Research Article

Supercyclicity and Hypercyclicity of an Isometry Plus a Nilpotent

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Suppose that X is a separable normed space and the operators A and Q are bounded on X . In this paper, it is shown that if $AQ = QA$, A is an isometry, and Q is a nilpotent then the operator $A + Q$ is neither supercyclic nor weakly hypercyclic. Moreover, if the underlying space is a Hilbert space and A is a co-isometric operator, then we give sufficient conditions under which the operator $A + Q$ satisfies the supercyclicity criterion.

1. Introduction

Let x be a vector in a separable normed space $\mathcal{X}$ and T an operator on $\mathcal{X}$. The orbit of x under T is defined by

$$\text{orb}(T, x) = \{T^n x : n = 0, 1, 2, \dots\}. \quad (1.1)$$

We recall that a vector x in $\mathcal{X}$ is cyclic for an operator T on $\mathcal{X}$ if the closed linear span of $\text{orb}(T, x)$ is $\mathcal{X}$; it is supercyclic, if the set of all scalar multiples of the elements of $\text{orb}(T, x)$ is dense in $\mathcal{X}$; also it is said to be (weakly) hypercyclic if $\text{orb}(T, x)$ is (weakly) dense in $\mathcal{X}$. An operator T is called cyclic, supercyclic, or (weakly) hypercyclic operator, respectively, if it has a cyclic, supercyclic, or (weakly) hypercyclic vector. Recently, the cyclicity of operators has attracted much attention from operator theorists. For a good source on this topic, see [1]. Hilden and Wallen in [2] proved that isometries on Hilbert spaces with dimension more than one are not supercyclic. Ansari and Bourdon in [3] and Miller in [4] independently proved

this fact on Banach spaces. Moreover, recently it is shown in [5] that m -isometric operators on Hilbert spaces, which forms a larger class than isometries, are neither supercyclic nor weakly hypercyclic. In this paper, it is shown that an isometry plus a nilpotent on normed spaces are neither supercyclic nor weakly hypercyclic if they commute. We also discuss this fact when the underlying space is a Hilbert space and the isometry is replaced by a co-isometry. We begin with some elementary properties of such operators. In what follows, as usual, for an operator T , $\sigma_{ap}(T)$, $\sigma_p(T)$, and $\sigma(T)$ are denoted, respectively, the approximate point spectrum, point spectrum, and spectrum of T . Also, $\mathbb{D}$ denotes the open unit disc. Recall that an operator Q on a normed space $\mathcal{X}$ is a nilpotent operator of order $N \geq 1$ if $Q^N = 0$ and $Q^{N-1} \neq 0$. From now on, we assume that Q is a nilpotent operator of order $N \geq 1$ unless stated otherwise.

Proposition 1.1. *Suppose that $\mathcal{X}$ is a normed space, and $A \in \mathcal{B}(\mathcal{X})$ is an isometry such that $AQ = QA$. If $T = A + Q$, then*

- (i) $\sigma(T) = \sigma(A)$,
- (ii) $\sigma_p(T) = \sigma_p(A)$,
- (iii) $\sigma_{ap}(T) = \sigma_{ap}(A)$.

Proof. (i) Suppose that $\lambda \notin \sigma(A)$. Then it is easily seen that

$$(T - \lambda)^{-1} = \sum_{k=1}^N (-1)^{k-1} (A - \lambda)^{-k} Q^{k-1} \quad (1.2)$$

which implies that $\lambda \notin \sigma(T)$. Consequently, $\sigma(T) \subseteq \sigma(A)$. Since $A = T - Q$, a similar argument shows that $\sigma(A) \subseteq \sigma(T)$.

(ii) If $\lambda \in \sigma_p(A)$, there exists $x \neq 0$ such that $Ax = \lambda x$. Therefore,

$$TQ^{N-1}x = AQ^{N-1}x = \lambda Q^{N-1}x. \quad (1.3)$$

Now, if $Q^{N-1}x \neq 0$, then $\lambda \in \sigma_p(T)$; otherwise,

$$TQ^{N-2}x = AQ^{N-2}x = \lambda Q^{N-2}x. \quad (1.4)$$

Also, if $Q^{N-2}x \neq 0$ then $\lambda \in \sigma_p(T)$; otherwise, consider $Q^{N-3}x$ and continue this process to conclude that $Tx = Ax = \lambda x$ which implies that $\lambda \in \sigma_p(T)$. Hence, $\sigma_p(A) \subseteq \sigma_p(T)$. Moreover, since $A = T - Q$, using a similar method, we get $\sigma_p(T) \subseteq \sigma_p(A)$.

