Research Article

Integral-Type Operators from Bloch-Type Spaces to $Q_K$ Spaces

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The boundedness and compactness of the integral-type operator $I_{\psi,g}^{(n)}f(z) = \int_0^zf^{(n)}(\psi(\zeta))g(\zeta)\,d\zeta$, where $n \in \mathbb{N}_0$, $\psi$ is a holomorphic self-map of the unit disk $D$, and $g$ is a holomorphic function on $D$, from $\alpha$-Bloch spaces to $Q_K$ spaces are characterized.

1. Introduction

Let $D$ be the open unit disk in the complex plane, $\partial D$ be its boundary, $D(w, r)$ be disk centered at $w$ of radius $r$, and let $H(D)$ be the class of all holomorphic functions on $D$. Let

$$\eta_a(z) = \frac{a - z}{1 - az}, \quad a \in D,$$

be the involutive Möbius transformation which interchanges points 0 and $a$. If $X$ is a Banach space, then by $B_X$ we will denote the closed unit ball in $X$.

The $\alpha$-Bloch space, $B^\alpha(D) = B^\alpha$, $\alpha > 0$, consists of all $f \in H(D)$ such that

$$\sup_{z \in D} \left(1 - |z|^2\right)^\alpha |f'(z)| < \infty.$$
The little $\alpha$-Bloch space $\mathcal{B}_0^\alpha(D) = \mathcal{B}_0^\alpha$ consists of all functions $f$ holomorphic on $D$ such that $\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0$. The norm on $\mathcal{B}_0^\alpha$ is defined by

$$\|f\|_{\mathcal{B}_0^\alpha} = |f(0)| + \sup_{z \in \overline{D}} \left(1 - |z|^2\right)^\alpha |f'(z)|.$$  (1.3)

With this norm, $\mathcal{B}_0^\alpha$ is a Banach space, and the little $\alpha$-Bloch space $\mathcal{B}_0^\alpha$ is a closed subspace of the $\alpha$-Bloch space. Note that $\mathcal{B}_1^1 = \mathcal{B}$ is the usual Bloch space.

Given a nonnegative Lebesgue measurable function $K$ on $(0, 1]$ the space $Q_K$ consists of those $f \in H(D)$ for which

$$b_{Q_K}^2(f) = \sup_{a \in D} \int_D |f'(z)|^2 K\left(1 - |\eta_a(z)|^2\right) dm(z) < \infty,$$  (1.4)

where $dm(z) = (1/\pi) dx \, dy = (1/\pi) r \, dr \, d\theta$ is the normalized area measure on $D$ [1]. It is known that $b_{Q_K}$ is a seminorm on $Q_K$ which is Mobius invariant, that is,

$$b_{Q_K}(f \circ \eta) = b_{Q_K}(f), \quad \eta \in \text{Aut}(D),$$  (1.5)

where Aut$(D)$ is the group of all automorphisms of the unit disk $D$. It is a Banach space with the norm defined by

$$\|f\|_{Q_K} = |f(0)| + b_{Q_K}(f).$$  (1.6)

The space $Q_{K,0}$ consists of all $f \in H(D)$ such that

$$\lim_{|a| \to 1} \int_D |f'(z)|^2 K\left(1 - |\eta_a(z)|^2\right) dm(z) = 0.$$  (1.7)

It is known that $Q_{K,0}$ is a closed subspace of $Q_K$. For classical $Q$ spaces, see [2].

It is clear that each $Q_K$ contains all constant functions. If $Q_K$ consists of just constant functions, we say that it is trivial. $Q_K$ is nontrivial if and only if

$$\sup_{t \in (0, 1)} \int_0^1 K(1 - r) \frac{(1 - t)^2}{(1 - tr^2)^3} r \, dr < \infty.$$  (1.8)

Throughout this paper, we assume that condition (1.8) is satisfied, so that the space $Q_K$ is nontrivial. An important tool in the study of $Q_K$ spaces is the auxiliary function $\lambda_K$ defined by

$$\lambda_K(s) = \sup_{0 \leq d \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty,$$  (1.9)

where the domain of $K$ is extended to $(0, \infty)$ by setting $K(t) = K(1)$ when $t > 1$. The next two conditions play important role in the study of $Q_K$ spaces.
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(a) There is a constant $C > 0$ such that for all $t > 0$

$$K(2t) \leq CK(t).$$  \hspace{1cm} (1.10)

(b) The auxiliary function $\lambda_K$ satisfies the following condition:

$$\int_0^1 \frac{\lambda_K(s)}{s} ds < \infty.$$  \hspace{1cm} (1.11)

Let $\Omega(0, \infty)$ denote the class of all nondecreasing continuous functions on $(0, \infty)$ satisfying conditions (1.8), (1.10), and (1.11).

A positive Borel measure $\mu$ on $\mathbb{D}$ is called a $K$-Carleson measure [3] if

$$\sup_{I} \int_{S(I)} K \left( \frac{1-|z|}{|I|} \right) d\mu(z) < \infty,$$

where the supermum is taken over all subarcs $I \subset \partial \mathbb{D}$, $|I|$ is the length of $I$, and $S(I)$ is the Carleson box defined by

$$S(I) = \left\{ z : 1 - |I| < |z| < 1, \frac{z}{|z|} \in I \right\}.$$  \hspace{1cm} (1.13)

A positive Borel measure $\mu$ is called a vanishing $K$-Carleson measure if

$$\lim_{|I| \to 0} \int_{S(I)} K \left( \frac{1-|z|}{|I|} \right) d\mu(z) = 0.$$  \hspace{1cm} (1.14)

We also need the following results of Wulan and Zhu in [3], in which $Q_K$ spaces are characterized in terms of $K$-Carleson measures.

