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Research Article

Positive Solutions for Singular Complementary Lidstone Boundary Value Problems

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By using fixed-point theorems of a cone, we investigate the existence and multiplicity of positive solutions for complementary Lidstone boundary value problems: $(-1)^n u^{(2n+1)}(t) = h(t)f(u(t))$, in $0 < t < 1$, $u(0) = 0$, $u^{(2i+1)}(0) = u^{(2i+1)}(1) = 0$, $0 \le i \le n-1$, where $n \in N$.

1. Introduction

In this paper, we are concerned with the existence of positive solutions for the following nonlinear differential equation:

$$\begin{gathered}(-1)^n u^{(2n+1)}(t) = h(t)f(u(t)), \quad \text{in } 0 < t < 1, \\ u(0) = 0, \qquad u^{(2i+1)}(0) = u^{(2i+1)}(1) = 0, \quad 0 \le i \le n-1,\end{gathered} \tag{1.1}$$

where $n \in N$, $h(t) \in C((0,1),[0,+\infty))$, and $h(t)$ may be singular at $t = 0$ or $t = 1$; $f \in C([0,+\infty),[0,+\infty))$.

Recently, on the boundary value problems of $2n$th-order ordinary differential equation (system)

$$(-1)^n u^{(2n)}(t) = \lambda h(t)f(u(t)), \tag{1.2}$$

many authors have established the existence and multiplicity of positive solutions of (1.2) by means of the method of upper and lower solutions and fixed point theorem, see [1–7] and references therein. More recently, the complementary Lidstone problem:

$$\begin{gathered}(-1)^n u^{(2n+1)}(t) = h\Big(t, u(t), \ldots, u^{(q)}(t)\Big), \quad n \geq 1,\ q \text{ fixed}, 0 \leq q \leq 2n \text{ in } 0 < t < 1, \\ u(0) = a_0, \quad u^{(2i+1)}(0) = a_i, \quad u^{(2i+1)}(1) = \beta_i, \quad 0 \leq i \leq n-1,\end{gathered} \tag{1.3}$$

was discussed in [8]. Here, $h : [0,1] \times R^{q+1} \to R$ is continuous at least in the interior of the domain of interest. Existence and uniqueness criteria for the above problem are proved by the complementary Lidstone interpolating polynomial of degree $2n$. In [9], the authors have studied the existence of positive solutions of singular complementary Lidstone problems on the basis of a fixed-point theorem of cone compression type. As far as we know, no papers are concerned with the multiplicity of positive solutions for (1.1). Therefore, inspired by the above references, we will show the existence and multiplicity of positive solutions of (1.1). The proof of our results is based on the following fixed-point theorems in a cone

Let E be a real Banach space with norm $\|\cdot\|$ and $P \subset E$ a cone in E, $P_r = \{x \in P : \|x\| < r\}(r > 0)$. Then $\overline{P_r} = \{x \in P : \|x\| \leq r\}$. A map α is said to be a nonnegative continuous concave functional on P if $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha\big(tx + (1-t)y\big) \geq t\alpha(x) + (1-t)\alpha\big(y\big) \tag{1.4}$$

for all $x, y \in P$ and $t \in [0,1]$. For numbers a, b such that $0 < a < b$ and α is a nonnegative continuous concave functional on P, we define the convex set

$$P(\alpha, a, b) = \{x \in P : a \leq \alpha(x), \|x\| \leq b\}. \tag{1.5}$$

Lemma 1.1 (see [10]). *Let $A : \overline{P_c} \to \overline{P_c}$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Suppose there exists $0 < d < a < b \leq c$ such that*

(i) *$\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in P(\alpha, a, b)$;*

(ii) *$\|Ax\| < d$ for $\|x\| \leq d$;*

(iii) *$\alpha(Ax) > a$ for $x \in P(\alpha, a, c)$ with $\|Ax\| > b$.*

Then A has at least three fixed points x_1, x_2, x_3 satisfying

$$\begin{aligned}\|x_1\| < d, \qquad & a < \alpha(x_2), \\ \|x_3\| > d, \qquad & \alpha(x_3) < a.\end{aligned} \tag{1.6}$$

Lemma 1.2 (see [10]). *Let E be a Banach space, and let $P \subset E$ be a closed, convex cone in E. Assume Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Au\| \le \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Au\| \ge \|u\|, u \in P \cap \partial\Omega_2$ *or*

(ii) $\|Au\| \ge \|u\|, u \in P \cap \partial\Omega_1$ *and* $\|Au\| \le \|u\|, u \in P \cap \partial\Omega_2$.