(iii) Let $\lambda \in \sigma_{ap}(T)$; then there exists a sequence $(x_n)_n$ in $\mathcal{X}$ such that $\|x_n\| = 1$ and

$$Tx_n - \lambda x_n \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \quad (1.5)$$

Therefore,

$$AQ^{N-1}x_n - \lambda Q^{N-1}x_n \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \quad (1.6)$$

Suppose that there is $c_1 > 0$ so that

$$\|Q^{N-1}x_n\| > c_1 \quad (1.7)$$

for all $n \geq 1$; then $Ay_n - \lambda y_n \rightarrow 0$ as $n \rightarrow +\infty$ where

$$y_n = \frac{Q^{N-1}x_n}{\|Q^{N-1}x_n\|} \quad (1.8)$$

which, in turn, implies that $\lambda \in \sigma_{ap}(A)$.

Now, if (1.7) does not hold, then we can assume, without loss of generality, that $(x_n)_n$ satisfies

$$Q^{N-1}x_n \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (1.9)$$

So by (1.5),

$$AQ^{N-2}x_n - \lambda Q^{N-2}x_n \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (1.10)$$

Now, if there is a constant $c_2 > 0$ such that

$$\|Q^{N-2}x_n\| > c_2 \quad (1.11)$$

for all n , then $Az_n - \lambda z_n \rightarrow 0$ where

$$z_n = \frac{Q^{N-2}x_n}{\|Q^{N-2}x_n\|} \quad (1.12)$$

which implies that $\lambda \in \sigma_{ap}(A)$. Otherwise, we can assume, without loss of generality, that $Q^{N-2}x_n \rightarrow 0$ as $n \rightarrow \infty$ and by (1.5)

$$AQ^{N-3}x_n - \lambda Q^{N-3}x_n \rightarrow 0 \quad (1.13)$$

as $n \rightarrow \infty$. The procedure continues to conclude that $\lambda \in \sigma_{ap}(A)$. Since $A = T - Q$, by a similar method $\sigma_{ap}(A) \subseteq \sigma_{ap}(T)$. $\square$

In the remaining results of this section, the operators A and T are as in Proposition 1.1.

Corollary 1.2. *Suppose that $\mathcal{X}$ is a normed space. Then $T - \lambda I$ is bounded below where $|\lambda| \neq 1$.*

Proof. Since A is an isometry, $\sigma_{ap}(T) = \sigma_{ap}(A) \subseteq \partial\mathbb{D}$. In fact, let $\lambda \in \sigma_{ap}(A)$; then $|\lambda| \leq \|A\| = 1$; moreover, there exists a sequence $(x_n)_n$ in $\mathcal{X}$ with $\|x_n\| = 1$ and so $(A - \lambda I)(x_n) \rightarrow 0$ if $n \rightarrow \infty$. Therefore,

$$0 \leq 1 - |\lambda| \leq \|(A - \lambda I)(x_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (1.14)$$

and so $|\lambda| = 1$. Now, if $|\lambda| \neq 1$, then $\lambda \notin \sigma_{ap}(T)$ and so $T - \lambda$ is bounded below. $\square$

Corollary 1.3. *Suppose that $\mathcal{X}$ is an infinite dimensional Banach space. Then the operator T on $\mathcal{X}$ is not a compact operator.*

Proof. If T is a compact operator, then $0 \in \sigma(T) = \sigma(A)$. Thus $\overline{\mathbb{D}} \subseteq \sigma(T)$ which contradicts the fact that the spectrum of a compact operator is at most countable. $\square$

Proposition 1.4. *If the operators T and A are defined on a normed space $\mathcal{X}$, then $\ker(T - \lambda) \subseteq \ker(A - \lambda)$ for every scalar λ .*

