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Research Article

Weyl-Titchmarsh Theory for Time Scale Symplectic Systems on Half Line

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We develop the Weyl-Titchmarsh theory for time scale symplectic systems. We introduce the $M(\lambda)$-function, study its properties, construct the corresponding Weyl disk and Weyl circle, and establish their geometric structure including the formulas for their center and matrix radii. Similar properties are then derived for the limiting Weyl disk. We discuss the notions of the system being in the limit point or limit circle case and prove several characterizations of the system in the limit point case and one condition for the limit circle case. We also define the Green function for the associated nonhomogeneous system and use its properties for deriving further results for the original system in the limit point or limit circle case. Our work directly generalizes the corresponding discrete time theory obtained recently by S. Clark and P. Zemánek (2010). It also unifies the results in many other papers on the Weyl-Titchmarsh theory for linear Hamiltonian differential, difference, and dynamic systems when the spectral parameter appears in the second equation. Some of our results are new even in the case of the second-order Sturm-Liouville equations on time scales.

1. Introduction

In this paper we develop systematically the Weyl-Titchmarsh theory for time scale symplectic systems. Such systems unify and extend the classical linear Hamiltonian differential systems and discrete symplectic and Hamiltonian systems, including the Sturm-Liouville differential and difference equations of arbitrary even order. As the research in the Weyl-Titchmarsh theory has been very active in the last years, we contribute to this development by presenting a theory which directly generalizes and unifies the results in several recent papers, such as [1–4] and partly in [5–14].

Historically, the theory nowadays called by Weyl and Titchmarsh started in [15] by the investigation of the second-order linear differential equation

$$\left(r(t)z'(t)\right) + q(t)z(t) = \lambda z(t), \quad t \in [0,\infty), \tag{1.1}$$

where $r, q : [0,\infty) \to \mathbb{R}$ are continuous, $r(t) > 0$, and $\lambda \in \mathbb{C}$, is a spectral parameter. By using a geometrical approach it was showed that (1.1) can be divided into two classes called the limit circle and limit point meaning that either all solutions of (1.1) are square integrable for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ or there is a unique (up to a multiplicative constant) square-integrable solution of (1.1) on $[0,\infty)$. Analytic methods for the investigation of (1.1) have been introduced in a series of papers starting with [16]; see also [17]. We refer to [18–20] for an overview of the original contributions to the Weyl-Titchmarsh theory for (1.1); see also [21]. Extensions of the Weyl-Titchmarsh theory to more general equations, namely, to the linear Hamiltonian differential systems

$$z'(t) = [\lambda A(t) + B(t)]z(t), \quad t \in [0,\infty), \tag{1.2}$$

was initiated in [22] and developed further in [6, 8, 10, 11, 23–38].

According to [19], the first paper dealing with the parallel discrete time Weyl theory for second-order difference equations appears to be the work mentioned in [39]. Since then a long time elapsed until the theory of difference equations attracted more attention. The Weyl-Titchmarsh theory for the second-order Sturm-Liouville difference equations was developed in [22, 40, 41]; see also the references in [19]. For higher-order Sturm-Liouville difference equations and linear Hamiltonian difference systems, such as

$$\Delta x_k = A_k x_{k+1} + \left(B_k + \lambda W_k^{[2]}\right)u_k, \quad \Delta u_k = \left(C_k - \lambda W_k^{[1]}\right)x_{k+1} - A_k^* u_k, \quad k \in [0,\infty)_{\mathbb{Z}}, \tag{1.3}$$

where A_k, B_k, C_k, $W_k^{[1]}$, $W_k^{[2]}$ are complex $n \times n$ matrices such that B_k and C_k are Hermitian and $W_k^{[1]}$ and $W_k^{[2]}$ are Hermitian and nonnegative definite, the Weyl-Titchmarsh theory was studied in [9, 14, 42]. Recently, the results for linear Hamiltonian difference systems were generalized in [1, 2] to discrete symplectic systems

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k + \lambda \mathcal{W}_k x_{k+1}, \quad k \in [0,\infty)_{\mathbb{Z}}, \tag{1.4}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$, $\mathcal{W}_k$ are complex $n \times n$ matrices such that $\mathcal{W}_k$ is Hermitian and nonnegative definite and the $2n \times 2n$ transition matrix in (1.4) is symplectic, that is,

$$\mathcal{S}_k := \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \quad \mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.5}$$

In the unifying theory for differential and difference equations—the theory of time scales—the classification of second-order Sturm-Liouville dynamic equations

$$y^{\Delta\Delta}(t) + q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{1.6}$$

to be of the limit point or limit circle type is given in [4, 43]. These two papers seem to be the only ones on time scales which are devoted to the Weyl-Titchmarsh theory for the second order dynamic equations. Another way of generalizing the Weyl-Titchmarsh theory for continuous and discrete Hamiltonian systems was presented in [3, 5]. In these references the authors consider the linear Hamiltonian system

$$\begin{aligned} x^{\Delta}(t) &= A(t)x^{\sigma}(t) + [B(t) + \lambda W_2(t)]u(t), \\ u^{\Delta}(t) &= [C(t) - \lambda W_1(t)]x^{\sigma}(t) - A^*(t)u(t), \quad t \in [a,\infty)_{\mathbb{T}}, \end{aligned} \tag{1.7}$$

on the so-called Sturmian or general time scales, respectively. Here $f^{\Delta}(t)$ is the time scale Δ-derivative and $f^{\sigma}(t) := f(\sigma(t))$, where $\sigma(t)$ is the forward jump at t; see the time scale notation in Section 2.

In the present paper we develop the Weyl-Titchmarsh theory for more general linear dynamic systems, namely, the time scale symplectic systems

$$\begin{aligned} x^{\Delta}(t) &= \mathcal{A}(t)x(t) + \mathcal{B}(t)u(t), \\ u^{\Delta}(t) &= \mathcal{C}(t)x(t) + \mathcal{D}(t)u(t) - \lambda\mathcal{W}(t)x^{\sigma}(t), \quad t \in [a,\infty)_{\mathbb{T}}, \end{aligned} \tag{$\mathcal{S}_\lambda$}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ are complex $n \times n$ matrix functions on $[a,\infty)_{\mathbb{T}}$, $\mathcal{W}(t)$ is Hermitian and nonnegative definite, $\lambda \in \mathbb{C}$, and the $2n \times 2n$ coefficient matrix in system $(\mathcal{S}_\lambda)$ satisfies

$$\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix}, \quad \mathcal{S}^*(t)\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\mathcal{S}^*(t)\mathcal{J}\mathcal{S}(t) = 0, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{1.8}$$