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

This paper is organized as follows: in Section 2, some preliminaries are given; in Section 3, we give the existence results.

2. Preliminaries

First, it is clear to see that the boundary value problem (1.1),

$$\begin{gathered}(-1)^n u^{(2n+1)}(t) = h(t) f(u(t)), \quad \text{in } 0 < t < 1, \\ u(0) = 0, \qquad u^{(2i+1)}(0) = u^{(2i+1)}(1) = 0, \quad 0 \le i \le n-1,\end{gathered} \tag{2.1}$$

is equivalent to the system

$$\begin{gathered}u'(t) = v, \quad \text{in } 0 < t < 1, \\ (-1)^n v^{(2n)}(t) = h(t) f(u(t)), \quad \text{in } 0 < t < 1, \\ u(0) = 0, \qquad v^{(2i)}(0) = v^{(2i)}(1) = 0, \quad 0 \le i \le n-1.\end{gathered} \tag{2.2}$$

Next, Problem (2.2) can be easily transformed into a nonlinear $2n$-order ordinary differential equation. Briefly, the initial value problem,

$$\begin{gathered}u'(t) = v, \quad \text{in } 0 < t < 1, \\ u(0) = 0,\end{gathered} \tag{2.3}$$

can be solved as

$$u(t) = \int_0^t v(s)ds. \tag{2.4}$$

Then, inserting (2.4) into the second equation of (1.1), we have

$$\begin{gathered}(-1)^n v^{(2n)}(t) = h(t) f\left(\int_0^t v(s)ds\right), \quad \text{in } 0 < t < 1, \\ v^{(2i)}(0) = v^{(2i)}(1) = 0, \quad 0 \le i \le n-1.\end{gathered} \tag{2.5}$$

Finally, we only need to consider the existence of positive solutions of (2.5). The function $v \in C[0,1]$ is a positive solution of (2.5), if v satisfies (2.5) and $v \ge 0$, $t \in [0,1]$, $v \ne 0$.

Let $G_n(t,s)$ be the Greens function of the following problem:

$$\begin{gathered}(-1)^n\omega^{(2n)}(t)=0,\quad \text{in } 0<t<1,\\ \omega^{(2i)}(0)=\omega^{(2i)}(1)=0,\quad 0\le i\le n-1.\end{gathered}\tag{2.6}$$

By induction, the Greens function $G_n(t,s)$ can be expressed as (see [2])

$$G_i(t,s)=\int_0^1 G(t,\xi)G_{i-1}(\xi,s)d\xi,\quad 2\le i\le n,\tag{2.7}$$

where

$$G_1(t,s)=G(t,s)=\begin{cases}t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases}\tag{2.8}$$

So it is easy to see that

$$G_n(t,s)>0,\quad (t,s)\in(0,1)\times(0,1).\tag{2.9}$$

Lemma 2.1 (see [2]). (I) *For any* $(t,s)\in[0,1]\times[0,1]$,

$$G_n(t,s)\le\frac{1}{6^{n-1}}s(1-s).\tag{2.10}$$

(II) *Let* $\delta\in(0,1/2)$, *then for any* $(t,s)\in[\delta,1-\delta]\times[0,1]$,

$$G_n(t,s)\ge\theta_n(\delta)s(1-s)\ge 6^{n-1}\theta_n(\delta)\max_{0\le t\le 1}G_n(t,s),\tag{2.11}$$

where $\theta_n(\delta)=\delta^n((4\delta^3-6\delta^2+1)/6)^{n-1}$.

