Research Article

On Alzer and Qiu's Conjecture for Complete Elliptic Integral and Inverse Hyperbolic Tangent Function

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We prove that the double inequality $(\pi/2)(\operatorname{arth} r/r)^{3/4+\alpha^*r} < \mathcal{K}(r) < (\pi/2)(\operatorname{arth} r/r)^{3/4+\beta^*r}$ holds for all $r \in (0, 1)$ with the best possible constants $\alpha^* = 0$ and $\beta^* = 1/4$, which answer to an open problem proposed by Alzer and Qiu. Here, $\mathcal{K}(r)$ is the complete elliptic integrals of the first kind, and arth is the inverse hyperbolic tangent function.

1. Introduction

For $r \in [0, 1]$, Langedre's complete elliptic integrals of the first and second kind [1] are defined by

$$\begin{aligned}\mathcal{K} &= \mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} d\theta, \\ \mathcal{K}' &= \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) &= \frac{\pi}{2}, \quad \mathcal{K}(1) = \infty, \\ \mathcal{E} &= \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta, \\ \mathcal{E}' &= \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) &= \frac{\pi}{2}, \quad \mathcal{E}(1) = 1,\end{aligned}\tag{1.1}$$

respectively. Here and in what follows, we set $r' = \sqrt{1-r^2}$. These integrals are special cases of Gaussian hypergeometric function

$$F_2(a, b; c; x) = F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!} \quad (-1 < x < 1), \quad (1.2)$$

where $(a, n) = \prod_{k=0}^{n-1} (a+k)$. Indeed, we have

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \quad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right). \quad (1.3)$$

It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory, quasiconformal analysis, theory of mean values, number theory, and other related fields [2–13].

Recently, the complete elliptic integrals have been the subject of intensive research. In particular, many remarkable properties and inequalities can be found in the literature [3, 10–18].

In 1992, Anderson et al. [15] discovered that $\mathcal{K}$ can be approximated by the inverse hyperbolic tangent function, arth, and proved that

$$\frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{1/2} < \mathcal{K}(r) < \frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right), \quad (1.4)$$

for $r \in (0, 1)$.

In [16], Alzer and Qiu proved that the double inequality

$$\frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{\alpha} < \mathcal{K}(r) < \frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{\beta}, \quad (1.5)$$

holds for all $r \in (0, 1)$ with the best possible constants $\alpha = 3/4$ and $\beta = 1$ and proposed an open problem as follows.

Open Problem #

The double inequality

$$\frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{3/4+\alpha^*r} < \mathcal{K}(r) < \frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{3/4+\beta^*r}, \quad (1.6)$$

holds for all $r \in (0, 1)$ with the best possible constants $\alpha^* = 0$ and $\beta^* = 1/4$.

It is the aim of this paper to give a positive answer to the open problem #.

2. Lemmas and Theorem

In order to establish our main result, we need several formulas and lemmas, which we present in this section.

For $0 < r < 1$, the following derivative formulas were presented in [4, Appendix E, pages 474-475]:

$$\begin{aligned} \frac{d\mathcal{K}}{dr} &= \frac{\mathcal{E} - r'^2 \mathcal{K}}{rr'^2}, & \frac{d\mathcal{E}}{dr} &= \frac{\mathcal{E} - \mathcal{K}}{r}, \\ \frac{d(\mathcal{E} - r'^2 \mathcal{K})}{dr} &= r\mathcal{K}, & \frac{d(\mathcal{K} - \mathcal{E})}{dr} &= \frac{r\mathcal{E}}{r'^2}. \end{aligned} \quad (2.1)$$

Lemma 2.1 (see [4, Theorem 1.25]). For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and be differentiable on (a, b) , let $g'(x) \neq 0$ be on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \quad \frac{f(x) - f(b)}{g(x) - g(b)}. \quad (2.2)$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

The following Lemma 2.2 can be found in [9, Lemma 3(1)] and [4, Theorem 3.21(1) and Exercise 3.43(30) and (46)].

