Differential Subordinations for Certain Meromorphically Multivalent Functions Defined by Dziok-Srivastava Operator

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By making use of the Dziok-Srivastava operator, we introduce a new class of meromorphically multivalent functions. Some inclusion properties of functions belonging to this class are derived.

1. Introduction

Let \( \Sigma(p) \) denote the class of functions of the form

\[
 f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}),
\]  

(1.1)

which are analytic in the punctured open unit disk \( \mathbb{U}_0 = \{ z : 0 < |z| < 1 \} \) with a pole at \( z = 0 \). Also let the Hadamard product (or convolution) of the following functions:

\[
 f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n,j} z^{n-p} \quad (j = 1, 2)
\]  

(1.2)

be given by

\[
 (f_1 * f_2)(z) := z^{-p} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^{n-p} = (f_2 * f_1)(z).
\]  

(1.3)
Given two functions \( f(z) \) and \( g(z) \), which are analytic in \( U = U_0 \cup \{0\} \), we say that the function \( g(z) \) is subordinate to \( f(z) \) and write \( g < f \) or (more precisely) \( g(z) < f(z) \) (\( z \in U \)), if there exists a Schwarz function \( \omega(z) \), analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) (\( z \in U \)) such that \( g(z) = f(\omega(z)) \) (\( z \in U \)). In particular, if \( f(z) \) is univalent in \( U \), we have the following equivalence:

\[
g(z) < f(z) \quad (z \in U) \iff g(0) = f(0), \quad g(U) \subset f(U). \tag{1.4}
\]

Let \( A \) be the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} b_n z^n,
\]

which are analytic in \( U \). A function \( f(z) \in A \) is said to be in the class \( S^*(\alpha) \) if

\[
\text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)
\]

for some \( \alpha \) (\( \alpha < 1 \)). When \( 0 \leq \alpha < 1 \), \( S^*(\alpha) \) is the class of starlike functions of order \( \alpha \) in \( U \). A function \( f(z) \in A \) is said to be prestarlike of order \( \alpha \) in \( U \) if

\[
\frac{z}{(1-z)^{2(1-\alpha)}} \ast f(z) \in S^*(\alpha) \quad (\alpha < 1),
\]

where the symbol \( \ast \) means the familiar Hadamard product (or convolution) of two analytic functions in \( U \). We denote this class by \( R(\alpha) \) (see [1]). Clearly a function \( f(z) \in A \) is in the class \( R(\alpha) \) if and only if \( f(z) \) is convex univalent in \( U \) and \( R(1/2) = S^*(1/2) \).

For complex parameters

\[
\alpha_1, \ldots, \alpha_q \text{ and } \beta_1, \ldots, \beta_s \quad (\beta_j \notin \mathbb{Z}_0^c := \{0, -1, -2, \ldots\}; \ j = 1, 2, \ldots, s),
\]

we define the generalized hypergeometric function \( {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \) by

\[
{}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}
\]

\[
(q \leq s + 1; \ q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \ z \in U),
\]

where \( (x)_n \) is the Pochhammer symbol defined, in terms of the Gamma function \( \Gamma(x) \), by

\[
(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0; x \in \mathbb{C} \setminus \{0\}) \\ x(x+1) \cdots (x+n-1) & (n \in \mathbb{N}; x \in \mathbb{C}). \end{cases}
\]
Corresponding to a function $h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by
\[
h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-\alpha_1} F_{q}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z), \tag{1.11}
\]
we now consider a linear operator
\[
H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \Sigma(p) \to \Sigma(p), \tag{1.12}
\]
defined by means of the Hadamard product (or convolution) as follows:
\[
H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) := h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \ast f(z). \tag{1.13}
\]
For convenience, we write
\[
H_{p,q,s}(\alpha_1) := H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s). \tag{1.14}
\]
Thus, after some calculations, we have
\[
z (H_{p,q,s}(\alpha_1) f(z))' = \alpha_1 H_{p,q,s}(\alpha_1 + 1) f(z) - (\alpha_1 + p) H_{p,q,s}(\alpha_1) f(z). \tag{1.15}
\]