where $\mu(t) := \sigma(t) - t$ is the graininess of the time scale. The spectral parameter λ is only in the second equation of system $(\mathcal{S}_\lambda)$. This system was introduced in [44], and it naturally unifies the previously mentioned continuous, discrete, and time scale linear Hamiltonian systems (having the spectral parameter in the second equation only) and discrete symplectic systems into one framework. Our main results are the properties of the $M(\lambda)$ function, the geometric description of the Weyl disks, and characterizations of the limit point and limit circle cases for the time scale symplectic system $(\mathcal{S}_\lambda)$. In addition, we give a formula for the $L^2_{\mathcal{W}}$ solutions of a nonhomogeneous time scale symplectic system in terms of its Green function. These results generalize and unify in particular all the results in [1–4] and some results from [5–14]. The theory of time scale symplectic systems or Hamiltonian systems is a topic with active research in recent years; see, for example, [44–51]. This paper can be regarded not only as a completion of these papers by establishing the Weyl-Titchmarsh theory for time scale symplectic systems but also as a comparison of the corresponding continuous and discrete time results. The references to particular statements in the literature are displayed throughout the text. Many results of this paper are new even for (1.6), being a special case of system $(\mathcal{S}_\lambda)$. An overview of these new results for (1.6) will be presented in our subsequent work.

This paper is organized as follows. In the next section we recall some basic notions from the theory of time scales and linear algebra. In Section 3 we present fundamental properties of time scale symplectic systems with complex coefficients, including the important Lagrange identity (Theorem 3.5) and other formulas involving their solutions.

In Section 4 we define the time scale $M(\lambda)$-function for system $(\mathcal{S}_\lambda)$ and establish its basic properties in the case of the regular spectral problem. In Section 5 we introduce the Weyl disks and circles for system $(\mathcal{S}_\lambda)$ and describe their geometric structure in terms of contractive matrices in $\mathbb{C}^{n\times n}$. The properties of the limiting Weyl disk and Weyl circle are then studied in Section 6, where we also prove that system $(\mathcal{S}_\lambda)$ has at least n linearly independent solutions in the space $L^2_{\mathcal{W}}$ (see Theorem 6.7). In Section 7 we define the system $(\mathcal{S}_\lambda)$ to be in the limit point and limit circle case and prove several characterizations of these properties. In the final section we consider the system $(\mathcal{S}_\lambda)$ with a nonhomogeneous term. We construct its Green function, discuss its properties, and characterize the $L^2_{\mathcal{W}}$ solutions of this nonhomogeneous system in terms of the Green function (Theorem 8.5). A certain uniqueness result is also proven for the limit point case.

2. Time Scales

Following [52, 53], a time scale $\mathbb{T}$ is any nonempty and closed subset of $\mathbb{R}$. A bounded time scale can be therefore identified as $[a,b]_{\mathbb{T}} := [a,b]\cap\mathbb{T}$ which we call the time scale interval, where $a := \min\mathbb{T}$ and $b := \max\mathbb{T}$. Similarly, a time scale which is unbounded above has the form $[a,\infty)_{\mathbb{T}} := [a,\infty)\cap\mathbb{T}$. The forward and backward jump operators on a time scale are denoted by $\sigma(t)$ and $\rho(t)$ and the graininess function by $\mu(t) := \sigma(t) - t$. If not otherwise stated, all functions in this paper are considered to be complex valued. A function f on $[a,b]_{\mathbb{T}}$ is called *piecewise rd-continuous*; we write $f \in \mathrm{C}_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$ if the right-hand limit $f(t^+)$ exists finite at all right-dense points $t \in [a,b)_{\mathbb{T}}$, and the left-hand limit $f(t^-)$ exists finite at all left-dense points $t \in (a,b]_{\mathbb{T}}$ and f is continuous in the topology of the given time scale at all but possibly finitely many right-dense points $t \in [a,b)_{\mathbb{T}}$. A function f on $[a,\infty)_{\mathbb{T}}$ is *piecewise rd-continuous*; we write $f \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\infty)_{\mathbb{T}}$ if $f \in \mathrm{C}_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$ for every $b \in (a,\infty)_{\mathbb{T}}$. An $n\times n$ matrix-valued function f is called *regressive* on a given time scale interval if $I + \mu(t)f(t)$ is invertible for all t in this interval.

The time scale Δ-derivative of a function f at a point t is denoted by $f^\Delta(t)$; see [52, Definition 1.10]. Whenever $f^\Delta(t)$ exists, the formula $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$ holds true. The product rule for the Δ-differentiation of the product of two functions has the form

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t). \tag{2.1}$$

A function f on $[a,b]_{\mathbb{T}}$ is called *piecewise rd-continuously Δ-differentiable*; we write $f \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$; if it is continuous on $[a,b]_{\mathbb{T}}$, then $f^\Delta(t)$ exists at all except for possibly finitely many points $t \in [a,\rho(b)]_{\mathbb{T}}$, and $f^\Delta \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\rho(b)]_{\mathbb{T}}$. As a consequence we have that the finitely many points t_i at which $f^\Delta(t_i)$ does not exist belong to $(a,b)_{\mathbb{T}}$ and these points t_i are necessarily right-dense and left-dense at the same time. Also, since at those points we know that $f^\Delta(t_i^+)$ and $f^\Delta(t_i^-)$ exist finite, we replace the quantity $f^\Delta(t_i)$ by $f^\Delta(t_i^\pm)$ in any formula involving $f^\Delta(t)$ for all $t \in [a,\rho(b)]_{\mathbb{T}}$. Similarly as above we define $f \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,\infty)_{\mathbb{T}}$. The time scale integral of a piecewise rd-continuous function f over $[a,b]_{\mathbb{T}}$ is denoted by $\int_a^b f(t)\Delta t$ and over $[a,\infty)_{\mathbb{T}}$ by $\int_a^\infty f(t)\Delta t$ provided this integral is convergent in the usual sense; see [52, Definitions 1.71 and 1.82].

Remark 2.1. As it is known in [52, Theorem 5.8] and discussed in [54, Remark 3.8], for a fixed $t_0 \in [a,b]_{\mathbb{T}}$ and a piecewise rd-continuous $n \times n$ matrix function $A(\cdot)$ on $[a,b]_{\mathbb{T}}$ which is regressive on $[a,t_0)_{\mathbb{T}}$, the initial value problem $y^{\Delta}(t) = A(t)y(t)$ for $t \in [a,\rho(b)]_{\mathbb{T}}$ with $y(t_0) = y_0$ has a unique solution $y(\cdot) \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$ for any $y_0 \in \mathbb{C}^n$. Similarly, this result holds on $[a,\infty)_{\mathbb{T}}$.