**Theorem 1.1.** Let $K \in \Omega(0, \infty)$. Then a positive Borel measure $\mu$ on $\mathbb{D}$ is a $K$-Carleson measure if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} K \left( 1 - |\eta_a(z)|^2 \right) d\mu(z) < \infty.$$  \hspace{1cm} (1.15)

Also, $\mu$ is a vanishing $K$-Carleson measure if and only if

$$\lim_{|a| \to 1} \int_{\mathbb{D}} K \left( 1 - |\eta_a(z)|^2 \right) d\mu(z) = 0.$$  \hspace{1cm} (1.16)

From Theorem 1.1 and the definition of the spaces $Q_K$ and $Q_{K,0}$, we see that when $K \in \Omega(0, \infty)$, then $f \in Q_K$ if and only if the measure $d\mu_f = |f'(z)|^2 dm(z)$ is a $K$-Carleson measure, while $f \in Q_{K,0}$ if and only if this measure is a vanishing $K$-Carleson measure.
Let \( \varphi \in S(\mathbb{D}) \) be the family of all holomorphic self-maps of \( \mathbb{D} \), \( g \in H(\mathbb{D}) \), and \( n \in \mathbb{N}_0 \). We define an integral-type operator as follows:

\[
I_n^{\varphi, g} f(z) = \int_0^z f^{(n)}(\varphi(\zeta)) g(\zeta) d\zeta, \quad z \in \mathbb{D}.
\] (1.17)

Operator (1.17) extends several operators which have been introduced and studied recently (see, e.g., [4–9]). For related operators in \( n \)-dimensional case, see, for example, [10–19]. For some classical operators see, for example, [20, 21] and the related references therein. For other product-type operators, see [22] and the references therein.

Motivated by [23, 24] (see also [25–29]), we characterize when \( \varphi \) and \( g \) induce bounded and/or compact operators in (1.17) from \( \alpha \)-Bloch to \( Q_K \) spaces.

Throughout this paper, constants are denoted by \( C \); they are positive and not necessarily the same at each occurrence. The notation \( A \approx B \) means that there is a positive constant \( C \) such that \( B/C \leq A \leq CB \).

2. Auxiliary Results

Here, we quote several lemmas which will be used in the proofs of the main results in this paper. The following lemma is folklore (see, e.g., [30]).

**Lemma 2.1.** For any \( f \in H(\mathbb{D}) \) and \( z \in \mathbb{D} \), the following inequalities hold

\[
|f(z)| \leq \begin{cases} 
\|f\|_{\mathcal{B}^\alpha} & \text{if } 0 < \alpha < 1, \\
\|f\|_{\mathcal{B}^\alpha} \ln \frac{e}{1 - |z|^2} & \text{if } \alpha = 1, \\
\|f\|_{\mathcal{B}^\alpha} \left( \frac{1}{1 - |z|^2} \right)^{\alpha-1} & \text{if } \alpha > 1,
\end{cases}
\] (2.1)

\[
|f^{(n)}(z)| \leq C \frac{\sup_{w \in D(z, (1-|z|)/2)} \left( 1 - |w|^2 \right)^n |f'(w)|}{\left( 1 - |z|^2 \right)^{a+n-1}} 
\leq C \frac{\|f\|_{\mathcal{B}^\alpha}}{\left( 1 - |z|^2 \right)^{a+n-1}}, \quad n \in \mathbb{N}.
\] (2.2)

The next lemma is obtained in [31, 32].

**Lemma 2.2.** Let \( \alpha > 0 \). Then there are two functions \( f_1, f_2 \in \mathcal{B}^\alpha \) such that

\[
|f_1(z)| + |f_2(z)| \geq \frac{C}{\left( 1 - |z|^2 \right)^\alpha}, \quad z \in \mathbb{D}.
\] (2.3)
Also, if \( \alpha \neq 1 \), then there are two functions \( f_3, f_4 \in \mathcal{B}^a \) and \( C > 0 \), such that

\[
|f_3(z)| + |f_4(z)| \geq \frac{C}{(1 - |z|^2)^{a-1}}, \quad z \in \mathbb{D}.
\]  

The next Schwartz-type lemma [33] is proved in a standard way, so we omit the proof.

**Lemma 2.3.** Let \( \alpha > 0, K \in \Omega(0, \infty), \varphi \in S(\mathbb{D}), g \in H(\mathbb{D}) \), and \( n \in \mathbb{N}_0 \). Then \( I_{\psi,g}^{(n)} : \mathcal{B}^a \) (or \( \mathcal{B}^a_n \)) \( \to Q_K \) is compact if and only if for any bounded sequence \( (f_m)_{m\in\mathbb{N}} \) in \( \mathcal{B}^a \) converging to zero on compacts of \( \mathbb{D} \), we have

\[
\lim_{m \to \infty} \|I_{\psi,g}^{(n)}(f_m)\|_{Q_K} = 0.
\]

**Proof.** By a known theorem \( I_{\psi,g}^{(n)} : \mathcal{B}^a_0 \to Q_K \) (or \( Q_{K,0} \)) is weakly compact if and only if \( (I_{\psi,g}^{(n)})^* : Q_K^* \) (or \( Q_{K,0}^* \)) \( \to \mathcal{B}^a_0^* \) is weakly compact. Since \( \mathcal{B}^a_0 \) is equivalent to \( \mathcal{B}^a_0^* \), is compact, which is equivalent to \( I_{\psi,g}^{(n)} : \mathcal{B}^a_0 \to Q_K \) (or \( Q_{K,0} \)), is compact, as claimed. \( \square \)

**Lemma 2.4.** Let \( \alpha > 0, K \in \Omega(0, \infty), \varphi \in S(\mathbb{D}), g \in H(\mathbb{D}) \), and \( n \in \mathbb{N}_0 \). Then \( I_{\psi,g}^{(n)} : \mathcal{B}^a_0 \to Q_K \) (or \( Q_{K,0} \)) is weakly compact if and only if it is compact.