Therefore, the solution of (2.5) can be expressed as

$$v(t)=\int_0^1 G_n(t,s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds.\tag{2.12}$$

We now define a mapping $T:C[0,1]\to C[0,1]$ by

$$Tv(t)=\int_0^1 G_n(t,s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds.\tag{2.13}$$

Set

$$P=\left\{v\in C[0,1]:v(t)\ge 0,\ \min_{t\in[1/4,3/4]}v(t)\ge\sigma\|v\|\right\},\tag{2.14}$$

where $\|v\| = \max_{t\in[0,1]}|v(t)|$, $\sigma = 6^{n-1}\theta_n(1/4) = 11^{n-1}/2^{6n-4}$. It is clear that $\sigma \in (0,1)$ by Lemma 2.1.

Lemma 2.2 (see [2]). *$T : P \to P$ is completely continuous.*

3. Main Result

Theorem 3.1. *Assume that the following conditions hold*

(H1) *$h(t) \in C((0,1),[0,+\infty))$ does not vanish identically on any subinterval of $[0,1]$,*

$$0 < \int_{2/4}^{3/4} s(1-s)h(s)ds \le \int_0^1 s(1-s)h(s)ds < +\infty \tag{3.1}$$

(H2) *$f : [0,+\infty) \to [0,+\infty)$ is nondecreasing, and*

$$\lim_{r\to 0^+}\frac{f(r)}{r} = 0, \qquad \lim_{r\to +\infty}\frac{f(r)}{r} = +\infty. \tag{3.2}$$

Then (2.5) *or* (1.1) *has at least one positive solution.*

Proof. Since $\lim_{r\to 0^+}(f(r))/r = 0$, there exists $\eta_1 > 0$ such that

$$\frac{f(r)}{r} \le \frac{6^{n-1}}{\int_0^1 s(1-s)h(s)ds}, \qquad \text{for } 0 < r \le \eta_1. \tag{3.3}$$

Take $R_1 = \eta_1$, and set $\Omega_1 = \{v \in E : \|v\| < R_1\}$. Then, for for all $v \in P \cap \partial\Omega_1$, we have

$$\begin{aligned} Tv(t) &= \int_0^1 G_n(t,s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\ &\le \frac{1}{6^{n-1}}\int_0^1 s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\ &\le \frac{1}{6^{n-1}}\int_0^1 s(1-s)h(s)f(R_1)ds \\ &\le \frac{1}{6^{n-1}}\frac{6^{n-1}R_1}{\int_0^1 s(1-s)h(s)ds}\int_0^1 s(1-s)h(s)ds \\ &\le R_1 = \|v\| \end{aligned} \tag{3.4}$$

by (H2) and Lemma 2.1.

Consequently,

$$\|Tv\| \le \|v\|, \quad \forall v \in \partial P \cap \partial\Omega_1. \tag{3.5}$$

On the other hand, since $\lim_{r\to+\infty}(f(r))/r = +\infty$, there exists $\widehat{R} > 0$ such that

$$\frac{f(r)}{r} \geq \frac{4}{\sigma\theta_n(1/4)\int_{2/4}^{3/4} s(1-s)h(s)ds}, \quad \text{for } r \geq \widehat{R}. \tag{3.6}$$

Choose $R_2 = \max\{R_1, 4\widehat{R}/\sigma\} + 1$, and set $\Omega_2 = \{v \in E : \|v\| < R_2\}$. Then, for $\forall v \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
Tv\left(\frac{1}{4}\right) &= \int_0^1 G_n\left(\frac{1}{4}, s\right)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{1/4}^{3/4} s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_0^{2/4} v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_{1/4}^{2/4} v(\tau)d\tau\right)ds
\end{aligned} \tag{3.7}$$

by (H2) and Lemma 2.1.