The operator $H_{p,q,s}(\alpha_1)$ is popularly known as the generalized Dziok-Srivastava operator. Many interesting subclasses of multivalent functions, associated with the operator $H_{p,q,s}(\alpha_1)$ and its various special cases, were investigated recently by (e.g.) Dziok and Srivastava [2-4], Liu [5], Liu and Srivastava [6, 7], Patel et al. [8], Wang et al. [9], and others.

Let $P$ be the class of functions $h(z)$ with $h(0) = 1$, which are analytic and convex univalent in $\mathbb{U}$.

**Definition 1.1.** A function $f(z) \in \Sigma(p)$ is said to be in the class $T_{p,q,s}(\alpha_1, \lambda; h)$ if it satisfies the subordination condition
\[
\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda}{p(p + 1)} z^{p+2} (H_{p,q,s}(\alpha_1) f(z))'' < h(z), \tag{1.16}
\]
where $\lambda$ is a complex number and $h(z) \in P$.

The main object of this paper is to present a systematic investigation of the class $T_{p,q,s}(\alpha_1, \lambda; h)$ defined above by means of the generalized Dziok-Srivastava operator $H_{p,q,s}(\alpha_1)$.

For our purpose, we shall need the following lemmas to derive our main results for the class $T_{p,q,s}(\alpha_1, \lambda; h)$.

**Lemma 1.2** (see [10]). Let $g(z)$ be analytic in $\mathbb{U}$ and $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If
\[
g(z) + \frac{1}{\mu} z g'(z) < h(z), \tag{1.17}
\]
where $\Re \mu > 0$, then

$$g(z) < \tilde{h}(z) = \mu z^{-\mu} \int_0^z t^\mu h(t) dt < h(z) \tag{1.18}$$

and $\tilde{h}(z)$ is the best dominant of (1.17).

**Lemma 1.3** (see [1]). Let $\alpha < 1$, $f(z) \in S^*(\alpha)$ and $g(z) \in R(\alpha)$. Then, for any analytic function $F(z)$ in $U$,

$$g \ast (fF)(U) \subset \overline{co}(F(U)), \tag{1.19}$$

where $\overline{co}(F(U))$ denotes the closed convex hull of $F(U)$.

### 2. Properties of the Class $T_{p,q,s}(\alpha_1, \alpha; h)$

**Theorem 2.1.** Let $\lambda_1 < \lambda_2 \leq 0$. Then $T_{p,q,s}(\alpha_1, \lambda_1; h) \subset T_{p,q,s}(\alpha_1, \lambda_2; h)$.

**Proof.** Let $\lambda_1 < \lambda_2 \leq 0$ and suppose that

$$g(z) = \frac{z^{\alpha+1} (H_{p,q,s}(\alpha_1) f(z))'}{p} \tag{2.1}$$

for $f(z) \in T_{p,q,s}(\alpha_1, \lambda_1; h)$. Then the function $g(z)$ is analytic in $U$ with $g(0) = 1$. Differentiating both sides of (2.1) with respect to $z$ and using (1.16), we have

$$\frac{(\lambda_1 - 1)}{p} z^{\alpha+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda_1}{p(p+1)} z^{\alpha+2} (H_{p,q,s}(\alpha_1) f(z))'' = g(z) - \frac{\lambda_1}{p+1} z g'(z) < h(z). \tag{2.2}$$