Proof. Fix $\lambda \in \mathbb{C}$ and suppose that $Tx = \lambda x$ for some nonzero vector x . By Proposition 1.1, $\lambda \in \sigma_p(A)$ which implies that $|\lambda| = 1$. Therefore, if $n > N - 1$, we have

$$\begin{aligned} \|x\|^2 &= \|T^n x\|^2 \\ &= \left\| A^{n-(N-1)} \sum_{k=0}^{N-1} \binom{n}{k} Q^k A^{N-1-k} x \right\|^2 \\ &= \left\| \sum_{k=0}^{N-1} \binom{n}{k} Q^k A^{N-1-k} x \right\|^2 \\ &= \left(\binom{n}{N-1} \right)^2 \left\| \sum_{k=0}^{N-1} \frac{(N-1)!(n-N+1)!}{k!(n-k)!} Q^k A^{N-1-k} x \right\|^2. \end{aligned} \quad (1.15)$$

Consequently,

$$\|x\| \geq \binom{n}{N-1} \left[\|Q^{N-1} x\| - \sum_{k=0}^{N-2} \frac{(N-1)!(n-N+1)!}{k!(n-k)!} \|Q^k A^{N-1-k} x\| \right]. \quad (1.16)$$

Since

$$\lim_{n \rightarrow \infty} \frac{(N-1)!(n-N+1)!}{k!(n-k)!} = 0 \quad (1.17)$$

for every $0 \leq k \leq N - 2$, we conclude that $\|Q^{N-1} x\| = 0$. Continue the above process to get $Qx = 0$, and so $Ax = \lambda x$. $\square$

Corollary 1.5. *If $\mathcal{X}$ is a Hilbert space, then the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.*

Proof. Let x and y be eigenvectors of T corresponding to distinct eigenvalues λ_1 and λ_2 . So, $Tx = \lambda_1 x$ and $Ty = \lambda_2 y$. By Proposition 1.4, $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ which implies that $|\lambda_1| = |\lambda_2| = 1$. Suppose that $\langle \cdot \rangle$ denotes the inner product of $\mathcal{X}$. Then

$$0 = \|\lambda_1 x + \lambda_2 y\|^2 - \|x + y\|^2 = 2 \operatorname{Re}(\lambda_1 \overline{\lambda_2} - 1) \langle x, y \rangle. \quad (1.18)$$

Replacing y by iy , we obtain $\operatorname{Im}(\lambda_1 \overline{\lambda_2} - 1) \langle x, y \rangle = 0$; consequently,

$$\left(\frac{\lambda_1}{\lambda_2} - 1 \right) \langle x, y \rangle = (\lambda_1 \overline{\lambda_2} - 1) \langle x, y \rangle = 0. \quad (1.19)$$

But $\lambda_1 \neq \lambda_2$, and so $\langle x, y \rangle = 0$. $\square$

Recall that an operator T is power bounded if there exists some constant $c > 0$ such that $\|T^n\| \leq c$ for all $n = 1, 2, 3, \dots$

Proposition 1.6. *Let $\mathcal{X}$ be a normed space and $x \in \mathcal{X}$. If there is a constant $c > 0$ such that $\|T^n x\| \leq c$ for all $n \geq 1$, then $Qx = 0$. In particular, if T is power bounded, then $Q = 0$.*

Proof. Since the sequence $(\|T^n x\|)_n$ is bounded, an argument similar to the proof of the Proposition 1.4 shows that $Qx = 0$. $\square$

2. Supercyclicity and Hypercyclicity

We begin this section with a useful lemma.

Lemma 2.1. *Let $\mathcal{X}$ be a normed space. For nonnegative integers k, n , if*

$$P_k(n) = x_0 + x_1 n + x_2 n^2 + \dots + x_k n^k \quad (2.1)$$

is a polynomial in n with coefficients in $\mathcal{X}$ of degree k , then the sequence $(\|P_k(n)\|)_n$ is eventually increasing.

Proof. We prove the lemma by induction on k , the degree of the polynomial $P_k(n)$. For $k = 1$, let $P_1(n) = x_0 + x_1 n$. It is easily seen that for every $n \geq 1$

$$\|P_1(n+1)\| \leq \frac{1}{2}(\|P_1(n)\| + \|P_1(n+2)\|). \quad (2.2)$$

Since $\lim_{n \rightarrow \infty} \|P_1(n)\| = +\infty$, there is a positive integer i such that

$$\|P_1(i)\| < \|P_1(i+1)\|. \quad (2.3)$$

This fact coupled with (2.2) implies that

$$0 < \|P_1(i+1)\| - \|P_1(i)\| \leq \|P_1(n+1)\| - \|P_1(n)\| \quad (2.4)$$

for every $n \geq i$. Therefore, the sequence $(\|P_1(n)\|)_{n \geq i}$ is increasing. Suppose that $(\|P_k(n)\|)_n$ is eventually increasing and let