**Proof.** By Lemma 2.4, \( I_{\psi,g}^{(n)} : \mathcal{B}^a_0 \to Q_{K,0} \) is compact if and only if it is weakly compact, which, by Gantmacher’s theorem ([34]), is equivalent to \( (I_{\psi,g}^{(n)})^{**} = (Q_{K,0}^*) \to \mathcal{B}^a_0^* \). Since \( \mathcal{B}^a_0^* = \mathcal{B}^a_0 \) and by a standard duality argument \( (I_{\psi,g}^{(n)})^{**} = I_{\psi,g}^{(n)} \) on \( \mathcal{B}^a_0 \), this can be written as \( I_{\psi,g}^{(n)}(\mathcal{B}^a_0) \subseteq Q_{K,0} \), which by the closed graph theorem is equivalent to \( I_{\psi,g}^{(n)} : \mathcal{B}^a \to Q_{K,0} \) is bounded. \( \square \)

For \( a \in \mathbb{D} \), set

\[
\Phi_{\psi,g,K}(a) = \int_{\mathbb{D}} K \left( 1 - |\eta_a(z)|^2 \right) |g(z)|^2 \left( 1 - |\varphi(z)|^2 \right)^{2(1-a-n)} dm(z).
\]

**Lemma 2.6.** Let \( \alpha > 0, K \in \Omega(0, \infty), \varphi \in S(\mathbb{D}), g \in H(\mathbb{D}) \), and \( n \in \mathbb{N}_0 \). If \( \Phi_{\psi,g,K} \) is finite at some point \( a \in \mathbb{D} \), then it is continuous on \( \mathbb{D} \).

**Proof.** We follow the lines of Lemma 2.3 in [24]. From the elementary inequality

\[
\frac{(1 - |a|)(1 - |a_1|)}{4} \leq \frac{1 - |\eta_a(z)|^2}{1 - |\eta_{a_1}(z)|^2} \leq \frac{4}{(1 - |a|)(1 - |a_1|)}, \quad a, a_1, z \in \mathbb{D},
\]

and since \( K \) is nondecreasing and satisfies (1.10), we easily get

\[
K \left( 1 - |\eta_a(z)|^2 \right) \leq C^{\log_2(4/(1-|a|)(1-|a_1|))}[1 + K \left( 1 - |\eta_a(z)|^2 \right)].
\]
Moreover, if $I_{\varphi,g,K}(a)$ is finite, it follows that $I_{\varphi,g,K}$ is finite at each point $a_1 \in \mathbb{D}$. Let $a \in \mathbb{D}$ be fixed, and let $(a_i)_{i \in \mathbb{N}} \subset \mathbb{D}$ be a sequence converging to $a$.

We have

$$|I_{\varphi,g,K}(a) - I_{\varphi,g,K}(a_i)| \leq \int_{\mathbb{D}} \frac{|g(z)|^2 |K| \left(1 - |\eta_a(z)|^2\right) - K \left(1 - |\eta_a(z)|^2\right)}{(1 - |\varphi(z)|^2)^{(a+n-1)}} \, dm(z). \tag{2.8}$$

From (2.6), we have that for $l$ such that $1 - |a_i| \geq (1 - |a|)/2$, say $l \geq l_0$, holds

$$1 - |\eta_{a_i}(z)|^2 \leq \frac{8}{(1 - |a|)^2} \left(1 - |\eta_a(z)|^2\right), \tag{2.9}$$

and consequently for $l \geq l_0$, it holds

$$\left|K \left(1 - |\eta_a(z)|^2\right) - K \left(1 - |\eta_{a_i}(z)|^2\right)\right| \leq \left(1 + C^{[\log_2(8/(1-|a|)^2)]+1}\right)K \left(1 - |\eta_a(z)|^2\right). \tag{2.10}$$

From this and since $I_{\varphi,g,K}$ is finite at $a$, by the Lebesgue dominated convergence theorem, we get that the integral in (2.8) converges to zero as $l \to \infty$ which implies that $I_{\varphi,g,K}(a) \to I_{\varphi,g,K}(a)$ as $l \to \infty$, from which the lemma follows. \hfill \Box

### 3. Boundedness and Compactness of $I_{\varphi,g}^{(n)} : B^a (\text{or } B^a_0) \to Q_K (\text{or } Q_{K,0})$

In this section, we characterize the boundedness and compactness of the operators $I_{\varphi,g}^{(n)} : B^a (\text{or } B^a_0) \to Q_K (\text{or } Q_{K,0})$. Let

$$d\mu_{\varphi,g,n,a}(z) = |g(z)|^2 \left(1 - |\varphi(z)|^2\right)^{(1-a-n)} \, dm(z). \tag{3.1}$$

**Theorem 3.1.** Let $\alpha > 0$, $K \in \Omega(0, \infty)$, $\varphi \in S(\mathbb{D})$, $g \in H(\mathbb{D})$, and $n \in \mathbb{N}$, or $n = 0$ and $\alpha > 1$. Then the following statements are equivalent.

(a) $I_{\varphi,g}^{(n)} : B^a \to Q_K$ is bounded.

(b) $I_{\varphi,g}^{(n)} : B^a_0 \to Q_K$ is bounded.

(c) $M := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} K \left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(1 - |\varphi(z)|^2\right)^{(1-a-n)} \, dm(z) < \infty$.

(d) $d\mu_{\varphi,g,n,a}(z)$ is a $K$-Carleson measure.

Moreover, if $I_{\varphi,g}^{(n)} : B^a \to Q_K$ is bounded, then the next asymptotic relations hold

$$\|I_{\varphi,g}^{(n)}\|_{B^a \to Q_K} \times \|I_{\varphi,g}^{(n)}\|_{B^a_0 \to Q_K} \times M^{1/2}. \tag{3.2}$$
Proof. By Theorem 1.1, it is clear that (c) and (d) are equivalent.