Hence an application of Lemma 1.2 yields

$$g(z) < h(z). \tag{2.3}$$

Noting that $0 < \lambda_2/\lambda_1 < 1$ and that $h(z)$ is convex univalent in $U$, it follows from (2.1) to (2.3) that

$$\frac{(\lambda_2 - 1)}{p} z^{\alpha+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda_2}{p(p+1)} z^{\alpha+2} (H_{p,q,s}(\alpha_1) f(z))''$$

$$= \frac{\lambda_2}{\lambda_1} \left( \frac{(\lambda_1 - 1)}{p} z^{\alpha+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda_1}{p(p+1)} z^{\alpha+2} (H_{p,q,s}(\alpha_1) f(z))'' \right)$$

$$+ \left( 1 - \frac{\lambda_2}{\lambda_1} \right) g(z) < h(z). \tag{2.4}$$

Thus $f(z) \in T_{p,q,s}(\alpha_1, \lambda_2; h)$ and the proof of Theorem 2.1 is completed. \qed
Let $0 < b_1 < b_2$. Then $T_{p,q,s}(b_2, \lambda; h) \subset T_{p,q,s}(b_1, \lambda; h)$.

**Proof.** Define a function $g(z)$ by

$$g(z) = z + \sum_{n=1}^{\infty} \frac{(b_1)^n}{(b_2)^n} z^{n+1} \quad (z \in \mathbb{U}; 0 < b_1 < b_2).$$

(2.5)

Then

$$z^{p+1} h_p(b_1, \alpha_2, \ldots, \alpha_s, 1; b_2, \alpha_2, \ldots, \alpha_s; z) = g(z) \in A,$$

(2.6)

where

$$h_p(b_1, \alpha_2, \ldots, \alpha_s, 1; b_2, \alpha_2, \ldots, \alpha_s; z)$$

is defined as in (1.11), and

$$\frac{z}{(1-z)^{b_2}} \ast g(z) = \frac{z}{(1-z)^{b_1}}.$$  

(2.8)

By (2.8), we see that

$$\frac{z}{(1-z)^{b_2}} \ast g(z) \in S^* \left(1 - \frac{b_1}{2}\right) \subset S^* \left(1 - \frac{b_2}{2}\right) \quad (0 < b_1 < b_2),$$

(2.9)

which implies that

$$g(z) \in R \left(1 - \frac{b_2}{2}\right).$$  

(2.10)

Let $f(z) \in T_{p,q,s}(b_2, \lambda; h)$. It is easy to verify that

$$z^{p+1} (H_{p,q,s}(b_1) f(z))^\prime = \left(z^p h_p(b_1, \alpha_2, \ldots, \alpha_s, 1; b_2, \alpha_2, \ldots, \alpha_s; z)\right) \ast \left(z^{p+1} (H_{p,q,s}(b_2) f(z))^\prime\right)$$

(2.11)

and

$$z^{p+2} (H_{p,q,s}(b_1) f(z))^\prime = \left(z^p h_p(b_1, \alpha_2, \ldots, \alpha_s, 1; b_2, \alpha_2, \ldots, \alpha_s; z)\right) \ast \left(z^{p+2} (H_{p,q,s}(b_2) f(z))^\prime\right).$$

(2.12)

From (2.11), (2.12), and (2.6), we deduce that

$$\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(b_1) f(z))^\prime + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(b_1) f(z))^\prime$$

(2.13)

$$= \frac{g(z)}{z} \ast w(z) = \frac{g(z) \ast (zw(z))}{g(z) \ast z},$$
where

\[
\omega(z) := \frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(b_2) f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(b_2) f(z))'' < h(z). \tag{2.14}
\]

Since the function \( z \) belongs to the function class \( S^*(1 - b_2/2) \) and \( h(z) \) is convex univalent in \( U \), it follows from (2.12), (2.13), (2.14), and Lemma 1.3 that

\[
\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(b_1) f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(b_1) f(z))'' < h(z). \tag{2.15}
\]

Thus \( f(z) \in T_{p,q,s}(b_1, \lambda; h) \) and the proof of Theorem 2.2 is completed. \( \square \)