Let us recall some matrix notations from linear algebra used in this paper. Given a complex square matrix M, by M^*, $M > 0$, $M \geq 0$, $M < 0$, $M \leq 0$, $\operatorname{rank} M$, $\operatorname{Ker} M$, $\operatorname{def} M$, we denote, respectively, the conjugate transpose, positive definiteness, positive semidefiniteness, negative definiteness, negative semidefiniteness, rank, kernel, and the defect (i.e., the dimension of the kernel) of the matrix M. Moreover, we will use the notation $\operatorname{Im}(M) := (M - M^*)/(2i)$ and $\operatorname{Re}(M) := (M + M^*)/2$ for the Hermitian components of the matrix M; see [55, pages 268-269] or [56, Fact 3.5.24]. This notation will be also used with $\lambda \in \mathbb{C}$, and in this case $\operatorname{Im}(\lambda)$ and $\operatorname{Re}(\lambda)$ represent the imaginary and real parts of λ.

Remark 2.2. If the matrix $\operatorname{Im}(M)$ is positive or negative definite, then the matrix M is necessarily invertible. The proof of this fact can be found, for example, in [2, Remark 2.6].

In order to simplify the notation we abbreviate $[f^{\sigma}(t)]^*$ and $[f^*(t)]^{\sigma}$ by $f^{\sigma *}(t)$. Similarly, instead of $[f^{\Delta}(t)]^*$ and $[f^*(t)]^{\Delta}$ we will use $f^{\Delta *}(t)$.

3. Time Scale Symplectic Systems

Let $\mathcal{A}(\cdot), \mathcal{B}(\cdot), \mathcal{C}(\cdot), \mathcal{D}(\cdot), \mathcal{W}(\cdot)$ be $n \times n$ piecewise rd-continuous functions on $[a,\infty)_{\mathbb{T}}$ such that $\mathcal{W}(t) \geq 0$ for all $t \in [a,\infty)_{\mathbb{T}}$; that is, $\mathcal{W}(t)$ is Hermitian and nonnegative definite, satisfying identity (1.8). In this paper we consider the linear system $(\mathcal{S}_\lambda)$ introduced in the previous section. This system can be written as

$$z^{\Delta}(t,\lambda) = \mathcal{S}(t)z(t,\lambda) + \lambda \mathcal{J}\widetilde{\mathcal{W}}(t)z^{\sigma}(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{$\mathcal{S}_\lambda$}$$

where the $2n \times 2n$ matrix $\widetilde{\mathcal{W}}(t)$ is defined and has the property

$$\widetilde{\mathcal{W}}(t) := \begin{pmatrix} \mathcal{W}(t) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{J}\widetilde{\mathcal{W}}(t) = \begin{pmatrix} 0 & 0 \\ -\mathcal{W}(t) & 0 \end{pmatrix}. \tag{3.1}$$

The system $(\mathcal{S}_\lambda)$ can be written in the equivalent form

$$z^{\Delta}(t,\lambda) = \mathcal{S}(t,\lambda)z(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{3.2}$$

where the matrix $\mathcal{S}(t,\lambda)$ is defined through the matrices $\mathcal{S}(t)$ and $\widetilde{\mathcal{W}}(t)$ from (1.8) and (3.1) by

$$\begin{aligned}\mathcal{S}(t,\lambda) &:= \mathcal{S}(t) + \lambda\mathcal{J}\widetilde{\mathcal{W}}(t)\left[I + \mu(t)\mathcal{S}(t)\right] \\ &= \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) - \lambda\mathcal{W}(t)\left[I + \mu(t)\mathcal{A}(t)\right] & \mathcal{D}(t) - \lambda\mu(t)\mathcal{W}(t)\mathcal{B}(t) \end{pmatrix}.\end{aligned} \tag{3.3}$$

By using the identity in (1.8), a direct calculation shows that the matrix function $\mathcal{S}(\cdot,\cdot)$ satisfies

$$\mathcal{S}^*(t,\lambda)\mathcal{J} + \mathcal{J}\mathcal{S}\left(t,\overline{\lambda}\right) + \mu(t)\mathcal{S}^*(t,\lambda)\mathcal{J}\mathcal{S}\left(t,\overline{\lambda}\right) = 0, \quad t \in [a,\infty)_{\mathbb{T}},\ \lambda \in \mathbb{C}. \tag{3.4}$$

Here $\mathcal{S}^*(t,\lambda) = [\mathcal{S}(t,\lambda)]^*$, and $\overline{\lambda}$ is the usual conjugate number to λ.

Remark 3.1. The name time scale *symplectic system* or *Hamiltonian system* has been reserved in the literature for the system of the form

$$z^{\Delta}(t) = \mathbb{S}(t)z(t), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{3.5}$$

in which the matrix function $\mathbb{S}(\cdot)$ satisfies the identity in (1.8); see [44–47, 57], and compare also, for example, with [58–61]. Since for a fixed $\lambda, \nu \in \mathbb{C}$ the matrix $\mathcal{S}(t,\lambda)$ from (3.3) satisfies

$$\mathcal{S}^*(t,\lambda)\mathcal{J} + \mathcal{J}\mathcal{S}(t,\nu) + \mu(t)\mathcal{S}^*(t,\lambda)\mathcal{J}\mathcal{S}(t,\nu) = \left(\overline{\lambda} - \nu\right)\left[I + \mu(t)\mathcal{S}^*(t)\right]\widetilde{\mathcal{W}}(t)\left[I + \mu(t)\mathcal{S}(t)\right], \tag{3.6}$$

it follows that the system $(\mathcal{S}_\lambda)$ is a true time scale symplectic system according to the above terminology only for $\lambda \in \mathbb{R}$, while strictly speaking $(\mathcal{S}_\lambda)$ *is not* a time scale symplectic system for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. However, since $(\mathcal{S}_\lambda)$ is a perturbation of the time scale symplectic system $(\mathcal{S}_0)$ and since the important properties of time scale symplectic systems needed in the presented Weyl-Titchmarsh theory, such as (3.4) or (3.8), are satisfied in an appropriate modification, we accept with the above understanding the same terminology for the system $(\mathcal{S}_\lambda)$ for any $\lambda \in \mathbb{C}$.

Equation (3.4) represents a fundamental identity for the theory of time scale symplectic systems $(\mathcal{S}_\lambda)$. Some important properties of the matrix $\mathcal{S}(t,\lambda)$ are displayed below. Note that formula (3.7) is a generalization of [46, equation (10.4)] to complex values of λ.