(c) $\Rightarrow$ (a). Let $f \in B^r$. First note that $I_{\psi, \delta}^{(n)}f(0) = 0$ for each $f \in H(\mathbb{D})$ and $n \in \mathbb{N}_0$. From this and by Lemma 2.1, we have

$$
\left\| I_{\psi, \delta}^{(n)}f \right\|_{Q_K}^2 = \sup_{a \in \mathbb{D}} \int_D \left| \left( I_{\psi, \delta}^{(n)}f \right)'(z) \right|^2 K \left( 1 - |\eta_a(z)|^2 \right) dm(z)
$$

$$
= \sup_{a \in \mathbb{D}} \int_D \left| f^{(n)}(\varphi(z)) \right|^2 |g(z)|^2 K \left( 1 - |\eta_a(z)|^2 \right) dm(z)
$$

$$
\leq C \left\| f \right\|_{B^a}^2 \sup_{a \in \mathbb{D}} \int_D K \left( 1 - |\eta_a(z)|^2 \right) |g(z)|^2 \left( 1 - |\varphi(z)|^2 \right)^{2(1 - \alpha - n)} dm(z),
$$

from which the boundedness of $I_{\psi, \delta}^{(n)} : B^a \rightarrow Q_K$ follows, and moreover

$$
\left\| I_{\psi, \delta}^{(n)} \right\|_{B^R \rightarrow Q_K} \leq CM^{1/2}.
$$

(a) $\Rightarrow$ (b). This implication is obvious.

(b) $\Rightarrow$ (c). By Lemma 2.2, if $n \in \mathbb{N}$, there are two functions $f_1, f_2 \in B^\alpha$ such that (2.3) holds, and if $n = 0$ and $\alpha > 1$ such that (2.4) holds. Let

$$
h_1(z) = f_1(z) - \sum_{k=1}^{n-1} \frac{f_1^{(k)}(0)}{k!} z^k, \quad h_2(z) = f_2(z) - \sum_{k=1}^{n-1} \frac{f_2^{(k)}(0)}{k!} z^k.
$$

It is known (see [30]) that for each $f \in H(\mathbb{D})$ and $n \in \mathbb{N}$, we have

$$
\left( 1 - |z|^2 \right)^{\alpha + n - 1} \left| f^{(n)}(z) \right| + \sum_{k=1}^{n-1} \left| f^{(k)}(0) \right| \times \left( 1 - |z|^2 \right)^\alpha |f'(z)|.
$$

From this, Lemma 2.2, and since $h_1^{(k)}(0) = h_2^{(k)}(0) = 0$, $k = 0, 1, \ldots, n - 1$, we have that there is a $\delta > 0$ such that

$$
C \left( 1 - |z|^2 \right)^{-\alpha + n - 1} \leq \left| h_1^{(n)}(z) \right| + \left| h_2^{(n)}(z) \right|, \quad \text{for} \ |z| > \delta.
$$

Now note that for any $f \in B^a$, the functions $f_r(z) = f(rz)$, $r \in (0, 1)$ belong to $B^a$, and moreover, $\sup_{0 < r < 1} \left\| f_r \right\|_{B^a} \leq \left\| f \right\|_{B^a}$. 
Applying (3.7), using an elementary inequality, the boundedness of $I_{\psi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$, and the last inequality, we obtain
\[
\int_{|\psi(z)|<\delta} r^{2n} K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(1 - |r| |\varphi(z)|^2\right)^{2(1-\alpha-n)} \, dm(z)
\] \[
\leq C \int_{\mathcal{D}} r^{2n} K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(\left|I_{\psi,g}^{(n)}(h_1)_r\right|^2 + \left|I_{\psi,g}^{(n)}(h_2)_r\right|^2\right) \, dm(z)
\]
\[
= C \int_{\mathcal{D}} K\left(1 - |\eta_a(z)|^2\right) \left|\left(I_{\psi,g}^{(n)}(h_1)_r\right)'(z)\right|^2 \, dm(z)
\] \[
+ C \int_{\mathcal{D}} K\left(1 - |\eta_a(z)|^2\right) \left|\left(I_{\psi,g}^{(n)}(h_2)_r\right)'(z)\right|^2 \, dm(z)
\]
\[
\leq \|I_{\psi,g}^{(n)}\|_{\mathcal{B}_0^\alpha \to Q_K}^2 \left(\|h_1\|^2_{\mathcal{B}^\alpha} + \|h_2\|^2_{\mathcal{B}^\alpha}\right).
\] (3.8)

Letting $r \to 1$ in (3.8) and using the monotone convergence theorem, we get
\[
\int_{|\psi(z)|<\delta} K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(1 - |\varphi(z)|^2\right)^{2(1-\alpha-n)} \, dm(z) \leq C \|I_{\psi,g}^{(n)}\|_{\mathcal{B}_0^\alpha \to Q_K}^2.
\] (3.9)

On the other hand, for $f_0(z) = z^n/n! \in \mathcal{B}_0^\alpha$, we get $I_{\psi,g}^{(n)} f_0 \in Q_K$ which implies
\[
\sup_{a \in \mathcal{D}} \int_{|\psi(z)|<\delta} K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(1 - |\varphi(z)|^2\right)^{2(1-\alpha-n)} \, dm(z) \leq \frac{\|I_{\psi,g}^{(n)}\|^2_{\mathcal{B}_0^\alpha \to Q_K} \|f_0\|^2_{\mathcal{B}^\alpha}}{(1 - \delta^2)^{2(\alpha+n-1)}}.
\] (3.10)

From (3.9) and (3.10), (c) follows. Moreover we get $M^{1/2} \leq C \|I_{\psi,g}^{(n)}\|_{\mathcal{B}_0^\alpha \to Q_K}$. From this, (3.4) and since $\|I_{\psi,g}^{(n)}\|_{\mathcal{B}_0^\alpha \to Q_K} \leq \|I_{\psi,g}^{\alpha}\|_{\mathcal{B}^\alpha \to Q_K}$, the asymptotic relations in (3.2) follow, finishing the proof of the theorem. \hfill \Box

**Theorem 3.2.** Let $\alpha > 0$, $K \in \Omega(0,\infty)$, $\psi \in S(\mathbb{D})$, $\varphi \in H(\mathbb{D})$, and $n \in \mathbb{N}$, or $n = 0$ and $\alpha > 1$. Let $I_{\psi,g}^{(n)} : \mathcal{B}^\alpha \to Q_K$ be bounded. Then the following statements are equivalent.