$$P_{k+1}(n) = x_0 + x_1 n + \dots + x_{k+1} n^{k+1}, \quad (2.5)$$

where $x_{k+1} \neq 0$. Since

$$\lim_{n \rightarrow \infty} \|x_1 + x_2(n+1) + \cdots + x_{k+1}(n+1)^k\| = +\infty, \quad (2.6)$$

using the induction hypothesis there is a positive integer j such that for every $n \geq j$

$$\|x_1 + x_2(n+1) + \cdots + x_{k+1}(n+1)^k\| \geq \max\{2\|x_0\|, \|x_1 + x_2n + \cdots + x_{k+1}n^k\|\}. \quad (2.7)$$

Therefore,

$$\frac{\|P_{k+1}(n+1)\|}{\|P_{k+1}(n)\|} \geq \frac{(n+1)\|x_1 + x_2(n+1) + \cdots + x_{k+1}(n+1)^k\| - \|x_0\|}{n\|x_1 + x_2n + \cdots + x_{k+1}n^k\| + \|x_0\|} \geq 1 \quad (2.8)$$

for every $n \geq j$. Hence, the sequence $(\|p_{k+1}(n)\|)_{n \geq j}$ is increasing. $\square$

Theorem 2.2. Suppose that $\mathcal{X}$ is a normed space, and $A \in B(\mathcal{X})$ is an isometry such that $AQ = QA$. If $T = A + Q$, then the operator T is neither supercyclic nor weakly hypercyclic.

Proof. Let $\tilde{\mathcal{X}}$ be the completion of $\mathcal{X}$ and $\tilde{T}$, $\tilde{A}$, and $\tilde{Q}$ the extensions of T , A , and Q on $\tilde{\mathcal{X}}$, respectively. Thus, $\tilde{T} = \tilde{A} + \tilde{Q}$ where $\tilde{A}$ is an isometry and $\tilde{Q}$ is a nilpotent operator; moreover, $\tilde{A}\tilde{Q} = \tilde{Q}\tilde{A}$. Also, note that the supercyclicity of the operator T implies the supercyclicity of $\tilde{T}$. So we can assume, without loss of generality, that $\mathcal{X}$ is a Banach space.

As we have seen in the proof of Proposition 1.4, if $x \in \mathcal{X}$ then

$$\|T^n x\| = \left\| \sum_{k=0}^{N-1} \binom{n}{k} Q^k A^{N-1-k} x \right\|, \quad (2.9)$$

and so by Lemma 2.1, the sequence $(\|T^n x\|)_n$ is eventually increasing. Suppose that x_0 is a supercyclic vector for T . Thus, for any $x \in \mathcal{X}$ there is a sequence $(n_i)_i$ of positive integers and a sequence $(\alpha_i)_i$ of scalars such that $\alpha_i T^{n_i} x_0 \rightarrow x$. Moreover, since the sequence $(\|T^n x_0\|)_n$ is eventually increasing, we have $\|\alpha_i T^{n_i} x_0\| \leq \|\alpha_i T^{n_i+1} x_0\|$ for large i . So letting $i \rightarrow \infty$, we conclude that $\|x\| \leq \|Tx\|$, for all x in $\mathcal{X}$. On the other hand, the supercyclicity of T implies that it has a dense range and so is invertible. Thus, in light of Proposition 1.1 we see that A is invertible. It is easy to see that

$$T^{-1} = A^{-1} + P, \quad (2.10)$$

where

$$P = \sum_{k=1}^{N-1} (-1)^k A^{-(k+1)} Q^k. \quad (2.11)$$

Since $P^N = 0$, by a similar argument the sequence $(\|T^{-n}x\|)_n$ is eventually increasing for every $x \in \mathcal{X}$. But T^{-1} is also supercyclic (see [1, Theorem 1.12]); therefore,

$$\|x\| \leq \|T^{-1}x\| \quad (2.12)$$

for every $x \in \mathcal{X}$. Thus, T is an isometry which implies that it is not a supercyclic operator.