Lemma 3.2. *Identity* (3.4) *is equivalent to the identity*

$$\mathcal{S}\left(t,\overline{\lambda}\right)\mathcal{J} + \mathcal{J}\mathcal{S}^*(t,\lambda) + \mu(t)\mathcal{S}\left(t,\overline{\lambda}\right)\mathcal{J}\mathcal{S}^*(t,\lambda) = 0, \quad t \in [a,\infty)_{\mathbb{T}},\ \lambda \in \mathbb{C}. \tag{3.7}$$

In this case for any $\lambda \in \mathbb{C}$ we have

$$[I + \mu(t)\mathcal{S}^*(t,\lambda)]\mathcal{J}\left[I + \mu(t)\mathcal{S}\left(t,\overline{\lambda}\right)\right] = \mathcal{J}, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{3.8}$$

$$\left[I + \mu(t)\mathcal{S}\left(t,\overline{\lambda}\right)\right]\mathcal{J}[I + \mu(t)\mathcal{S}^*(t,\lambda)] = \mathcal{J}, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{3.9}$$

and the matrices $I + \mu(t)\mathcal{S}(t,\lambda)$ and $I + \mu(t)\mathcal{S}(t,\overline{\lambda})$ are invertible with

$$[I + \mu(t)\mathcal{S}(t,\lambda)]^{-1} = -\mathcal{J}\left[I + \mu(t)\mathcal{S}^*\left(t,\overline{\lambda}\right)\right]\mathcal{J}, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{3.10}$$

Proof. Let $t \in [a,\infty)_{\mathbb{T}}$ and $\lambda \in \mathbb{C}$ be fixed. If t is right-dense, that is, $\mu(t) = 0$, then identity (3.4) reduces to $\mathcal{S}^*(t,\lambda)\mathcal{J} + \mathcal{J}\mathcal{S}(t,\overline{\lambda}) = 0$. Upon multiplying this equation by $\mathcal{J}$ from the left and right side, we get identity (3.7) with $\mu(t) = 0$. If t is right scattered, that is, $\mu(t) > 0$, then (3.4) is equivalent to (3.8). It follows that the determinants of $I + \mu(t)\mathcal{S}(t,\lambda)$ and $I + \mu(t)\mathcal{S}(t,\overline{\lambda})$ are nonzero proving that these matrices are invertible with the inverse given by (3.10). Upon multiplying (3.8) by the invertible matrices $[I + \mu(t)\mathcal{S}(t,\overline{\lambda})]\mathcal{J}$ from the left and $-[I + \mu(t)\mathcal{S}(t,\overline{\lambda})]^{-1}\mathcal{J}$ from the right and by using $\mathcal{J}^2 = -I$, we get formula (3.9), which is equivalent to (3.7) due to $\mu(t) > 0$. □

Remark 3.3. Equation (3.10) allows writing the system $(\mathcal{S}_\lambda)$ in the equivalent adjoint form

$$z^\Delta(t,\lambda) = \mathcal{J}\mathcal{S}^*\left(t,\overline{\lambda}\right)\mathcal{J}z^\sigma(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}. \tag{3.11}$$

System (3.11) can be found, for example, in [47, Remark 3.1(iii)] or [50, equation (3.2)] in the connection with optimality conditions for variational problems over time scales.

In the following result we show that (3.4) guarantees, among other properties, the existence and uniqueness of solutions of the initial value problems associated with $(\mathcal{S}_\lambda)$.

Theorem 3.4 (existence and uniqueness theorem). *Let $\lambda \in \mathbb{C}$, $t_0 \in [a,\infty)_{\mathbb{T}}$, and $z_0 \in \mathbb{C}^{2n}$ be given. Then the initial value problem $(\mathcal{S}_\lambda)$ with $z(t_0) = z_0$ has a unique solution $z(\cdot,\lambda) \in \mathrm{C}^1_{\mathrm{prd}}$ on the interval $[a,\infty)_{\mathbb{T}}$.*

Proof. The coefficient matrix of system $(\mathcal{S}_\lambda)$, or equivalently of system (3.2), is piecewise rd-continuous on $[a,\infty)_{\mathbb{T}}$. By Lemma 3.2, the matrix $I+\mu(t)\mathcal{S}(t,\lambda)$ is invertible for all $t \in [a,\infty)_{\mathbb{T}}$, which proves that the function $\mathcal{S}(\cdot,\lambda)$ is regressive on $[a,\infty)_{\mathbb{T}}$. Hence, the result follows from Remark 2.1. □

If not specified otherwise, we use a common agreement that $2n$-vector solutions of system $(\mathcal{S}_\lambda)$ and $2n \times n$-matrix solutions of system $(\mathcal{S}_\lambda)$ are denoted by small letters and capital letters, respectively, typically by $z(\cdot,\lambda)$ or $\tilde{z}(\cdot,\lambda)$ and $Z(\cdot,\lambda)$ or $\tilde{Z}(\cdot,\lambda)$.

Next we establish several identities involving solutions of system $(\mathcal{S}_\lambda)$ or solutions of two such systems with different spectral parameters. The first result is the Lagrange identity known in the special cases of continuous time linear Hamiltonian systems in [11, Theorem 4.1] or [8, equation (2.23)], discrete linear Hamiltonian systems in [9, equation (2.55)]

or [14, Lemma 2.2], discrete symplectic systems in [1, Lemma 2.6] or [2, Lemma 2.3], and time scale linear Hamiltonian systems in [3, Lemma 3.5] and [5, Theorem 2.2].

Theorem 3.5 (Lagrange identity). *Let $\lambda, \nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If $z(\cdot,\lambda)$ and $z(\cdot,\nu)$ are $2n\times m$ solutions of systems $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, respectively, then*

$$[z^*(t,\lambda)\mathcal{J}z(t,\nu)]^\Delta = \left(\overline{\lambda}-\nu\right)z^{\sigma*}(t,\lambda)\widetilde{\mathcal{W}}(t)z^\sigma(t,\nu), \quad t\in[a,\infty)_{\mathbb{T}}. \tag{3.12}$$

Proof. Formula (3.12) follows from the time scales product rule (2.1) by using the relation $z^\sigma(t,\lambda) = [I+\mu(t)\mathcal{S}(t,\lambda)]z(t,\lambda)$ and identity (3.6). □

As consequences of Theorem 3.5, we obtain the following.

Corollary 3.6. *Let $\lambda, \nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If $z(\cdot,\lambda)$ and $z(\cdot,\nu)$ are $2n\times m$ solutions of systems $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, respectively, then for all $t\in[a,\infty)_{\mathbb{T}}$ we have*

$$z^*(t,\lambda)\mathcal{J}z(t,\nu) = z^*(a,\lambda)\mathcal{J}z(a,\nu) + \left(\overline{\lambda}-\nu\right)\int_a^t z^{\sigma*}(s,\lambda)\widetilde{\mathcal{W}}(s)z^\sigma(s,\nu)\Delta s. \tag{3.13}$$

One can easily see that if $z(\cdot,\lambda)$ is a solution of system $(\mathcal{S}_\lambda)$, then $z(\cdot,\overline{\lambda})$ is a solution of system $(\mathcal{S}_{\overline{\lambda}})$. Therefore, Theorem 3.5 with $\nu=\overline{\lambda}$ yields a Wronskian-type property of solutions of system $(\mathcal{S}_\lambda)$.