Lemma 2.2. (1) $[(r')^c \operatorname{arth} r]/r$ is strictly decreasing in $(0, 1)$ if and only if $c \geq 2/3$;
 (2) $(\mathcal{E} - r'^2 \mathcal{K})/r^2$ is strictly increasing from $(0, 1)$ onto $(\pi/4, 1)$;
 (3) $(\mathcal{E} - r'^2 \mathcal{K})/(r^2 \mathcal{K})$ is strictly decreasing from $(0, 1)$ onto $(0, 1/2)$;
 (4) $r\mathcal{K}/\operatorname{arth} r$ is strictly decreasing from $(0, 1)$ onto $(1, \pi/2)$.

Lemma 2.3. (1) $f_1(r) = [r - r'^2 \operatorname{arth} r]/r^3$ is strictly increasing from $(0, 1)$ onto $(2/3, 1)$;
 (2) $f_2(r) = (\log[\operatorname{arth}(r)/r])/r^2$ is strictly increasing from $(0, 1)$ onto $(1/3, \infty)$;
 (3) $f_3(r) = [\mathcal{E} \operatorname{arth} r - r'^2 \mathcal{K} \operatorname{arth}(r)/4 - 3r\mathcal{K}/4]/r^5$ is strictly increasing from $(0, 1)$ onto $(\pi/480, \infty)$;
 (4) $f_4(r) = (3/4 + r/4)(r - r'^2 \operatorname{arth} r)\mathcal{K} - (\mathcal{E} - r'^2 \mathcal{K}) \operatorname{arth} r$ is positive and strictly increasing in $(\sqrt{2}/2, 1)$;
 (5) $f_5(r) = (3/4 + r^2) \log[\operatorname{arth}(r)/r] - \log(2\mathcal{K}/\pi)$ is positive and strictly increasing on $(0, 1/4)$.

Proof. For part (1), let $h_1(r) = r - r'^2 \operatorname{arth} r$ and $h_2(r) = r^3$. Then $f_1(r) = h_1(r)/h_2(r)$, $h_1(0) = h_2(0) = 0$ and

$$\frac{h'_1(r)}{h'_2(r)} = \frac{2}{3} \frac{\operatorname{arth} r}{r}. \quad (2.3)$$

It is well known that the function $r \mapsto \operatorname{arth}(r)/r$ is strictly increasing from $(0, 1)$ onto $(1, \infty)$. Therefore, from (2.3) and Lemma 2.1 together with l'Hôpital's rule, we know that $f_1(r)$ is strictly increasing in $(0, 1)$, $f_1(0^+) = 2/3$ and $f_1(1^-) = 1$.

For part (2), clearly $f_2(1^-) = +\infty$. Let $h_3(r) = \log[\operatorname{arth}(r)/r]$ and $h_4(r) = r^2$, then $f_2(r) = h_3(r)/h_4(r)$, $h_3(0) = h_4(0) = 0$, and

$$\frac{h'_3(r)}{h'_4(r)} = \frac{r - r'^2 \operatorname{arth} r}{2r^2 r'^2 \operatorname{arth} r} = \frac{1}{2} \frac{r - r'^2 \operatorname{arth} r}{r^3} \frac{r}{r'^2 \operatorname{arth} r}. \quad (2.4)$$

It follows from Lemma 2.1, Lemma 2.2(1), part (1), (2.4), and l'Hôpital's rule that $f_2(r)$ is strictly increasing in $(0, 1)$ and $f_2(0^+) = 1/3$.