To show that the operator T is not weakly hypercyclic, note that

$$\|T^{*n}x^*\| = \left\| \sum_{k=0}^{N-1} \binom{n}{k} Q^{*k} A^{*n-k} x^* \right\| \quad (2.13)$$

for every $x^* \in \mathcal{X}^*$ and every positive integer n . If $\ker Q^* \neq \{0\}$, then there is a nonzero $x^* \in \mathcal{X}^*$ such that $\|T^{*n}x^*\| = \|A^{*n}x^*\| \leq \|x^*\|$ because $\|A^*\| = \|A\| = 1$. Now, suppose that x_0 is a weakly hypercyclic vector for T . Since $\text{orb}(T, x_0)$ is weakly dense in $\mathcal{X}$ and x^* is nonzero, the set $\{x^*(T^n x_0) : n \geq 0\}$ is dense in $\mathbb{C}$. But

$$\|x^*(T^n x_0)\| = \|(T^{*n}x^*)(x_0)\| \leq \|T^{*n}x^*\| \|x_0\| \leq \|x^*\| \|x_0\| \quad (2.14)$$

for all $n \geq 0$, which is a contradiction. If $\ker Q^* = \{0\}$, then $Q^* = 0$ and so $T = A$ is not a weakly hypercyclic operator. $\square$

We remark that there are Banach space isometries which are also weakly supercyclic. Indeed, the unweighted bilateral weighted shift on the space $l^p(\mathbb{Z})$ where $p > 2$ is weakly supercyclic (see [1, Corollary 10.32]). However, the question that whether an isometry plus a nonzero nilpotent which commute with each other, are weakly supercyclic or not is still an open question.

The following examples show that the commutativity of A and Q is essential in the preceding theorem.

Example 2.3. Let $(e_n)_{n=-\infty}^{+\infty}$ be the standard orthonormal basis for $l^2(\mathbb{Z})$. Define the isometric operator A by $Ae_n = e_{n+1}$ for all $n \in \mathbb{Z}$ and the weighted shift operator Q by $Qe_n = w_n e_{n+1}$, where $w_{2n} = 0$ for all integers n , $w_{2n-1} = 1/(2n-1)^2$ for all $n \geq 1$, and $w_{2n-1} = 1/(1-2n)$ for all $n \leq 0$. Note that $Q^2 = 0$ and $AQ \neq QA$. Moreover, since $1 \leq \inf_n (1 + w_n) \leq \sup_n (1 + w_n) \leq 2$, the weighted shift operator $T = A + Q$ is invertible. To see that T is supercyclic by Theorem 3.4 of [6], it is sufficient to show that

$$\lim_{n \rightarrow \infty} \prod_{j=1}^n (1 + w_j) \prod_{j=1}^n \frac{1}{1 + w_{-j}} = 0. \quad (2.15)$$

But $\prod_{j=1}^{\infty} (1 + w_j)$ is finite, because $\sum_{j=1}^{\infty} w_j < \infty$. Furthermore, $\prod_{j=1}^{\infty} 1/(1 + w_{-j}) = 0$, because

$$\begin{aligned} \sum_{j=1}^{\infty} \left(1 - \frac{1}{1 + w_{-j}} \right) &= \sum_{j=1}^{\infty} \frac{w_{-j}}{1 + w_{-j}} = \sum_{j=1}^{\infty} \frac{w_{-(2j-1)}}{1 + w_{-(2j-1)}} \\ &= \sum_{j=1}^{\infty} \frac{1/(2j-1)}{1 + 1/(2j-1)} = \sum_{j=1}^{\infty} \frac{1}{2j} = \infty \end{aligned} \quad (2.16)$$

(see [7, pages 299 and 300]). Therefore, (2.15) holds.

Example 2.4. Consider the isometric operator A on $\ell^2(\mathbb{Z})$ defined by $Ae_n = e_{n-1}$ and the weighted shift operator Q defined by $Qe_n = w_n e_{n-1}$, where $w_{2n} = 0$ for all $n \in \mathbb{Z}$, $w_{2n-1} = 1/(2n-1)$, for $n \geq 1$, and $w_{2n-1} = 1/(2n-1)^2$ for $n \leq 0$. Note that $Q^2 = 0$ and $AQ \neq QA$. Also, since

$$(1 + w_1)(1 + w_2) \cdots (1 + w_n) \geq w_1 + w_2 + \cdots + w_n \quad (2.17)$$

for all $n \geq 1$, and $\sum_{n=1}^{\infty} w_n = \infty$, we conclude that

$$\lim_{n \rightarrow \infty} (1 + w_1)(1 + w_2) \cdots (1 + w_n) = \infty. \quad (2.18)$$

Furthermore,

$$\lim_{n \rightarrow \infty} (1 + w_{-1})(1 + w_{-2}) \cdots (1 + w_{-n}) < \infty, \quad (2.19)$$

because

$$\sum_{n=1}^{\infty} w_{-n} < \infty. \quad (2.20)$$

Hence, using Corollary 10.27 of [1], we observe that the operator $A + Q$ is weakly hypercyclic.

