $\square$ on $\mathcal{H}''$ is specified by

$$(\Phi, \mu) \square (\Psi, \nu) = (\Phi \Psi, \Phi \cdot \nu + \mu \cdot \Psi).$$

It is straightforward to check that $(\Phi, \mu) \in Z_1^1(\mathcal{H}'')$ if and only if

(a) $\Phi \in Z_1^1(\mathcal{A}'')$,

(b) $\nu \mapsto \Phi \cdot \nu : X'' \to X''$ is $\omega^*rame$ continuous,

(c) $\Psi \mapsto \mu \cdot \Psi : \mathcal{A}'' \to X''$ is $\omega^*rame$ continuous, (see [5, 8]).

If $\mathcal{A}''$ has the second Arens product $\diamond$, then $X''$ is an $\mathcal{A}''$-bimodule in the same way. We denote this module action by the symbol $" \cdot \"$. The second Arens product $\diamond$ on $\mathcal{H}''$ and second topological centre $Z_1^2(\mathcal{H}'')$ of $\mathcal{H}''$ can be defined analogously. Thus, the Banach algebra $\mathcal{H}$ is Arens regular if and only if $\mathcal{A}$ is Arens regular and

$$\Phi \cdot \nu = \Phi \cdot \nu, \quad \nu \cdot \Phi = \nu \cdot \Phi \quad (\Phi \in \mathcal{A}''; \nu \in X'').$$

We consider $\mathcal{A}$ as a Banach $\mathcal{A}$-bimodule equipped with its own multiplication. Then $\mathcal{A} = \mathcal{A}^{(0)}, \mathcal{A}', \mathcal{A}'', \ldots, \mathcal{A}^{(n)}$ can be made into a Banach $\mathcal{A}$-bimodule in a natural fashion [4]. Clearly, regularity of $\mathcal{H} = \mathcal{A} \oplus \mathcal{A}^{(n)}$ implies that of $\mathcal{A}$ but the converse is not true in general. For example, let $\mathcal{A}$ be a nonreflexive Banach space and let $\varphi$ be a nonzero element of $\mathcal{A}'$ such that $\|\varphi\| \leq 1$. Then the product $a \cdot b = \varphi(a)b$ turns $\mathcal{A}$ into a Banach algebra [6], such that $\mathcal{A}^{(2n)}$ is Arens regular for all $n \in \mathbb{N}$.

Now we consider the bilinear mappings

$$\pi_1 : \mathcal{A} \times \mathcal{A}^{(2n-1)} \to \mathcal{A}^{(2n-1)}, \quad \pi_2 : \mathcal{A}^{(2n-1)} \times \mathcal{A} \to \mathcal{A}^{(2n-1)}.$$  

One can verify that $\pi_2$ is Arens regular for all $n \in \mathbb{N}$ but $\pi_1$ is not regular for each $n \in \mathbb{N}$. This shows that $\mathcal{H} = \mathcal{A} \oplus \mathcal{A}^{(2n-1)}$ is not regular. However, $\mathcal{H} = \mathcal{A} \oplus \mathcal{A}^{(2n)}$ is Arens regular.

We commence with the next result which studies Arens regularity of the left and right module actions of $\mathcal{A}$ on $\mathcal{A}^{(2n-1)}$.

**Theorem 2.1.** Let $\mathcal{A}$ be a Banach algebra and $n \in \mathbb{N}$.

(i) If $\pi^{(3n)}(\mathcal{A}^{(2n)}, \mathcal{A}^{(2n)}) \subseteq \mathcal{A}^{(2n-2)}$, then the right module action of $\mathcal{A}$ on $\mathcal{A}^{(2n-1)}$ is Arens regular.

(ii) If $\pi^{(3n)}(\mathcal{A}^{(2n)}, \mathcal{A}^{(2n)}) \subseteq \mathcal{A}^{(2n-2)}$, then the left module action of $\mathcal{A}$ on $\mathcal{A}^{(2n-1)}$ is Arens regular.