For part (3), from Lemma 2.2(4), we clearly see that $f_3(1^-) = +\infty$. Let $h_5(r) = \xi \operatorname{arth} r - r'^2 \mathcal{K} \operatorname{arth}(r)/4 - 3r \mathcal{K}(r)/4$, $h_6(r) = r^5$, $h_7(r) = (\xi - r'^2 \mathcal{K})/(4r'^2) - r \mathcal{K} \operatorname{arth}(r)/2 + 3 \operatorname{arth}(r)(\xi - r'^2 \mathcal{K})/(4r)$, and $h_8(r) = r^4$, then $f_3(r) = h_5(r)/h_6(r)$, $h_5(0) = h_6(0) = h_7(0) = h_8(0) = 0$,

$$\begin{aligned} \frac{h'_5(r)}{h'_6(r)} &= \frac{1}{5} \frac{h_7(r)}{h_8(r)}, \\ \frac{h'_7(r)}{h'_8(r)} &= \frac{1}{4r'^4} \frac{r - r'^2 \operatorname{arth} r}{r^3} \left[\frac{3}{4} \frac{\xi(r) - r'^2 \mathcal{K}(r)}{r^2} - \frac{1}{4} \xi(r) \right]. \end{aligned} \quad (2.5)$$

From Lemma 2.2(2) and part (1), we clearly see that $h'_7(r)/h'_8(r)$ is strictly increasing in $(0, 1)$. Thus, the monotonicity of $f_3(r)$ can be obtained from (2.5) and Lemma 2.1. Moreover, making use of l'Hôpital's rule, we have $f_3(0^+) = \pi/480$.

For part (4), let $h_9(r) = 2(1 + r) - \xi/(r \mathcal{K}) - 3(\xi - r'^2 \mathcal{K})/(r^2 \mathcal{K})$. Then, Lemma 2.2(3) leads to the conclusion that $h_9(r)$ is strictly increasing in $(0, 1)$. Note that

$$h_9\left(\frac{\sqrt{2}}{2}\right) = 1.013 \dots > 0, \quad (2.6)$$

$$f_4\left(\frac{\sqrt{2}}{2}\right) = 0.084 \dots > 0, \quad (2.7)$$

$$f'_4(r) = \frac{(\mathcal{K} - \xi) + r \mathcal{K}(r)}{4(1 + r)} + \frac{r \mathcal{K} \operatorname{arth} r}{4} h_9(r) > \frac{r \mathcal{K} \operatorname{arth} r}{4} h_9\left(\frac{\sqrt{2}}{2}\right) > 0 \quad (2.8)$$

for $r \in (\sqrt{2}/2, 1)$.

Therefore, part (4) follows from (2.7) and (2.8).

For part (5), simple computations lead to

$$\lim_{r \rightarrow 0^+} f_5(r) = 0, \quad (2.9)$$

$$f'_5(r) = 2r \log\left(\frac{\operatorname{arth} r}{r}\right) + \left(\frac{3}{4} + r^2\right) \frac{r - r'^2 \operatorname{arth} r}{r r'^2 \operatorname{arth} r} - \frac{\xi - r'^2 \mathcal{K}}{r r'^2 \mathcal{K}}. \quad (2.10)$$

Making use of parts (1)–(4), one has

$$\begin{aligned} \frac{r'^2 \mathcal{K} \operatorname{arth} r}{r^4} f_5'(r) &= \frac{2r'^2 \mathcal{K} \operatorname{arth} r}{r} f_2(r) + \mathcal{K} f_1(r) - f_3(r) \\ &> K f_1(r) - f_3(r) > \frac{\pi}{3} - f_3\left(\frac{1}{4}\right) = 1.040 \dots > 0 \end{aligned} \quad (2.11)$$

for $r \in (0, 1/4)$.

Therefore, part (5) follows from (2.9) and (2.11). $\square$

Lemma 2.4. *Let*

$$g_c(r) = \left(\frac{3}{4} + cr\right) \log \left[\frac{\operatorname{arth}(r)}{r} \right] - \log \left(\frac{2\mathcal{K}}{\pi} \right) \quad (c \in \mathbb{R}), \quad (2.12)$$

then the following statements are true:

- (1) $g_c(r) > 0$ for all $r \in (0, 1)$ if and only if $c \in [1/4, \infty)$;
- (2) $g_c(r) < 0$ for all $r \in (0, 1)$ if and only if $c \in (-\infty, 0]$.