Corollary 3.7. *Let $\lambda\in\mathbb{C}$ and $m\in\mathbb{N}$ be given. For any $2n\times m$ solution $z(\cdot,\lambda)$ of systems $(\mathcal{S}_\lambda)$*

$$z^*(t,\lambda)\mathcal{J}z\left(t,\overline{\lambda}\right) \equiv z^*(a,\lambda)\mathcal{J}z\left(a,\overline{\lambda}\right), \quad \text{is constant on } [a,\infty)_{\mathbb{T}}. \tag{3.14}$$

The following result gives another interesting property of solutions of system $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_{\overline{\lambda}})$.

Lemma 3.8. *Let $\lambda\in\mathbb{C}$ and $m\in\mathbb{N}$ be given. For any $2n\times m$ solutions $z(\cdot,\lambda)$ and $\tilde{z}(\cdot,\lambda)$ of system $(\mathcal{S}_\lambda)$, the $2n\times 2n$ matrix function $K(\cdot,\lambda)$ defined by*

$$K(t,\lambda) := z(t,\lambda)\tilde{z}^*\left(t,\overline{\lambda}\right) - \tilde{z}(t,\lambda)z^*\left(t,\overline{\lambda}\right), \quad t\in[a,\infty)_{\mathbb{T}}, \tag{3.15}$$

satisfies the dynamic equation

$$K^\Delta(t,\lambda) = \mathcal{S}(t,\lambda)K(t,\lambda) + \left[I+\mu(t)\mathcal{S}(t,\lambda)\right]K(t,\lambda)\mathcal{S}^*\left(t,\overline{\lambda}\right), \quad t\in[a,\infty)_{\mathbb{T}}, \tag{3.16}$$

and the identities $K^*(t,\lambda) = -K(t,\overline{\lambda})$ and

$$K^\sigma(t,\lambda) = \left[I+\mu(t)\mathcal{S}(t,\lambda)\right]K(t,\lambda)\left[I+\mu(t)\mathcal{S}^*\left(t,\overline{\lambda}\right)\right], \quad t\in[a,\infty)_{\mathbb{T}}. \tag{3.17}$$

Proof. Having that $z(\cdot,\lambda)$ and $\tilde{z}(\cdot,\lambda)$ are solutions of system $(\mathcal{S}_\lambda)$, it follows that $z(\cdot,\overline{\lambda})$ and $\tilde{z}(\cdot,\overline{\lambda})$ are solutions of system $(\mathcal{S}_{\overline{\lambda}})$. The results then follow by direct calculations. □

Remark 3.9. The content of Lemma 3.8 appears to be new both in the continuous and discrete time cases. Moreover, when the matrix function $K(\cdot,\lambda) \equiv K(\lambda)$ is constant, identity (3.17) yields for any right-scattered $t \in [a,\infty)_{\mathbb{T}}$ that

$$\mathcal{S}(t,\lambda)K(\lambda) + K(\lambda)\mathcal{S}^*\left(t,\overline{\lambda}\right) + \mu(t)\mathcal{S}(t,\lambda)K(\lambda)\mathcal{S}^*\left(t,\overline{\lambda}\right) = 0. \tag{3.18}$$

It is interesting to note that this formula is very much like (3.7). More precisely, identity (3.7) is a consequence of (3.18) for the case of $K(\lambda) \equiv \mathcal{J}$.

Next we present properties of certain fundamental matrices $\Psi(\cdot,\lambda)$ of system $(\mathcal{S}_\lambda)$, which are generalizations of the corresponding results in [46, Section 10.2] to complex λ. Some of these results can be proven under the weaker condition that the initial value of $\Psi(a,\lambda)$ does depend on λ and satisfies $\Psi^*(a,\lambda)\mathcal{J}\Psi(a,\overline{\lambda}) = \mathcal{J}$. However, these more general results will not be needed in this paper.

Lemma 3.10. *Let $\lambda \in \mathbb{C}$ be fixed. If $\Psi(\cdot,\lambda)$ is a fundamental matrix of system $(\mathcal{S}_\lambda)$ such that $\Psi(a,\lambda)$ is symplectic and independent of λ, then for any $t \in [a,\infty)_{\mathbb{T}}$ we have*

$$\Psi^*(t,\lambda)\mathcal{J}\Psi\left(t,\overline{\lambda}\right) = \mathcal{J}, \qquad \Psi^{-1}(t,\lambda) = -\mathcal{J}\Psi^*\left(t,\overline{\lambda}\right)\mathcal{J}, \qquad \Psi(t,\lambda)\mathcal{J}\Psi^*\left(t,\overline{\lambda}\right) = \mathcal{J}. \tag{3.19}$$

Proof. Identity (3.19)(i) is a consequence of Corollary 3.7, in which we use the fact that $\Psi(a,\lambda)$ is symplectic and independent of λ. The second identity in (3.19) follows from the first one, while the third identity is obtained from the equation $\Psi(t,\lambda)\Psi^{-1}(t,\lambda) = I$. □

Remark 3.11. If the fundamental matrix $\Psi(\cdot,\lambda) = (Z(\cdot,\lambda) \quad \tilde{Z}(\cdot,\lambda))$ in Lemma 3.10 is partitioned into two $2n \times n$ blocks, then (3.19)(i) and (3.19)(iii) have, respectively, the form

$$Z^*(t,\lambda)\mathcal{J}Z\left(t,\overline{\lambda}\right) = 0, \qquad Z^*(t,\lambda)\mathcal{J}\tilde{Z}\left(t,\overline{\lambda}\right) = I, \qquad \tilde{Z}^*(t,\lambda)\mathcal{J}\tilde{Z}\left(t,\overline{\lambda}\right) = 0, \tag{3.20}$$

$$Z(t,\lambda)\tilde{Z}^*\left(t,\overline{\lambda}\right) - \tilde{Z}(t,\lambda)Z^*\left(t,\overline{\lambda}\right) = \mathcal{J}. \tag{3.21}$$

Observe that the matrix on the left-hand side of (3.21) represents a constant matrix $K(t,\lambda)$ from Lemma 3.8 and Remark 3.9.