**Proof.** We prove (i) that the assertion (ii) can be proved similarly.

Since $\mathcal{A}^{(n+2)} = \mathcal{A}^{(n)} \oplus (\mathcal{A}^{(n-1)})^\perp$ [7], as a direct sum of $\mathcal{A}$-bimodules, it is enough to show that the result is valid for $n = 1$, and it can be deduced for $n \geq 2$, analogously. To this end let $\Phi \in \mathcal{A}''$ and let $(a_i)$ be bounded net in $\mathcal{A}$ that is $\omega^*$-convergent to $\Phi$. Since $\mathcal{A}'' = \mathcal{A} \oplus \mathcal{A}^\perp$,

...
for each $\mu \in \mathcal{A}''$ there exist $f \in \mathcal{A}'$ and $\rho \in \mathcal{A}^\perp$ such that $\mu = f + \rho$. It follows that, for each $\Psi \in \mathcal{A}'', \pi''(a, \Psi) \to \pi''(\Phi, \Psi)$ in the weak topology. So we have that

$$
\langle \mu \cdot \Phi, \Psi \rangle = \langle \Phi, \Psi \cdot \mu \rangle = \lim_i \langle \tilde{a}_i, \Psi \cdot \mu \rangle = \lim_i \langle \mu, \pi''(a_i, \Psi) \rangle = \langle \mu, \pi''(\Phi, \Psi) \rangle = \langle \mu \cdot \Phi, \Psi \rangle.
$$

(2.6)

Therefore the right module action of $\mathcal{A}$ on $\mathcal{A}'$ is regular, as required. □

The corollary below follows from Theorem 3.1 of [5] and Theorem 2.1.

**Corollary 2.2.** Let $\pi_1$ be the left module action of a Banach algebra $\mathcal{A}$ on $\mathcal{A}'$. If $\pi_1''$ is onto and $\mathcal{A}'' \cap \mathcal{A}'' \subseteq \mathcal{A}$, then $\mathcal{A}$ is Arens regular.

The following theorem, which is the main one in the paper, characterizes Arens regularity of $\mathcal{A}''$.  

**Theorem 2.3.** Let $\mathcal{A}$ be an Arens regular Banach algebra. If $\mathcal{A}'' \cap \mathcal{A}'' \subseteq \mathcal{A}$, then, for all $n \in \mathbb{N}$, $\mathcal{A}''$ is Arens regular and

$$
\pi^{(3n+3)} \left( \mathcal{A}^{(2n+2)}, \mathcal{A}^{(2n+2)} \right) \subseteq \mathcal{A}^{(2n)}.
$$

(2.7)

**Proof.** Since $\mathcal{A}$ is Arens regular, $\mathcal{A}''$ is a dual Banach algebra with predual space $E = \mathcal{A}'$ [4]. Let $\mu \in \mathcal{A}''$ and $\Phi \in \mathcal{A}''$. Then the inclusion $\mathcal{A}'' \cap \mathcal{A}'' \subseteq \mathcal{A}$ shows that $\mu \cdot \Phi$ is $\omega^*$-continuous linear functional on $\mathcal{A}''$ and so it must be in $\mathcal{A}'$. It follows that $\beta \cdot \mu = 0$ for all $\beta \in E^\perp$, and hence $\pi^{(6)}(\alpha, \beta) = 0$ for each $\alpha \in E^\perp$. Similarly, we obtain $\pi^{(6)}(\alpha, \beta) = 0 (\alpha, \beta \in E^\perp)$. Then by Proposition 2.16 of [4] $\mathcal{A}''$ is Arens regular and

$$
\pi^{(6)} \left( (\Phi, \alpha), (\Psi, \beta) \right) = (\Phi \square \Psi, \Phi \cdot \beta + \alpha \cdot \Psi), \quad (\Phi, \Psi \in \mathcal{A}'', \alpha, \beta \in E^\perp).
$$