Since $v \in P \cap \partial\Omega_2$, then we have

$$\int_{1/4}^{2/4} v(\tau)d\tau \geq \int_{1/4}^{2/4}\left(\min_{\tau\in[1/4,3/4]} v(\tau)\right)d\tau \geq \frac{\sigma}{4}\|v\| \geq \frac{\sigma}{4}R_2 > \widehat{R}. \tag{3.8}$$

So from (3.7), we get

$$\begin{aligned}
Tv\left(\frac{1}{4}\right) &\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_{1/4}^{2/4} v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\frac{\sigma}{4}R_2\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)ds\frac{4}{\sigma\theta_n(1/4)\int_{2/4}^{3/4} s(1-s)h(s)ds}\frac{\sigma R_2}{4} \\
&\geq R_2 = \|v\|.
\end{aligned} \tag{3.9}$$

Consequently,

$$\|Tv\| \geq \|v\|, \quad \forall v \in \partial P \cap \partial\Omega_2. \tag{3.10}$$

Therefore, by Lemma 1.2, (1.1) has at least one positive solution. □

Theorem 3.2. *Assume (H1) holds. In addition, suppose that the following conditions hold:*

(H3) $f : [0,+\infty) \to [0,+\infty)$ *is nondecreasing,*

$$\lim_{r\to 0^+} \frac{f(r)}{r} = +\infty, \qquad \lim_{r\to +\infty} \frac{f(r)}{r} = 0. \tag{3.11}$$

Then (2.5) *or* (1.1) *has at least one positive solution.*

Proof. Since $\lim_{r\to 0^+}(f(r))/r = +\infty$, there exists $\eta > 0$ such that

$$\frac{f(r)}{r} \geq \frac{4}{\sigma\theta_n(1/4)\int_{2/4}^{3/4} s(1-s)h(s)ds}, \quad \text{for } 0 < r \leq \eta. \tag{3.12}$$

Take $R_1 \in (0,\eta)$, and set $\Omega_1 = \{v \in E : \|v\| < R_1\}$. Then, for for all$v \in P \cap \partial\Omega_1$, we have

$$\begin{aligned}
Tv\left(\frac{1}{4}\right) &= \int_0^1 G_n\left(\frac{1}{4}, s\right)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{1/4}^{3/4} s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_0^{2/4} v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_{1/4}^{2/4} v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_{1/4}^{2/4}\left(\min_{\tau\in[1/4,3/4]} v(\tau)\right)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\frac{\sigma}{4}R_1\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)ds\frac{4}{\sigma\theta_n(1/4)\int_{2/4}^{3/4} s(1-s)h(s)ds}\frac{\sigma R_1}{4} \\
&\geq R_1 = \|v\|
\end{aligned} \tag{3.13}$$

by (H3) and Lemma 2.1.

Consequently,

$$\|Tv\| \geq \|v\|, \quad \forall v \in \partial P \cap \partial\Omega_1. \tag{3.14}$$

On the other hand, since $\lim_{r\to+\infty}(f(r))/r = 0$, there exists $\overline{R} > 0$ such that

$$\frac{f(r)}{r} \leq \frac{6^{n-1}}{\int_0^1 s(1-s)h(s)ds}, \quad \text{for } r \geq \overline{R}. \tag{3.15}$$

Choose $R_2 = \max\{R_1, \overline{R}\} + 1$, and set $\Omega_2 = \{v \in E : \|v\| < R_2\}$. Then, for for all $v \in P \cap \partial\Omega_2$, we have

$$\begin{aligned} Tv(t) &= \int_0^1 G_n(t,s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\ &\leq \frac{1}{6^{n-1}}\int_0^1 s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\ &\leq \frac{1}{6^{n-1}}\int_0^1 s(1-s)h(s)f(R_2)ds \\ &\leq \frac{1}{6^{n-1}}\frac{6^{n-1}R_2}{\int_0^1 s(1-s)h(s)ds}\int_0^1 s(1-s)h(s)ds \\ &\leq R_2 = \|v\|. \end{aligned} \tag{3.16}$$

Consequently,

$$\|Tv\| \leq \|v\|, \quad \forall v \in \partial P \cap \partial\Omega_2. \tag{3.17}$$

Therefore, by Lemma 1.2, (1.1) has at least one positive solution. □

Theorem 3.3. *Assume that (H1) holds. In addition, the function f is nondecreasing and satisfies the following growth conditions:*

(H4)

$$\lim_{r\to\infty}\sup\frac{f(r)}{r} < \frac{1}{2}\frac{6^{n-1}}{\int_0^1 s(1-s)h(s)ds}; \tag{3.18}$$