**Theorem 2.3.** Let \( f(z) \in T_{p,q,s}(\alpha_1, \lambda; h) \), \( g(z) \in \Sigma(p) \) and

\[
\Re\{z^p g(z)\} > \frac{1}{2}, \quad (z \in U). \tag{2.16}
\]

Then

\[
(f * g)(z) \in T_{p,q,s}(\alpha_1, \lambda; h). \tag{2.17}
\]

**Proof.** For \( f(z) \in T_{p,q,s}(\alpha_1, \lambda; h) \) and \( g(z) \in \Sigma(p) \), we have

\[
\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) (f * g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(\alpha_1) (f * g)(z))''
\]

\[
= \frac{(\lambda - 1)}{p} (z^p g(z))' (z^{p+1} (H_{p,q,s}(\alpha_1) f(z))') + \frac{\lambda}{p(p+1)} (z^p g(z))' (z^{p+2} (H_{p,q,s}(\alpha_1) f(z))'')
\]

\[
= (z^p g(z))' \psi(z), \tag{2.18}
\]

where

\[
\psi(z) = \frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (H_{p,q,s}(\alpha_1) f(z))''. \tag{2.19}
\]

In view of (2.16), the function \( z^p g(z) \) has the Herglotz representation

\[
z^p g(z) = \int_{|x|=1} \frac{d\mu(x)}{1 - xz}, \quad (z \in U), \tag{2.20}
\]

where \( \mu(x) \) is a probability measure defined on the unit circle \( |x| = 1 \) and

\[
\int_{|x|=1} d\mu(x) = 1. \tag{2.21}
\]
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Since $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (2.18) to (2.20) that

$$
\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(\alpha_1) (f \ast g)(z))' + \frac{\lambda}{p(p + 1)} z^{p+2} (H_{p,q,s}(\alpha_1)(f \ast g)(z))''
$$

$$
= \int_{|z|=1} q(z) d\mu(x) < h(z).
$$

(2.22)

This shows that $(f \ast g)(z) \in T_{p,q,s}(\alpha_1, \lambda; h)$ and the theorem is proved.

\[\square\]

**Theorem 2.4.** Let $f(z) \in T_{p,q,s}(\alpha_1, \lambda; h)$, $g(z) \in \Sigma(p)$ and

$$
 z^{p+1}g(z) \in R(\alpha) \quad (\alpha < 1).
$$

Then

$$
(f \ast g)(z) \in T_{p,q,s}(\alpha_1, \lambda; h).
$$

(2.24)

**Proof.** For $f(z) \in T_{p,q,s}(\alpha_1, \lambda; h)$ and $g(z) \in \Sigma(p)$, from (2.18) we have

$$
\frac{(\lambda - 1)}{p} z^{p+1} (H_{p,q,s}(f \ast g)(z))' + \frac{\lambda}{p(p + 1)} z^{p+2} (H_{p,q,s}(f \ast g)(z))''
$$

$$
= \frac{(z^{p+1}g(z)) \ast (zq(z))}{(z^{p+1}g(z)) \ast z} \quad (z \in \mathbb{U}),
$$

(2.25)

where $q(z)$ is defined as in (2.19).

Since $h(z)$ is convex univalent in $\mathbb{U}$,

$$
q(z) < h(z), \quad z^{p+1}g(z) \in R(\alpha), \quad z \in S^*(\alpha) \quad (\alpha < 1),
$$

(2.26)

it follows from (2.25) and Lemma 1.3 the desired result.

\[\square\]

**Theorem 2.5.** Let $\lambda < 0$, $\beta > 0$ and $f(z) \in T_{p,q,s}(\lambda; \beta h + 1 - \beta)$. If $\beta \leq \beta_0$, where

$$
\beta_0 = \frac{1}{2} \left(1 + \frac{p + 1}{\lambda} \int_0^1 \frac{u^{-(p+1)/\lambda-1}}{1 + u} du \right)^{-1},
$$

(2.27)

then $f(z) \in T_{p,q,s}(0; h)$. The bound $\beta_0$ is sharp when $h(z) = 1/(1 - z)$.