Corollary 3.12. *Under the conditions of Lemma 3.10, for any $t \in [a,\infty)_{\mathbb{T}}$, we have*

$$\Psi^\sigma(t,\lambda)\mathcal{J}\Psi^*\left(t,\overline{\lambda}\right) = \left[I + \mu(t)\mathcal{S}(t,\lambda)\right]\mathcal{J}, \tag{3.22}$$

which in the notation of Remark 3.11 has the form

$$Z^\sigma(t,\lambda)\tilde{Z}^*\left(t,\overline{\lambda}\right) - \tilde{Z}^\sigma(t,\lambda)Z^*\left(t,\overline{\lambda}\right) = \left[I + \mu(t)\mathcal{S}(t,\lambda)\right]\mathcal{J}. \tag{3.23}$$

Proof. Identity (3.22) follows from the equation $\Psi^{\sigma}(t,\lambda) = [I + \mu(t)\mathcal{S}(t,\lambda)]\Psi(t,\lambda)$ by applying formula (3.19)(ii). □

4. $M(\lambda)$-Function for Regular Spectral Problem

In this section we consider the regular spectral problem on the time scale interval $[a,b]_{\mathbb{T}}$ with some fixed $b \in (a,\infty)_{\mathbb{T}}$. We will specify the corresponding boundary conditions in terms of complex $n \times 2n$ matrices from the set

$$\Gamma := \left\{\alpha \in \mathbb{C}^{n\times 2n},\ \alpha\alpha^* = I,\ \alpha\mathcal{J}\alpha^* = 0\right\}. \tag{4.1}$$

The two defining conditions for $\alpha \in \mathbb{C}^{n\times 2n}$ in (4.1) imply that the $2n \times 2n$ matrix $(\alpha^* \quad -\mathcal{J}\alpha^*)$ is unitary and symplectic. This yields the identity

$$(\alpha^* \quad -\mathcal{J}\alpha^*)\begin{pmatrix}\alpha\\ \alpha\mathcal{J}\end{pmatrix} = I \in \mathbb{C}^{2n\times 2n}, \quad \text{that is,}\ \alpha^*\alpha - \mathcal{J}\alpha^*\alpha\mathcal{J} = I. \tag{4.2}$$

The last equation also implies, compare with [60, Remark 2.1.2], that

$$\operatorname{Ker}\alpha = \operatorname{Im}\mathcal{J}\alpha^*. \tag{4.3}$$

Let $\alpha,\beta \in \Gamma$ be fixed and consider the boundary value problem

$$(\mathcal{S}_\lambda), \quad \alpha z(a,\lambda) = 0, \quad \beta z(b,\lambda) = 0. \tag{4.4}$$

Our first result shows that the boundary conditions in (4.4) are equivalent with the boundary conditions phrased in terms of the images of the $2n \times 2n$ matrices

$$R_a := (\mathcal{J}\alpha^* \quad 0), \qquad R_b := (0 \quad -\mathcal{J}\beta^*), \tag{4.5}$$

which satisfy $R_a^*\mathcal{J}R_a = 0$, $R_b^*\mathcal{J}R_b = 0$, and $\operatorname{rank}(R_a^* \quad R_b^*) = 2n$.

Lemma 4.1. *Let $\alpha,\beta \in \Gamma$ and $\lambda \in \mathbb{C}$ be fixed. A solution $z(\cdot,\lambda)$ of system $(\mathcal{S}_\lambda)$ satisfies the boundary conditions in* (4.4) *if and only if there exists a unique vector $\xi \in \mathbb{C}^{2n}$ such that*

$$z(a,\lambda) = R_a\xi, \qquad z(b,\lambda) = R_b\xi. \tag{4.6}$$

Proof. Assume that (4.4) holds. Identity (4.3) implies the existence of vectors $\xi_a,\xi_b \in \mathbb{C}^n$ such that $z(a,\lambda) = -\mathcal{J}\alpha^*\xi_a$ and $z(b,\lambda) = -\mathcal{J}\beta^*\xi_b$. It follows that $z(\cdot,\lambda)$ satisfies (4.6) with $\xi := (-\xi_a^* \quad \xi_b^*)^*$. It remains to prove that ξ is unique such a vector. If $z(\cdot,\lambda)$ satisfies (4.6) and also $z(a,\lambda) = R_a\zeta$ and $z(b,\lambda) = R_b\zeta$ for some $\xi,\zeta \in \mathbb{C}^{2n}$, then $R_a(\xi-\zeta) = 0$ and $R_b(\xi-\zeta) = 0$. Hence, $\mathcal{J}\alpha^*(I \quad 0)(\xi-\zeta) = 0$ and $-\mathcal{J}\beta^*(0 \quad I)(\xi-\zeta) = 0$. If we multiply the latter two equalities by $\alpha\mathcal{J}$ and $\beta\mathcal{J}$, respectively, and use $\alpha\alpha^* = I = \beta\beta^*$, then we obtain $(I \quad 0)(\xi-\zeta) = 0$ and $(0 \quad I)(\xi-\zeta) = 0$.

This yields $\xi - \zeta = 0$, which shows that the vector ξ in (4.6) is unique. The opposite direction, that is, that (4.6) implies (4.4), is trivial. □

Following the standard terminology, see, for example, [62, 63], a number $\lambda \in \mathbb{C}$ is an *eigenvalue* of (4.4) if this boundary value problem has a solution $z(\cdot,\lambda) \not\equiv 0$. In this case the function $z(\cdot,\lambda)$ is called the *eigenfunction* corresponding to the eigenvalue λ, and the dimension of the space of all eigenfunctions corresponding to λ (together with the zero function) is called the *geometric multiplicity* of λ.

Given $\alpha \in \Gamma$, we will utilize from now on the fundamental matrix $\Psi(\cdot,\lambda,\alpha)$ of system $(\mathcal{S}_\lambda)$ satisfying the initial condition from (4.4), that is,

$$\Psi^{\Delta}(t,\lambda,\alpha) = \mathcal{S}(t,\lambda)\Psi(t,\lambda,\alpha), \quad t \in [a,\rho(b)]_{\mathbb{T}}, \quad \Psi(a,\lambda,\alpha) = \begin{pmatrix} \alpha^* & -\mathcal{J}\alpha^* \end{pmatrix}. \tag{4.7}$$

Then $\Psi(a,\lambda,\alpha)$ does not depend on λ, and it is symplectic and unitary with the inverse $\Psi^{-1}(a,\lambda,\alpha) = \Psi^*(a,\lambda,\alpha)$. Hence, the properties of fundamental matrices derived earlier in Lemma 3.10, Remark 3.11, and Corollary 3.12 apply for the matrix function $\Psi(\cdot,\lambda,\alpha)$.

The following assumption will be imposed in this section when studying the regular spectral problem.

Hypothesis 4.2. For every $\lambda \in \mathbb{C}$, we have

$$\int_a^b \Psi^{\sigma *}(t,\lambda,\alpha)\widetilde{\mathcal{W}}(t)\Psi^{\sigma}(t,\lambda,\alpha)\Delta t > 0. \tag{4.8}$$

Condition (4.8) can be written in the equivalent form as

$$\int_a^b z^{\sigma *}(t,\lambda)\widetilde{\mathcal{W}}(t)z^{\sigma}(t,\lambda)\Delta t > 0, \tag{4.9}$$

for every nontrivial solution $z(\cdot,\lambda)$ of system $(\mathcal{S}_\lambda)$. Assumptions (4.8) and (4.9) are equivalent by a simple argument using the uniqueness of solutions of system $(\mathcal{S}_\lambda)$. The latter form (4.9) has been widely used in the literature, such as in the continuous time case in [8, Hypothesis 2.2], [30, equation (1.3)], [26, equation (2.3)], in the discrete time case in [9, Condition (2.16)], [14, equation (1.7)], [1, Assumption 2.2], [2, Hypothesis 2.4], and in the time scale Hamiltonian case in [3, Assumption 3] and [5, Condition (3.9)].