(H5)

$$\lim_{r\to 0}\sup\frac{f(r)}{r} < \frac{6^{n-1}}{\int_0^1 s(1-s)h(s)ds}; \tag{3.19}$$

(H6) *there exists a constant* $a > 0$ *such that*

$$f(r) > \frac{a}{\theta_n(1/4)\int_{2/4}^{3/4} s(1-s)h(s)ds}, \quad \text{for } r \in \left[\frac{a}{4}, \frac{a}{4\sigma}\right]. \tag{3.20}$$

Then (1.1) *has at least three positive solutions.*

Proof. For the sake of applying the Leggett-Williams fixed-point theorem, define a functional $\alpha(u)$ on cone P by

$$\alpha(v) = \min_{1/4 \le t \le 3/4} v(t), \quad \forall v \in P. \tag{3.21}$$

Evidently, $\alpha : P \to R^+$ is a nonnegative continuous and concave. Moreover, $\alpha(v) \le \|v\|$ for each $v \in P$.

Now we verify that the assumption of Lemma 1.1 is satisfied.

Firstly, it can verify that there exists a positive number c with $c \ge b = a/\sigma$ such that $T : \overline{P_c} \to P_c$.

By (H4), it is easy to see that there exists $\tau > 0$ such that

$$\frac{f(r)}{r} < \frac{1}{2}\frac{6^{n-1}}{\int_0^1 s(1-s)h(s)ds}, \quad \forall r \ge \tau. \tag{3.22}$$

Set $M_1 = f(\tau)$, and take

$$c > \max\left\{b, \frac{2M_1}{6^{n-1}}\int_0^1 s(1-s)h(s)ds\right\}. \tag{3.23}$$

If $v \in \overline{P_c}$, then

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \int_0^1 G_n(t,s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\le \frac{1}{6^{n-1}}\int_0^1 s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\le \frac{1}{6^{n-1}}\int_0^1 s(1-s)h(s)f(\|v\|)ds \\
&< \frac{1}{6^{n-1}}\int_0^1 s(1-s)h(s)ds\left[\frac{6^{n-1}\|v\|}{2\int_0^1 s(1-s)h(s)ds} + M_1\right] \\
&< c
\end{aligned} \tag{3.24}$$

by (H1) and (H3).

Next, from (H5), there exists $d \in (0, a)$ such that

$$\frac{f(r)}{r} < \frac{6^{n-1}}{\int_0^1 s(1-s)h(s)ds}, \ \forall r \in [0, d]. \tag{3.25}$$

Then for each $v \in \overline{P_d}$, we have

$$\begin{aligned}
\|Tv\| &= \max_{t\in[0,1]} \int_0^1 G_n(t,s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\le \frac{1}{6^{n-1}} \int_0^1 s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\le \frac{1}{6^{n-1}} \int_0^1 s(1-s)h(s)f(\|v\|)ds \\
&< \frac{1}{6^{n-1}} \int_0^1 s(1-s)h(s)ds \frac{6^{n-1}\|v\|}{\int_0^1 s(1-s)h(s)ds} \\
&\le d.
\end{aligned} \tag{3.26}$$

Finally, we will show that $\{v \in P(\alpha, a, b) : \alpha(v) > a\} \neq \emptyset$ and $\alpha(Tv) > a$ for all $v \in P(\alpha, a, b)$.

In fact,

$$v(t) = \frac{a+b}{2} \in \{v \in P(\alpha, a, b) : \alpha(v) > a\}. \tag{3.27}$$