(a) $I_{\psi,g}^{(n)} : \mathcal{B}^\alpha \to Q_K$ is compact.

(b) $I_{\psi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ is compact.

(c) $I_{\psi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ is weakly compact.

(d) $\sup_{a \in \mathcal{D}} \int_{\mathcal{D}} K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \, dm(z) < \infty$, and

\[
\lim_{r \to 1} \sup_{a \in \mathcal{D}} \int_{|\psi(z)|<r} K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(1 - |\varphi(z)|^2\right)^{2(1-\alpha-n)} \, dm(z) = 0.
\] (3.11)
Proof. By Lemma 2.4, we have that (b) is equivalent to (c).

(d) ⇒ (a). Let \((f_l)_{l \in \mathbb{N}}\) be a bounded sequence in \(\mathcal{B}^a\), say by \(L\), converging to zero uniformly on compacts of \(\mathbb{D}\). Then \(f_l^{(n)}\) also converges to zero uniformly on compacts of \(\mathbb{D}\). From (3.11) we have that for every \(\varepsilon > 0\) there is an \(r_1 \in (0, 1)\) such that for \(r \in (r_1, 1)\)

\[
\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} K \left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(1 - |\varphi(z)|^2\right)^{2(1-\alpha-n)} \, dm(z) < \varepsilon. \quad (3.12)
\]

Therefore, by Lemma 2.1 and (3.12), we have that for \(r \in (r_1, 1)\)

\[
\left\|I_{\varphi,g}^{(n)} f_l\right\|_{Q_K}^2 = \left(\int_{|\varphi(z)| \leq r} + \int_{|\varphi(z)| > r}\right) |f_l^{(n)}(\varphi(z))|^2 K \left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \, dm(z) < \varepsilon. \quad (3.13)
\]

Letting \(l \to \infty\) in (3.13), using the first condition in (d) and \(\sup_{|w| \leq r}|f_l^{(n)}(w)| \to 0\) as \(l \to \infty\), it follows that \(\lim_{l \to \infty} \left\|I_{\varphi,g}^{(n)} f_l\right\|_{Q_K} = 0\). Thus, by Lemma 2.3, \(I_{\varphi,g}^{(n)} : \mathcal{B}^a \to Q_K\) is compact.

(a) ⇒ (b). The implication is trivial since \(\mathcal{B}_{0}^a \subset \mathcal{B}^a\).

(b) ⇒ (d). By choosing \(f(z) = z^n / n! \in \mathcal{B}_{0}^a, n \in \mathbb{N}\), we have that the first condition in (d) holds. Let \(f_l(z) = z^l / l \in \mathbb{N}\). It is easy to see that \((f_l)_{l \in \mathbb{N}}\) is a bounded sequence in \(\mathcal{B}_{0}^a\) converging to zero uniformly on compacts of \(\mathbb{D}\). Hence, by Lemma 2.3, it follows that \(\left\|I_{\varphi,g}^{(n)} f_l\right\|_{Q_K} \to 0\) as \(l \to \infty\). Thus, for every \(\varepsilon > 0\), there is an \(l_0 \in \mathbb{N}\), \(l_0 > n\) such that for \(l \geq l_0\)

\[
\left(\prod_{j=1}^{n-1} (l - j)\right)^2 \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} \left|\varphi(z)\right|^{2l(n-\alpha)} K \left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \, dm(z) < \varepsilon. \quad (3.14)
\]

From (3.14) we have that for each \(r \in (0, 1)\) and \(l \geq l_0\)

\[
r^{2(l-n)} \left(\prod_{j=1}^{n-1} (l - j)\right)^2 \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} K \left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \, dm(z) < \varepsilon. \quad (3.15)
\]

Hence, for \(r \in [(\prod_{j=1}^{n-1} (l_0 - j))^{-1/(l_0-n)}, 1)\), we have that

\[
\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} K \left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \, dm(z) < \varepsilon. \quad (3.16)
\]
Let $f \in B^\eta_0$, and let $f_t(z) = f(tz)$, $0 < t < 1$. Then $\sup_{0 < t < 1} \| f_t \|_{B^\eta} \leq \| f \|_{B^\eta}$, $f_t \in B^\eta_0$, $t \in (0, 1)$, and $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. The compactness of $I_{\psi, g}^{(n)} : B^\eta_0 \to Q_k$ implies

$$
\lim_{t \to 1} \| I_{\psi, g}^{(n)} f_t - I_{\psi, g}^{(n)} f \|_{Q_k} = 0.
$$

(3.17)

Hence, for every $\varepsilon > 0$, there is a $t \in (0, 1)$ such that

$$
\sup_{a \in \mathbb{D}} \int_{|\phi(a)| > r} \left| f_t^{(n)}(\phi(z)) - f^{(n)}(\phi(z)) \right|^2 K \left( 1 - |\alpha(\phi(z))|^2 \right) |g(z)|^2 dm(z) < \varepsilon.
$$

(3.18)

From this and (3.16), we have that for such $t$ and each $r \in [(\prod_{j=1}^{n-1} (l_0 - j))^{-1/(l_0 - n)}$, 1)

$$
\sup_{a \in \mathbb{D}} \int_{|\phi(a)| > r} \left| f_t^{(n)}(\phi(z)) - f^{(n)}(\phi(z)) \right|^2 K \left( 1 - |\alpha(\phi(z))|^2 \right) |g(z)|^2 dm(z)
\leq 2 \sup_{a \in \mathbb{D}} \int_{|\phi(a)| > r} \left| f_t^{(n)}(\phi(z)) - f^{(n)}(\phi(z)) \right|^2 K \left( 1 - |\alpha(\phi(z))|^2 \right) |g(z)|^2 dm(z)
+ 2 \sup_{a \in \mathbb{D}} \int_{|\phi(a)| > r} \left| f_t^{(n)}(\phi(z)) \right|^2 K \left( 1 - |\alpha(\phi(z))|^2 \right) |g(z)|^2 dm(z)
< 2\varepsilon \left( 1 + \left\| f_t^{(n)} \right\|_\infty^2 \right).
$$