Following Remark 3.11, we partition the fundamental matrix $\Psi(\cdot,\lambda,\alpha)$ as

$$\Psi(\cdot,\lambda,\alpha) = \begin{pmatrix} Z(\cdot,\lambda,\alpha) & \widetilde{Z}(\cdot,\lambda,\alpha) \end{pmatrix}, \tag{4.10}$$

where $Z(\cdot,\lambda,\alpha)$ and $\widetilde{Z}(\cdot,\lambda,\alpha)$ are the $2n \times n$ solutions of system $(\mathcal{S}_\lambda)$ satisfying $Z(a,\lambda,\alpha) = \alpha^*$ and $\widetilde{Z}(a,\lambda,\alpha) = -\mathcal{J}\alpha^*$. With the notation

$$\Lambda(\lambda,\alpha,\beta) := \Psi(b,\lambda,\alpha)\Psi^*(a,\lambda,\alpha)R_a - R_b = \begin{pmatrix} -\widetilde{Z}(b,\lambda,\alpha) & \mathcal{J}\beta^* \end{pmatrix}, \tag{4.11}$$

we have the classical characterization of the eigenvalues of (4.4); see, for example, the continuous time in [64, Chapter 4], the discrete time in [14, Theorem 2.3, Lemma 2.4], [2, Lemma 2.9, Theorem 2.11], and the time scale case in [62, Lemma 3], [63, Corollary 1].

Proposition 4.3. *For $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C}$, we have with notation* (4.11) *the following.*

(i) *The number λ is an eigenvalue of* (4.4) *if and only if* $\det \Lambda(\lambda, \alpha, \beta) = 0$.

(ii) *The algebraic multiplicity of the eigenvalue λ, that is, the number* $\operatorname{def} \Lambda(\lambda, \alpha, \beta)$, *is equal to the geometric multiplicity of λ.*

(iii) *Under Hypothesis 4.2, the eigenvalues of* (4.4) *are real, and the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the semi-inner product*

$$\langle z(\cdot, \lambda), z(\cdot, \nu) \rangle_{\mathcal{W},b} := \int_a^b z^{\sigma *}(t, \lambda) \widetilde{\mathcal{W}}(t) z^{\sigma}(t, \nu) \Delta t. \tag{4.12}$$

Proof. The arguments are here standard, and we refer to [44, Section 5], [63, Corollary 1], [3, Theorem 3.6]. □

The next algebraic characterization of the eigenvalues of (4.4) is more appropriate for the development of the Weyl-Titchmarsh theory for (4.4), since it uses the matrix $\beta \widetilde{Z}(b, \lambda, \alpha)$ which has dimension n instead of using the matrix $\Lambda(\lambda, \alpha, \beta)$ which has dimension $2n$. Results of this type can be found in special cases of system $(\mathcal{S}_\lambda)$ in [8, Lemma 2.5], [11, Theorem 4.1], [9, Lemma 2.8], [14, Lemma 3.1], [1, Lemma 2.5], [3, Theorem 3.4], and [2, Lemma 3.1].

Lemma 4.4. *Let $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C}$ be fixed. Then λ is an eigenvalue of* (4.4) *if and only if* $\det \beta \widetilde{Z}(b, \lambda, \alpha) = 0$. *In this case the algebraic and geometric multiplicities of λ are equal to* $\operatorname{def} \beta \widetilde{Z}(b, \lambda, \alpha)$.

Proof. One can follow the same arguments as in the proof of the corresponding discrete symplectic case in [2, Lemma 3.1]. However, having the result of Proposition 4.3, we can proceed directly by the methods of linear algebra. In this proof we abbreviate $\Lambda := \Lambda(\lambda, \alpha, \beta)$ and $\widetilde{Z} := \widetilde{Z}(b, \lambda, \alpha)$. Assume that Λ is singular, that is, $-\widetilde{Z}c + \mathcal{J}\beta^* d = 0$ for some vectors $c, d \in \mathbb{C}^n$, not both zero. Then $\widetilde{Z}c = \mathcal{J}\beta^* d$, which yields that $\beta \widetilde{Z}c = 0$. If $c = 0$, then $\mathcal{J}\beta^* d = 0$, which implies upon the multiplication by $\beta\mathcal{J}$ from the left that $d = 0$. Since not both c and d can be zero, it follows that $c \neq 0$ and the matrix $\beta \widetilde{Z}$ is singular. Conversely, if $\beta \widetilde{Z}c = 0$ for some nonzero vector $c \in \mathbb{C}^n$, then $-\widetilde{Z}c + \mathcal{J}\beta^* d = 0$; that is, Λ is singular, with the vector $d := -\beta\mathcal{J}\widetilde{Z}c$. Indeed, by using identity (4.2) we have $\mathcal{J}\beta^* d = -\mathcal{J}\beta^*\beta\mathcal{J}\widetilde{Z}c = (I - \beta^*\beta)\widetilde{Z}c = \widetilde{Z}c$. From the above we can also see that the number of linearly independent vectors in $\operatorname{Ker} \beta \widetilde{Z}$ is the same as the number of linearly independent vectors in $\operatorname{Ker} \Lambda$. Therefore, by Proposition 4.3(ii), the algebraic and geometric multiplicities of λ as an eigenvalue of (4.4) are equal to $\operatorname{def} \beta \widetilde{Z}$. □

Since the eigenvalues of (4.4) are real, it follows that the matrix $\beta \widetilde{Z}(b, \lambda, \alpha)$ is invertible for every $\lambda \in \mathbb{C}$ except for at most n real numbers. This motivates the definition of the $M(\lambda)$-function for the regular spectral problem.

Definition 4.5 ($M(\lambda)$-function). Let $\alpha, \beta \in \Gamma$. Whenever the matrix $\beta\tilde{Z}(b,\lambda,\alpha)$ is invertible for some value $\lambda \in \mathbb{C}$, we define the *Weyl-Titchmarsh $M(\lambda)$-function* as the $n \times n$ matrix

$$M(\lambda) = M(\lambda, b) = M(\lambda, b, \alpha, \beta) := -\left[\beta\tilde{Z}(b,\lambda,\alpha)\right]^{-1}\beta Z(b,\lambda,\alpha). \tag{4.13}$$

The above definition of the $M(\lambda)$-function is a generalization of the corresponding definitions for the continuous and discrete linear Hamiltonian and symplectic systems in [8, Definition 2.6], [9, Definition 2.9], [14, equation (3.10)], [1, page 2859], [2, Definition 3.2] and time scale linear Hamiltonian systems in [3, equation (4.1)]. The dependence of the $M(\lambda)$-function on b, α, and β will be suppressed in the notation, and $M(\lambda, b)$ or $M(\lambda, b, \alpha, \beta)$ will be used only in few situations when we emphasize the dependence on b (such as at the end of Section 5) or on α and β (as in Lemma 4.14). By [65, Corollary 4.5], see also [44, Remark 2.2], the $M(\cdot)$-function is an entire function in λ. Another important property of the $M(\lambda)$-function is established in the following.