3. A Co-isometry Plus a Nilpotent

From now on, we assume that $\mathcal{H}$ is a separable Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Recall that the unilateral shift operator $S : \mathcal{H} \rightarrow \mathcal{H}$ is given by $Se_n = e_{n+1}$ for all n and the backward shift operator $B : \mathcal{H} \rightarrow \mathcal{H}$ is defined by $Be_0 = 0$ and $Be_n = e_{n-1}$ for all $n \geq 1$. It is known that the operator B is supercyclic (see [1, page 9]). It follows that a co-isometry can be supercyclic. In this section, we give sufficient conditions such that the sum of a co-isometry and a nilpotent is supercyclic on $\mathcal{H}$.

Theorem 3.1. Suppose that A is a co-isometric operator on a Hilbert space $\mathcal{H}$. Then A is supercyclic if and only if $\cap_{n \geq 0} A^{*n} \mathcal{H} = (0)$.

Proof. First assume that $\cap_{n \geq 0} A^{*n} \mathcal{H} = (0)$. Then by the von Neumann-Wold decomposition, $A^* = S^m$ for some positive integer m (see [8]). Therefore, $A = B^m$ which is a positive

power of a supercyclic operator and so is supercyclic [9]. For the converse, assume that $M = \cap_{n \geq 0} A^{*n} \mathcal{H} \neq (0)$, and let P_M denote the orthogonal projection on M . By the von Neumann-Wold decomposition, M is a reducing subspace for A and $A^*|_M$ is unitary. Since $(A|_M)^* = P_M A^*|_M$ is also unitary, the operator $A|_M$ is not supercyclic. Assume that A is supercyclic, and $h = g \oplus k$ is a supercyclic vector for A , where $g \in M$ and $k \in M^\perp$. If $g = 0$, then $\mathcal{H} = M^\perp$ which is impossible; so $g \neq 0$. Take $f \in M$, and let $\varepsilon > 0$ be arbitrary. Then there is a nonnegative integer n and a scalar $\alpha \in \mathbb{C}$ such that

$$\|\alpha A^n g - f\| \leq \|\alpha A^n (g \oplus k) - (f \oplus 0)\| < \varepsilon. \quad (3.1)$$

Hence, g is a supercyclic vector for $A|_M$ which is impossible. $\square$

To prove the next theorem, we need the supercyclicity criterion due to H. N. Salas (see [10], or more generally [11]).

Supercyclicity Criterion

Suppose that X is a separable Banach space and T is a bounded operator on X . If there is an increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$, and two dense sets Y and Z of X such that

- (1) there exists a function $S : Z \rightarrow Z$ satisfying $TSx = x$ for all $x \in Z$,
- (2) $\|T^{n_k} x\| \cdot \|S^{n_k} y\| \rightarrow 0$ for every $x \in Y$ and $y \in Z$,

then T is supercyclic.

Theorem 3.2. *Suppose that A is a co-isometry on a Hilbert space $\mathcal{H}$ such that $\cap_{n \geq 0} A^{*n} \mathcal{H} = (0)$. If $AQ = QA$, then the operator $T = A + Q$ satisfies the supercyclicity criterion.*

Proof. By Corollary 1.2, the operator T^* is bounded below and so is left invertible. Consequently, T is a right invertible operator. Let $x \in \cap_{n \geq 0} T^{*n} \mathcal{H}$. For every $i \geq 0$, there is a vector x_{N+i} in $\mathcal{H}$ such that $T^{*N+i} x_{N+i} = x$. Since $Q^N = 0$, we have

$$\begin{aligned} x &= T^{*N+i} x_{N+i} \\ &= \sum_{k=0}^{N+i} \binom{N+i}{k} (A^{*k} Q^{*N+i-k})(x_{N+i}) \\ &= \sum_{k=i+1}^{N+i} \binom{N+i}{k} (A^{*k} Q^{*N+i-k})(x_{N+i}) \end{aligned} \quad (3.2)$$