(3.19)

From (3.19) we conclude that for every $f \in B^\eta_0$, there is a $\delta_0 \in (0, 1)$ and $\delta_0 = \delta_0(f, \varepsilon)$ such that for $r \in (\delta_0, 1)$

$$
\sup_{a \in \mathbb{D}} \int_{|\phi(a)| > r} \left| f^{(n)}(\phi(z)) \right|^2 K \left( 1 - |\alpha(\phi(z))|^2 \right) |g(z)|^2 dm(z) < \varepsilon.
$$

(3.20)

Since $I_{\psi, g}^{(n)} : B^\eta_0 \to Q_k$ is compact, we have that for every $\varepsilon > 0$ there is a finite collection of functions $f_1, f_2, \ldots, f_k \in B^\eta_0$ such that, for each $f \in B^\eta_0$, there is a $j \in \{1, \ldots, k\}$, such that

$$
\sup_{a \in \mathbb{D}} \int_{|\phi(a)| > r} \left| f^{(n)}(\phi(z)) - f_j^{(n)}(\phi(z)) \right|^2 K \left( 1 - |\alpha(\phi(z))|^2 \right) |g(z)|^2 dm(z) < \varepsilon.
$$

(3.21)

On the other hand, from (3.20), it follows that if $\delta := \max_{1 \leq j \leq k} \delta_j(f_j, \varepsilon)$, then for $r \in (\delta, 1)$ and all $j \in \{1, \ldots, k\}$, we have

$$
\sup_{a \in \mathbb{D}} \int_{|\phi(a)| > r} \left| f_j^{(n)}(\phi(z)) \right|^2 K \left( 1 - |\alpha(\phi(z))|^2 \right) |g(z)|^2 dm(z) < \varepsilon.
$$

(3.22)
From (3.21) and (3.22), we have that for \( r \in (\delta, 1) \) and every \( f \in B_{\mathcal{B}^*} \)
\[
\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |f^{(n)}(\varphi(z))|^2 K \left( 1 - |\eta_a(z)|^2 \right)^2 |g(z)|^2 \, dm(z) < 4\varepsilon. \tag{3.23}
\]
If we apply (3.23) to the delays of the functions in (3.5) which are normalized so that they belong to \( B_{\mathcal{B}^*} \), and then use the monotone convergence theorem, we easily get that for \( r > \max\{\delta, \hat{\delta}\} \) where \( \delta \) is chosen as in (3.7)
\[
\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} K \left( 1 - |\eta_a(z)|^2 \right)^2 |g(z)|^2 \left( 1 - |\psi(z)|^2 \right)^{2(1-a-n)} \, dm(z) < C \varepsilon, \tag{3.24}
\]
from which (3.11) follows, as desired.

**Theorem 3.3.** Let \( \alpha > 0, K \in \Omega(0, \infty), \, \varphi \in S(\mathbb{D}), \, g \in H(\mathbb{D}) \) and \( n \in \mathbb{N}, \) or \( n = 0 \) and \( \alpha > 1. \) Then the next statements are equivalent.

(a) \( I_{\varphi,g}^{(n)} : \mathcal{B}_1 \to Q_{K,0} \) is bounded.

(b) \( I_{\varphi,g}^{(n)} : \mathcal{B} \to Q_{K,0} \) is compact.

(c) \( I_{\varphi,g}^{(n)} : \mathcal{B}_0 \to Q_{K,0} \) is compact.

(d) \( I_{\varphi,g}^{(n)} : \mathcal{B}_0 \to Q_{K,0} \) is weakly compact.

(e) \( \lim_{|a| \to 0} \int_{\mathbb{D}} K \left( 1 - |\eta_a(z)|^2 \right)^2 |g(z)|^2 \left( 1 - |\varphi(z)|^2 \right)^{2(1-a-n)} \, dm(z) = 0. \)

(f) \( d\mu_{\varphi,g,n,a}(z) \) is a vanishing \( K \)-Carleson measure.

**Proof.** By Theorem 1.1, (e) and (f) are equivalent; by Lemma 2.4, (c) is equivalent to (d), while, by Lemma 2.5, (a) is equivalent to (c). Also (b) obviously implies (a).

(a) \( \Rightarrow \) (e) Let \( h_1 \) and \( h_2 \) be as in (3.5). Then from (3.7) and an elementary inequality, we get
\[
\int \mathbb{D} K \left( 1 - |\eta_a(z)|^2 \right)^2 \left( 1 - |\varphi(z)|^2 \right)^{2(1-a-n)} |g(z)|^2 \, dm(z)
\leq C \int \mathbb{D} K \left( 1 - |\eta_a(z)|^2 \right) \left| \left( I_{\varphi,g}^{(n)} h_1 \right)(z) \right|^2 \, dm(z) \tag{3.25}
+ C \int \mathbb{D} K \left( 1 - |\eta_a(z)|^2 \right) \left| \left( I_{\varphi,g}^{(n)} h_2 \right)'(z) \right|^2 \, dm(z).
\]
For \( f_0(z) = z^n / n! \in \mathcal{B}_1 \), we get \( I_{\varphi,g}^{(n)} f_0 \in Q_{K,0} \). From this and since \( I_{\varphi,g}^{(n)} (h_j) \in Q_{K,0}, \ j = 1, 2 \), by letting \( |a| \to 1 \), we get that (e) holds.