Proof. Firstly, we prove that $g_c(r) > 0$ for $c \in [1/4, \infty)$. Since $g_c(r)$ is continuous and strictly increasing with respect to $c \in \mathbb{R}$ for fixed $r \in (0, 1)$, it suffices to prove that $g_{1/4}(r) > 0$ for all $r \in (0, 1)$. Note that

$$\lim_{r \rightarrow 0^+} g_{1/4}(r) = 0, \quad (2.13)$$

$$g'_{1/4}(r) = \frac{1}{4} \log \left(\frac{\operatorname{arth} r}{r} \right) + \left(\frac{3}{4} + \frac{1}{4}r \right) \frac{r - r'^2 \operatorname{arth} r}{rr'^2 \operatorname{arth} r} - \frac{\mathcal{E} - r'^2 \mathcal{K}}{rr'^2 \mathcal{K}}. \quad (2.14)$$

We divide the proof into two cases.

Case 1 ($r \in (0, \sqrt{2}/2]$). Then, making use of Lemma 2.3(1)–(3) and (2.14), we have

$$\begin{aligned} \frac{r'^2 \mathcal{K} \operatorname{arth} r}{r^3} g'_{1/4}(r) &= \frac{r'^2 \mathcal{K} \operatorname{arth} r}{4r} f_2(r) + \frac{1}{4} \mathcal{K}(r) f_1(r) - r f_3(r) \\ &> \frac{1}{4} \mathcal{K}(r) f_1(r) - r f_3(r) > \frac{\pi}{12} - \frac{\sqrt{2}}{2} f_3\left(\frac{\sqrt{2}}{2}\right) \\ &= 0.250 \dots > 0. \end{aligned} \quad (2.15)$$

Case 2 ($r \in (\sqrt{2}/2, 1)$). Then, making use of Lemma 2.3(4) and (2.14), we get

$$\frac{g'_{1/4}(r)}{\log[\operatorname{arth}(r)/r]} = \frac{1}{4} + \frac{f_4(r)}{rr'^2 \mathcal{K} \operatorname{arth} r \log[\operatorname{arth}(r)/r]} > 0. \quad (2.16)$$

Inequalities (2.15) and (2.16) imply that $g_{1/4}(r)$ is strictly increasing in $(0, 1)$. Therefore, $g_{1/4}(r) > 0$ follows from (2.13) and the monotonicity of $g_{1/4}(r)$.

On the other hand, inequality (1.5) leads to the conclusion that $g_c(r) < 0$ for all $r \in (0, 1)$ and $c \in (-\infty, 0]$.

Next, we prove that the parameters $1/4$ and 0 are the best possible parameters in Lemma 2.4(1) and (2), respectively.

If $c \in (0, 1/4)$, then $g_c(c) = f_5(c) > 0$ follows from Lemma 2.3(5). Moreover, let

$$F(r) = \frac{g_c(r)}{\log[\operatorname{arth}(r)/r]} = \frac{3}{4} + cr - \frac{\log(2\mathcal{K}/\pi)}{\log[\operatorname{arth}(r)/r]}, \quad (2.17)$$

then, using l'Hôpital's rule and Lemma 2.2(4), we get

$$\lim_{r \rightarrow 1^+} F(r) = c - \frac{1}{4} < 0. \quad (2.18)$$

Inequality (2.18) implies that there exists $\delta = \delta(c) > 0$ such that $F(r) < 0$ for all $r \in (1 - \delta, 1)$. Therefore, $g_c(r) < 0$ for $r \in (1 - \delta, 1)$ follows from (2.17). $\square$

From Lemma 2.4, we clearly see that the following Theorem 2.5 holds, which give a positive answer to the open problem #.

Theorem 2.5. *The double inequality*

$$\frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{3/4+\alpha^*r} < \mathcal{K}(r) < \frac{\pi}{2} \left(\frac{\operatorname{arth} r}{r} \right)^{3/4+\beta^*r} \quad (2.19)$$

holds for all $r \in (0, 1)$ with the best possible constants $\alpha^* = 0$ and $\beta^* = 1/4$.

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