which implies that $x \in A^{*i+1} \mathcal{H}$. Hence, $x \in \cap_{n \geq 0} T^{*n} \mathcal{H} = (0)$ and so the operator T admits a complete set of eigenvectors that is, $\mathcal{H} = \bigvee_{\mu \in \mathbb{D}_r} \ker(T - \mu)$ for every positive real number r , where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ (see [12], Part (A) of the lemma). Since T^* is bounded below, TT^* is invertible. Take $S = T^*(TT^*)^{-1}$. Choose $r > 0$ so that $r < 1/\|S\|$, and let

$$Y = Z = \text{span}\{\ker(T - \mu) : \mu \in \mathbb{D}_r\}. \quad (3.3)$$

Now, if $h \in Y = Z$, then

$$\|T^n h\| \|S^n h\| \leq |\mu|^n \|S\|^n \|h\| \leq (r\|S\|)^n \|h\| \longrightarrow 0 \quad (3.4)$$

as $n \rightarrow \infty$. Finally, $T^n S^n h = h$ for every $h \in \mathcal{H}$ and every $n \geq 0$. Thus, the operator T satisfies the supercyclicity criterion. $\square$

The Hilbert-Schmidt class, $\mathcal{C}_2(\mathcal{H})$, is the class of all bounded operators S defined on a Hilbert space $\mathcal{H}$, satisfying

$$\|S\|_2^2 = \sum_{n=1}^{\infty} \|S e_n\|^2 < +\infty, \quad (3.5)$$

where $\|\cdot\|$ is the norm on $\mathcal{H}$ induced by its inner product. We recall that $\mathcal{C}_2(\mathcal{H})$ is a Hilbert space equipped with the inner product $\langle S, T \rangle = \text{tr}(ST^*)$ in which $\text{tr}(S^*T)$ denotes the trace of S^*T . Furthermore, $\mathcal{C}_2(\mathcal{H})$ is an ideal of the algebra of all bounded operators on $\mathcal{H}$, see [8]. For any bounded operator B on a Hilbert space $\mathcal{H}$, the left multiplication operator L_B and the right multiplication operator R_B on $\mathcal{C}_2(\mathcal{H})$ are defined by $L_B(S) = BS$ and $R_B(S) = SB$ for all $S \in \mathcal{C}_2(\mathcal{H})$. It is known that an operator B satisfies the supercyclicity criterion if and only if L_B is supercyclic on the space $\mathcal{C}_2(\mathcal{H})$ (see [13, page 37]). In the following proposition, we see that an operator T may satisfy the supercyclicity criterion although R_T is not a supercyclic operator on $\mathcal{C}_2(\mathcal{H})$.

Proposition 3.3. *Suppose that $\mathcal{H}$ is a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ is a co-isometry such that $\cap_{n \geq 0} A^{*n} \mathcal{H} = (0)$ and $AQ = QA$. Then the operator $T = A + Q$ satisfies the supercyclicity criterion but the operator R_T is not supercyclic on $\mathcal{C}_2(\mathcal{H})$.*

Proof. By Theorem 3.2, the operator T satisfies the supercyclicity criterion. Moreover, for every $S \in \mathcal{C}_2(\mathcal{H})$ we have

$$\begin{aligned} \|R_A(S)\|_2^2 &= \|SA\|_2^2 = \|(SA)^*\|_2^2 = \|A^*S^*\|_2^2 \\ &= \sum_{n=1}^{\infty} \|A^*S^*e_n\|^2 = \sum_{n=1}^{\infty} \|S^*e_n\|^2 = \|S\|_2^2, \end{aligned} \quad (3.6)$$

which implies that R_A is an isometry. Also, if $S \in \mathcal{C}_2(\mathcal{H})$, then $R_Q^N(S) = 0$. Since $R_T(S) = R_A(S) + R_Q(S)$, Theorem 2.2 implies that R_T is not supercyclic. $\square$

The proof of the following proposition is similar to the proof of the second part of Theorem 2.2, and we omit it.

Proposition 3.4. *Suppose that $\mathcal{X}$ is a normed space and $A \in \mathcal{B}(\mathcal{X})$ is a co-isometry such that $AQ = QA$. Then the operator $T = A + Q$ is not weakly hypercyclic.*

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