(2.8)

One may verify that $\Phi \cdot \beta = \alpha \cdot \Psi = 0$ and, since $\mathcal{A}^{(4)} = \mathcal{A}'' \oplus E^\perp$, that we have $\pi^{(6)}(\mathcal{A}^{(4)}, \mathcal{A}^{(4)}) \subseteq \mathcal{A}''$. Thus the result is established for $n = 1$. An easy induction argument now finishes the proof. □

As a consequence of Theorems 2.1 and 2.3, we have the next result.

**Corollary 2.4.** Let $\mathcal{A}$ be an Arens regular Banach algebra. If $\mathcal{A}'' \cap \mathcal{A}'' \subseteq \mathcal{A}$, then the following assertions hold for all $n \in \mathbb{N}$.

(i) $\mathcal{H} = \mathcal{A} \oplus \mathcal{A}^{(n)}$ is Arens regular.

(ii) $\mathcal{A}^{(2n-1)}$ is an $\mathcal{A}^{(2n)}$-submodule of $\mathcal{A}^{(2n+1)}$. 

Let $A = l^1$, with pointwise product. Then $A$ is an Arens regular Banach algebra which is not reflexive but satisfies $A'' = A'$. Therefore by the preceding corollary $U = A ⊕ A''$ is Arens regular.

It is easy to verify that regularity of the left and right module actions of $A$ on $A''$ are equivalent for each Arens regular Banach algebra $A$ which is commutative.

Remark 2.5. It is well known that each $C^*$-algebra $A$ is Arens regular and $A''$ is also a $C^*$-algebra [3], and therefore $A''$ itself are Arens regular. This shows that for each $n \in \mathbb{N}$, $A_{(2n)}$ and hence $U = A ⊕ A_{(2n)}$ is Arens regular. But in general, $U = A ⊕ A_{(2n-1)}$ is not Arens regular. Indeed, it is Arens regular if and only if $A$ is reflexive [5].

3. Quotient Arens Regularity of Module Extension Banach Algebras

Let $A$ be a Banach algebra with a bounded approximate identity and let $X = A' · A$, the subspace of $A'$ consisting of the functionals of the form $f · a$, for all $f \in A'$ and $a \in A$. By Cohen's factorization theorem [9], $X$ is a closed $A$-submodule of $A'$. It is also left introverted in $A'$; that is, $\Phi · λ ∈ X$ for each $λ ∈ X$ and $\Phi ∈ A''$. Then $X'$ is a Banach algebra by the following (first Arens type) product:

\[
(\Phi · Ψ, λ) = (\Phi, Ψ · λ) \quad (\Phi, Ψ ∈ X', λ ∈ X).
\] (3.1)

As in [10], the Banach algebra $A$ is said to be left quotient Arens regular if $Z_l(X') = X'$, where

\[
Z_l(X') = \{ \Phi ∈ X' : Ψ → \Phi · Ψ is w*-w* continuous on X' \}. \] (3.2)

Similarly, $X = A · A′$ is an $A$-module and is right introverted in $A'$. As mentioned above, the second Arens product on $A''$ induces naturally a Banach algebra product on $X'$, which is denoted by $\cdot$. The topological centre $Z_l(X')$ and right quotient Arens regularity can be defined analogously. Obviously, every Arens regular Banach algebra is quotient Arens regular but the converse does not hold; see example 38 of [10]. Also a direct proof shows that, if $A$ is an ideal in $A''$, then $A$ is quotient Arens regular.

Proposition 3.1. Suppose that the Banach algebra $A$ is a left ideal in $A''$. Then $U = A ⊕ A_{(2n)}$ is a left ideal in $U'$ for all $n ∈ \mathbb{N}$.