(e) \( \Rightarrow \) (b). We have that for every \( \varepsilon > 0 \) there is a \( \delta \in (0, 1) \) so that for \( |a| > \delta \)
\[
\Phi_{\varphi,g,K}(a) < \varepsilon. \tag{3.26}
\]
On the other hand, by Lemma 2.6, $\Phi_{q,g,K}$ is continuous on $|a| \leq \delta$, so uniformly bounded therein, which along with (3.26) gives the boundedness of $\Phi_{q,g,K}$ on $\mathbb{D}$. Hence, by Theorem 3.1, $I_{q,g}^{(n)} : B^a \rightarrow Q_K$ is bounded. By Lemma 2.1, we have

$$\lim_{|a| \rightarrow 1} \sup_{\|f\|_{B^a} \leq 1} \int_\mathbb{D} \left| \left( I_{q,g}^{(n)} f(z) \right)'(z) \right|^2 K \left( 1 - |\eta_a(z)|^2 \right) dm(z) \leq C \sup_{\|f\|_{B^a} \leq 1} \|f\|_{B^a}^2 \lim_{|a| \rightarrow 1} \Phi_{q,g,K}^{(n)}(a) = C \lim_{|a| \rightarrow 1} \Phi_{q,g,K}(a) = 0,$$

so $I_{q,g}^{(n)} : B^a \rightarrow Q_{K,0}$ is bounded.

Now assume that $(f_i)_{i \in \mathbb{N}}$ is a bounded sequence in $B^a$, say by $L$, converging to zero uniformly on compacta of $\mathbb{D}$ as $i \rightarrow \infty$. To show that the operator $I_{q,g}^{(n)} : B^a \rightarrow Q_{K,0}$ is compact, it is enough to prove that there is a subsequence $(f_{i_k})_{k \in \mathbb{N}}$ of $(f_i)_{i \in \mathbb{N}}$ such that $I_{q,g}^{(n)} f_{i_k}$ converges in $Q_{K,0}$ as $k \rightarrow \infty$. By Lemma 2.1 and Montel’s theorem, it follows that there is a subsequence, which we may denote again by $(f_i)_{i \in \mathbb{N}}$ converging uniformly on compacta of $\mathbb{D}$ to an $f \in B^a$, such that $\|f\|_{B^a} \leq L$. Since $I_{q,g}^{(n)}(B^a) \subseteq Q_{K,0}$, then clearly $I_{q,g}^{(n)} f \in Q_{K,0}$. We show that

$$\lim_{i \rightarrow \infty} \|I_{q,g}^{(n)} f_i - I_{q,g}^{(n)} f\|_{Q_K} = 0.$$

From (3.26), Lemma 2.1, and some simple calculation, we obtain

$$\sup_{\delta \leq |a| < 1} \int_{\mathbb{D}} \left| \left( I_{q,g}^{(n)} f_i(z) - I_{q,g}^{(n)} f(z) \right)' \right|^2 K \left( 1 - |\eta_a(z)|^2 \right) dm(z) < 4CL^2 \varepsilon.$$  (3.29)

For $a \in \mathbb{D}$ and $t \in (0, 1)$, let

$$\Psi_t(a) = \int_{\mathbb{D} \setminus t \mathbb{D}} K \left( 1 - |\eta_a(z)|^2 \right) |g(z)|^2 \left( 1 - |\varphi(z)|^2 \right)^{2(1-a-n)} dm(z).$$  (3.30)

Lemma 2.6 essentially shows that $\Psi_t$ is continuous on $\mathbb{D}$. Hence, for each $a \in \mathbb{D}$, there is a $t(a) \in (r, 1)$ such that $\Psi_{t(a)}(a) < \varepsilon/2$. Moreover, there is a neighborhood $\mathcal{O}(a)$ of $a$ such that, for every $b \in \mathcal{O}(a)$, $\Psi_{t(a)}(b) < \varepsilon$. From this and since the set $|a| \leq \delta$ is compact, it follows that there is a $t_0 \in (0, 1)$ such that $\Psi_{t_0}(a) < \varepsilon$ when $|a| \leq \delta$. This along with Lemma 2.1 implies that

$$\sup_{|a| \leq \delta} \int_{\mathbb{D} \setminus t_0 \mathbb{D}} \left| \left( I_{q,g}^{(n)} f_i(z) - I_{q,g}^{(n)} f(z) \right)' \right|^2 K \left( 1 - |\eta_a(z)|^2 \right) dm(z) \leq C \|f_i - f\|_{B^a}^2 \sup_{|a| \leq \delta} \Psi_{t_0}(a) < 4CL^2 \varepsilon.$$  (3.31)
By the Weierstrass theorem $f_{l}^{(n)} \rightarrow f^{(n)}$ uniformly on compacta as $l \rightarrow \infty$, from which along with (2.2) and since $\varphi((0,\infty))$ is compact, for $r = \sup_{w \in \varphi((0,\infty))} |w|$, it follows that

\[
\sup_{|a| \leq \delta} \int_{D} \left| \left( I_{\varphi,g}^{(n)} f_{l}(z) - I_{\varphi,g}^{(n)} f(z) \right) \right|^{2} K \left( 1 - |\eta_{a}(z)|^{2} \right) dm(z)
\]

\[
\leq C \sup_{|a| \leq \delta} \left| (f_{l} - f)^{(n)}(z) \right|^{2} \sup_{|a| \leq \delta} \Phi_{\varphi,g,k}(a) \rightarrow 0, \quad \text{as } l \rightarrow \infty.
\]  

(3.32)

From (3.29)–(3.32) and since $I_{\varphi,g}^{(n)} f(0) = 0$ for each $f \in H(D)$, we easily get (3.28), from which (b) follows, finishing the proof of this theorem. \qed

**Theorem 3.4.** Let $\alpha > 0$, $K \in \Omega(0, \infty)$, $\varphi \in S(D)$, $g \in H(D)$, and $n \in \mathbb{N}$ or $n = 0$ and $\alpha > 1$. Then the following statements are equivalent:

(a) $I_{\varphi,g}^{(n)} : \mathcal{B}_{0}^{s} \rightarrow Q_{K,0}$ is bounded,

(b) $\sup_{a \in D} \int_{D} |g(z)|^{2} K \left( 1 - |\eta_{a}(z)|^{2} \right) \left( 1 - |\varphi(z)|^{2} \right)^{2(1-\alpha-n)} dm(z) < \infty$, and

\[
\lim_{|a| \rightarrow 1} \int_{D} |g(z)|^{2} K \left( 1 - |\eta_{a}(z)|^{2} \right) dm(z) = 0.
\]  

(3.33)