**Proof.** Let us define

$$
g(z) = -\frac{z^{p+1} (H_{p,q,s}(\alpha_1)f(z))'}{p}
$$

(2.28)
for \( f(z) \in T_{p,q,s}(\lambda; \beta h + 1 - \beta) \) with \( \lambda < 0 \) and \( \beta > 0 \). Then we have

\[
g(z) - \frac{\lambda}{p + 1} z g'(z) = \frac{(\lambda - 1)}{p} z^{p + 1} (H_{p,q,s}(\alpha_1)f(z))' + \frac{\lambda}{p(p + 1)} z^{p + 2} (H_{p,q,s}(\alpha_1)f(z))'' < \beta h(z) + 1 - \beta.
\]

Hence an application of Lemma 1.2 yields

\[
g(z) < -\beta \frac{(p + 1)}{\lambda} z^{(p + 1)/\lambda} \int_0^z t^{-(p + 1)/\lambda - 1} h(t) dt + 1 - \beta
\]

where

\[
\varphi(z) = -\beta \frac{(p + 1)}{\lambda} z^{(p + 1)/\lambda} \int_0^z t^{-(p + 1)/\lambda - 1} \frac{1 - u}{1 - tu} du + 1 - \beta.
\]

If \( 0 < \beta \leq \beta_0 \), where \( \beta_0 > 1 \) is given by (2.27), then it follows from (2.31) that

\[
\text{Re } \varphi(z) = -\beta \frac{(p + 1)}{\lambda} \int_0^1 u^{-(p + 1)/\lambda - 1} \text{Re} \left( \frac{1}{1 - uz} \right) du + 1 - \beta
\]

\[
> -\beta \frac{(p + 1)}{\lambda} \int_0^1 1 + u - \beta
\]

\[
\geq \frac{1}{z}\quad (z \in \mathbb{U}; \lambda < 0).
\]

Now, by using the Herglotz representation for \( \varphi(z) \), from (2.28) and (2.30), we arrive at

\[
-\frac{z^{p + 1} (H_{p,q,s}(\alpha_1)f(z))'}{p} \times (h * \varphi)(z) < h(z)
\]

because \( h(z) \) is convex univalent in \( \mathbb{U} \). This shows that \( f(z) \in T_{p,q,s}(0; h) \).

For \( h(z) = 1/(1 - z) \) and \( f(z) \in \Sigma(p) \) defined by

\[
-\frac{z^{p + 1} (H_{p,q,s}(\alpha_1)f(z))'}{p} = -\beta \frac{(p + 1)}{\lambda} z^{(p + 1)/\lambda} \int_0^z t^{-(p + 1)/\lambda - 1} dt + 1 - \beta,
\]

it is easy to verify that

\[
\frac{(\lambda - 1)}{p} z^{p + 1} (H_{p,q,s}(\alpha_1)f(z))' + \frac{\lambda}{p(p + 1)} z^{p + 2} (H_{p,q,s}(\alpha_1)f(z))'' = \beta h(z) + 1 - \beta.
\]
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Thus \( f(z) \in T_{p,q,s}(\lambda; \beta h + 1 - \beta) \). Also, for \( \beta > \beta_0 \), we have

\[
\text{Re}\left\{-\frac{z^{p+1}}{p}(H_{p,q,s}(a_1) f(z))'\right\} \rightarrow -\frac{\beta(p + 1)}{\lambda} \int_0^1 \frac{u^{-(p+1)/\lambda)-1}}{1 + u} du + 1 - \beta < \frac{1}{2} \quad (z \rightarrow -1),
\]

(2.36)

which implies that \( f(z) \notin T_{p,q,s}(0; h) \). Hence the bound \( \beta_0 \) cannot be increased when \( h(z) = 1/(1 - z) \).

\[\square\]

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