For $v \in P(\alpha, a, b)$, we have

$$b \ge \|v\| \ge v \ge \min_{t\in[1/4,3/4]} v(t) \ge a, \tag{3.28}$$

for all $t \in [1/4, 3/4]$. Then we have

$$\begin{aligned}
\min_{t\in[1/4,3/4]} Tv(t) &= \min_{t\in[1/4,3/4]} \int_0^1 G_n(t,s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\ge \theta_n\left(\frac{1}{4}\right) \int_{1/4}^{3/4} s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds \\
&\ge \theta_n\left(\frac{1}{4}\right) \int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_0^s v(\tau)d\tau\right)ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_0^{2/4} v(\tau)d\tau\right)ds \\
&\geq \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)f\left(\int_{1/4}^{2/4} v(\tau)d\tau\right)ds \\
&> \theta_n\left(\frac{1}{4}\right)\int_{2/4}^{3/4} s(1-s)h(s)ds\frac{a}{\theta_n(1/4)\int_{2/4}^{3/4} s(1-s)h(s)ds} \\
&= a
\end{aligned}
\tag{3.29}
$$

by (H6). In addition, for each $v \in P(\alpha, a, c)$ with $\|Tv\| > b$, we have

$$
\min_{t\in[1/4,3/4]}(Tv)(t) \geq \sigma\|Tv\| > \sigma b \geq a. \tag{3.30}
$$

Above all, we know that the conditions of Lemma 1.1 are satisfied. By Lemma 1.1, the operator T has at least three fixed points v_i $(i = 1, 2, 3)$ such that

$$
\begin{gathered}
\|v_1\| < d, \\
a < \min_{t\in[1/4,3/4]} v_2(t) \\
\|v_3\| > d \text{ with } \min_{t\in[1/4,3/4]} v_3(t) < a.
\end{gathered}
\tag{3.31}
$$

The proof is complete. □

Example 3.4. If $n = 1$, then consider the boundary value problem:

$$
\begin{gathered}
-u'''(t) = \frac{u^2}{t(1-t)}, \quad \text{in } 0 < t < 1, \\
u(0) = 0, \qquad u'(0) = u'(1) = 0.
\end{gathered}
\tag{3.32}
$$

Example 3.5. If $n = 2$, then consider the boundary value problem:

$$
\begin{gathered}
u^{(5)}(t) = \frac{u^{1/3}}{t(1-t)}, \quad \text{in } 0 < t < 1, \\
u(0) = 0, \qquad u'(0) = u'(1) = u'''(0) = u''' \ (1) = 0.
\end{gathered}
\tag{3.33}
$$

It is obvious to see that Examples 3.4 and 3.5 satisfy the assumptions of Theorems 3.1 and 3.2.

Example 3.6. Let $n = 1$, $h(t) = 1/(t(1-t))$. Then let us consider the following problem:

$$\begin{gathered} -u^{(3)}(t) = \frac{1}{t(1-t)} f(u), \quad \text{in } 0 < t < 1, \\ u(0) = 0, \qquad u'(0) = u'(1) = 0, \end{gathered} \tag{3.34}$$

where the function f is defined as follows:

$$f(u) = \begin{cases} (u-1)^{1/2} + 65, & u \geq 1, \\ \dfrac{195u^2}{2+u}, & 0 \leq u < 1. \end{cases} \tag{3.35}$$

It is obvious that f is continuous and (H1) holds. On the other hand, since $[u^2/(2+u)]' = (u^2+4u)/(2+u)^2 \geq 0$, for $0 \leq u$, it is clear to see that $195u^2/(2+u)$ is nondecreasing for $0 \leq u < 1$, and $(u-1)^{1/2} + 65$ is also nondecreasing for $u \geq 1$. In addition,

$$\begin{gathered} \limsup_{r\to\infty} \frac{f(r)}{r} = \limsup_{r\to\infty} \frac{(r-1)^{1/2}+65}{r} = 0 < \frac{1}{2} \frac{6^{n-1}}{\int_0^1 s(1-s)(1/s(1-s))ds} = \frac{1}{2}, \\ \limsup_{r\to 0} \frac{f(r)}{r} = \limsup_{r\to 0} \frac{195r^2}{r(2+r)} = 0 < \frac{6^{n-1}}{\int_0^1 s(1-s)(1/s(1-s))ds} = 1. \end{gathered} \tag{3.36}$$

So (H4) and (H5) hold.

Finally, choosing $a = 4$, then for $r \in [1, 1/\sigma]$, we have

$$f(r) \geq 65 > 64 = \frac{a}{\theta_n(1/4) \int_{2/4}^{3/4} s(1-s)h(s)ds}. \tag{3.37}$$

Therefore (H6) hold.

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