**Proof.** Suppose (b) holds and $f \in \mathcal{B}_{0}^{s}$. Then by Theorem 3.1, $I_{\varphi,g}^{(n)} : \mathcal{B}_{0}^{s} \rightarrow Q_{K}$ is bounded. We show $I_{\varphi,g}^{(n)} f \in Q_{K,0}$, for every $f \in \mathcal{B}_{0}^{s}$. Since $f \in \mathcal{B}_{0}^{s}$, we have that, for every $\varepsilon > 0$, there is an $r \in (0, 1)$ such that (see, e.g., the idea in [35, Lemma 2.4])

\[
\left| f^{(n)}(\varphi(z)) \right|^{2} \left( 1 - |\varphi(z)|^{2} \right)^{2(\alpha+n-1)} < \varepsilon \quad \text{for } |\varphi(z)| > r.
\]  

(3.34)

Thus,

\[
\sup_{a \in D} \int_{|\varphi(z)| > r} \left| \left( I_{\varphi,g} f(z) \right) \right|^{2} K \left( 1 - |\eta_{a}(z)|^{2} \right) dm(z)
\]

\[
< \varepsilon \sup_{a \in D} \int_{D} K \left( 1 - |\eta_{a}(z)|^{2} \right) \left( 1 - |\varphi(z)|^{2} \right)^{2(1-\alpha-n)} |g(z)|^{2} dm(z).
\]  

(3.35)

We also have

\[
\lim_{|a| \rightarrow 1} \int_{|\varphi(z)| \leq r} \left| \left( I_{\varphi,g} f(z) \right) \right|^{2} K \left( 1 - |\eta_{a}(z)|^{2} \right) dm(z)
\]

\[
\leq C \frac{\|f\|^{2}_{H^{s}}}{(1 - r^{2})^{2(\alpha+n-1)}} \lim_{|a| \rightarrow 1} \int_{|\varphi(z)| \leq r} K \left( 1 - |\eta_{a}(z)|^{2} \right) |g(z)|^{2} dm(z)
\]

\[
\leq C \frac{\|f\|^{2}_{H^{s}}}{(1 - r^{2})^{2(\alpha+n-1)}} \lim_{|a| \rightarrow 1} \int_{D} K \left( 1 - |\eta_{a}(z)|^{2} \right) |g(z)|^{2} dm(z) = 0.
\]  

(3.36)
Combining (3.35) and (3.36), we get $I^{(n)}_{\varphi,g} f \in Q_{K,0}$. Hence, $I^{(n)}_{\varphi,g} : B^a_0 \to Q_{K,0}$ is bounded.

Conversely, if $I^{(n)}_{\varphi,g} : B^a_0 \to Q_{K,0}$ is bounded, then $I^{(n)}_{\varphi,g} : B^a_0 \to Q_K$ is bounded too. Thus, by Theorem 3.1, we get the first condition in (b). For $f_0(z) = z^n/n! \in B^a_0$, we get $I^{(n)}_{\varphi,g} f_0 \in Q_{K,0}$, which is equivalent to (3.33), finishing the proof of the theorem. \hfill \Box

If $n = 0$, we simply denote the operator $I^{(0)}_{\varphi,g}$ by $I_{\varphi,g}$.

**Theorem 3.5.** Let $\alpha \in (0,1)$, $K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, and $g \in H(\mathbb{D})$. Then the following statements are equivalent.

(a) $I_{\varphi,g} : B^a \to Q_K$ is bounded.

(b) $I_{\varphi,g} : B^a_0 \to Q_K$ is bounded.

(c) $M_1 := \sup_{a \in \mathbb{D}} \int_D K(1 - |\eta_a(z)|^2) |g(z)|^2 dm(z) < \infty$.

(d) $d\mu_1(z) = |g(z)|^2 dm(z)$ is a $K$-Carleson measure.

(e) $I_{\varphi,g} : B^a \to Q_K$ is compact.

(f) $I_{\varphi,g} : B^a_0 \to Q_K$ is compact.

(g) $I_{\varphi,g} : B^a_0 \to Q_K$ is weakly compact.

Moreover, if $I_{\varphi,g} : B^a \to Q_K$ is bounded, then the next asymptotic relations hold

$$\|I_{\varphi,g}\|_{B^a \to Q_K} \asymp \|I_{\varphi,g}\|_{B^a_0 \to Q_K} \approx M_1^{1/2}. \quad (3.37)$$

**Proof.** The proof of the equivalence of statements (a)–(d) of this theorem is similar to the proof of Theorem 3.1; moreover, the implication (b) $\Rightarrow$ (c) is much simpler since it follows by using the test function $f_0(z) \equiv 1$. That (c) is equivalent to (e)--(g) is proved similarly as in Theorem 3.2, by using the well-known fact that if a bounded sequence $(f_l)_{l \in \mathbb{N}}$ in $B^a$, $\alpha \in (0,1)$ converges to zero uniformly on compacts of $\mathbb{D}$, then it converges to zero uniformly on the whole $\mathbb{D}$. The details are omitted. \hfill \Box

The proof of the next theorem is similar to the proofs of Theorems 3.3 and 3.4 and will be omitted.

**Theorem 3.6.** Let $\alpha \in (0,1), K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, and $g \in H(\mathbb{D})$. Then the following statements are equivalent.

(a) $I_{\varphi,g} : B^a_0 \to Q_{K,0}$ is bounded.

(b) $I_{\varphi,g} : B^a \to Q_{K,0}$ is bounded.

(c) $I_{\varphi,g} : B^a \to Q_{K,0}$ is compact.

(d) $I_{\varphi,g} : B^a_0 \to Q_{K,0}$ is compact.

(e) $I_{\varphi,g} : B^a_0 \to Q_{K,0}$ is weakly compact.

(f) $\lim_{|\alpha| \to 1} \int_D K(1 - |\eta_{a}(z)|^2) |g(z)|^2 dm(z) = 0$.

(g) $d\mu_1(z) = |g(z)|^2 dm(z)$ is a vanishing $K$-Carleson measure.
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