Proof. We first show that, if $A$ is a left ideal in $A''$, then it is also a left ideal in $A_{(2n)}$ for each $n ∈ \mathbb{N}$. So let $a ∈ A$ and $α ∈ A_{(4)}$. Then, for all $μ ∈ A''$, there exist $f ∈ A'$ and $ρ ∈ A^⊥$ such that $μ = f + ρ$. By assumption $a · ρ = 0$, and therefore $a · μ = a · f$. This shows that $a · μ$ is $w^*$-continuous linear functional on $A''$ and so $a · μ ∈ A'$. Since $A_{(4)} = A'' ⊕ (A')^⊥$, $α = Φ + σ$ for some $Φ ∈ A''$ and $σ ∈ (A')^⊥$. Then we have that

\[
(α · a, μ) = (α, a · μ) = (Φ + σ, a · μ) = (Φ, a · μ) = (Φ · a, μ).
\] (3.3)

It follows that $α · a = Φ · a$, and thus $A$ is a left ideal in $A_{(4)}$. An easy induction argument now finishes our claim. Therefore by definition $U$ is a left ideal in $U'$ for each $n ∈ \mathbb{N}$. □
In general, the above result is not valid if we replace $2n$ with $2n - 1$. For example, let $\mathcal{A}$ be the group algebra of an infinite compact group $G$. Then $\mathcal{A}$ is an ideal in $\mathcal{A}''$, as is well known, but $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(2n-1)}$ is not ideal in $\mathcal{U}''$. By additional hypothesis we have the next result.

**Theorem 3.2.** If the Banach algebra $\mathcal{A}$ is a left ideal in $\mathcal{A}''$ and the right module action of $\mathcal{A}$ on $\mathcal{A}^{(2n-2)}$ is regular, then $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(2n-1)}$ is a left ideal in $\mathcal{U}''$.

**Proof.** The result is straightforward for the case $n = 1$. So we give the proof for $n = 2$. Let $(a, \mu) \in \mathcal{U}$ and $(\Phi, \Lambda) \in \mathcal{U}''$. Then a similar argument to what has been used in the proof of the preceding proposition shows that $\Lambda \cdot \tilde{a} \in \mathcal{A}''$. On the other hand, regularity of the right module action of $\mathcal{A}$ on $\mathcal{A}''$ implies that $\Phi \cdot \tilde{\mu}$ is $\tau^*$-continuous linear functional on $\mathcal{A}^{(4)}$ and so it must be in $\mathcal{A}''$. Thus, $\Phi \cdot \tilde{\mu} + \Lambda \cdot \tilde{a} \in \mathcal{A}''$. Therefore by definition we have that $(\Phi, \Lambda) \in \mathcal{U}$, and hence $\mathcal{U}$ is a left ideal in $\mathcal{U}''$. A similar discussion reveals that the result will be established for $n > 2$.

Recall that the right version of Proposition 3.1 and Theorem 3.2 holds. Therefore, we have the following results.

**Corollary 3.3.** Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}$ is an ideal in $\mathcal{A}''$. Suppose that the left and right module actions of $\mathcal{A}$ on $\mathcal{A}^{(2n-2)}$ are regular. Then $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(n)}$ is quotient Arens regular.

**Corollary 3.4.** Let $\mathcal{A}$ be a C$^*$-algebra and $n \in \mathbb{N}$. If $\mathcal{A}$ is an ideal in $\mathcal{A}''$, then $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(n)}$ is quotient Arens regular.

**Example 3.5.** Let $\mathcal{A} = c_0$, with pointwise product and $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}^{(n)}$. Then $\mathcal{A}$ is a commutative C$^*$-algebra which is an ideal in $\mathcal{A}''$. Therefore by the above corollary $\mathcal{U}$ is quotient Arens regular for all $n \in \mathbb{N}$. Note that, by Remark 2.5, $\mathcal{U}$ is not Arens regular for the odd case $n$.

**References**


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