Lemma 4.6. *Let $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$M^*(\lambda) = M\left(\overline{\lambda}\right). \tag{4.14}$$

Proof. We abbreviate $Z(\lambda) := Z(b,\lambda,\alpha)$ and $\tilde{Z}(\lambda) := \tilde{Z}(b,\lambda,\alpha)$. By using the definition of $M(\lambda)$ in (4.13) and identity (3.21), we have

$$\begin{aligned} M^*(\lambda) - M\left(\overline{\lambda}\right) &= \left[\beta\tilde{Z}\left(\overline{\lambda}\right)\right]^{-1}\beta\left[Z\left(\overline{\lambda}\right)\tilde{Z}^*(\lambda) - \tilde{Z}\left(\overline{\lambda}\right)Z^*(\lambda)\right]\beta^*\left[\beta\tilde{Z}(\lambda)\right]^{*-1} \\ &\overset{(3.21)}{=} \left[\beta\tilde{Z}\left(\overline{\lambda}\right)\right]^{-1}\beta\mathcal{J}\beta^*\left[\beta\tilde{Z}(\lambda)\right]^{*-1} = 0, \end{aligned} \tag{4.15}$$

because $\beta \in \Gamma$. Hence, equality (4.14) holds true. □

The following solution plays an important role in particular in the results concerning the square integrable solutions of system $(\mathcal{S}_\lambda)$.

Definition 4.7 (Weyl solution). For any matrix $M \in \mathbb{C}^{n\times n}$, we define the so-called *Weyl solution* of system $(\mathcal{S}_\lambda)$ by

$$\mathcal{X}(\cdot,\lambda,\alpha,M) := \Psi(\cdot,\lambda,\alpha)\begin{pmatrix} I & M^* \end{pmatrix}^* = Z(\cdot,\lambda,\alpha) + \tilde{Z}(\cdot,\lambda,\alpha)M, \tag{4.16}$$

where $Z(\cdot,\lambda,\alpha)$ and $\tilde{Z}(\cdot,\lambda,\alpha)$ are defined in (4.10).

The function $\mathcal{X}(\cdot,\lambda,\alpha,M)$, being a linear combination of two solutions of system $(\mathcal{S}_\lambda)$, is also a solution of this system. Moreover, $\alpha\mathcal{X}(a,\lambda,\alpha,M) = I$, and, if $\beta\tilde{Z}(b,\lambda,\alpha)$ is invertible, then $\beta\tilde{\mathcal{X}}(b,\lambda,\alpha,M) = \beta\tilde{Z}(b,\lambda,\alpha)[M - M(\lambda)]$. Consequently, if we take $M := M(\lambda)$ in Definition 4.7, then $\beta\mathcal{X}(b,\lambda,\alpha,M(\lambda)) = 0$; that is, the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M(\lambda))$ satisfies the right endpoint boundary condition in (4.4).

Following the corresponding notions in [8, equation (2.18)], [9, equation (2.51)], [14, page 471], [1, page 2859], [2, equation (3.13)], [3, equation (4.2)], we define the Hermitian $n \times n$ matrix function $\mathcal{E}(M)$ for system $(\mathcal{S}_\lambda)$.

Definition 4.8. For a fixed $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we define the matrix function

$$\mathcal{E} : \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}, \qquad \mathcal{E}(M) = \mathcal{E}(M, b) := i\delta(\lambda)\mathcal{X}^*(b, \lambda, \alpha, M)\mathcal{J}\mathcal{X}(b, \lambda, \alpha, M), \tag{4.17}$$

where $\delta(\lambda) := \operatorname{sgn} \operatorname{Im}(\lambda)$.

For brevity we suppress the dependence of the function $\mathcal{E}(\cdot)$ on b and λ. In few cases we will need $\mathcal{E}(M)$ depending on b (as in Theorem 5.1 and Definition 6.2) and in such situations we will use the notation $\mathcal{E}(M, b)$. Since $(i\mathcal{J})^* = i\mathcal{J}$, it follows that $\mathcal{E}(M)$ is a Hermitian matrix for any $M \in \mathbb{C}^{n\times n}$. Moreover, from Corollary 3.6, we obtain the identity

$$\mathcal{E}(M) = -2\delta(\lambda) \operatorname{Im}(M) + 2|\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma*}(t, \lambda, \alpha, M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t, \lambda, \alpha, M)\Delta t, \tag{4.18}$$

where we used the fact that

$$\mathcal{X}^*(a, \lambda, \alpha, M)\mathcal{J}\mathcal{X}(a, \lambda, \alpha, M) \overset{(4.7)}{=} M - M^* = 2i \operatorname{Im}(M). \tag{4.19}$$

Next we define the Weyl disk and Weyl circle for the regular spectral problem. The geometric characterizations of the Weyl disk and Weyl circle in terms of the contractive or unitary matrices which justify the terminology "disk" or "circle" will be presented in Section 5.

Definition 4.9 (Weyl disk and Weyl circle). For a fixed $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the set

$$D(\lambda) = D(\lambda, b) := \{M \in \mathbb{C}^{n\times n},\ \mathcal{E}(M) \le 0\}, \tag{4.20}$$

is called the *Weyl disk*, and the set

$$C(\lambda) = C(\lambda, b) := \partial D(\lambda) = \{M \in \mathbb{C}^{n\times n},\ \mathcal{E}(M) = 0\}, \tag{4.21}$$

is called the *Weyl circle*.

The dependence of the Weyl disk and Weyl circle on b will be again suppressed. In the following result we show that the Weyl circle consists of precisely those matrices $M(\lambda)$ with $\beta \in \Gamma$. This result generalizes the corresponding statements in [8, Lemma 2.8], [9, Lemma 2.13], [14, Lemma 3.3], [1, Theorem 3.1], [2, Theorem 3.6], and [3, Theorem 4.2].

Theorem 4.10. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M belongs to the Weyl circle $C(\lambda)$ if and only if there exists $\beta \in \Gamma$ such that $\beta\mathcal{X}(b, \lambda, \alpha, M) = 0$. In this case and under Hypothesis 4.2, we have with such a matrix β that $M = M(\lambda)$ as defined in* (4.13).

Proof. Assume that $M \in C(\lambda)$, that is, $\mathcal{E}(M) = 0$. Then, with the vector

$$\beta := \mathcal{K}^*(b)\mathcal{J} = \begin{pmatrix} I & M^* \end{pmatrix} \Psi^*(b,\lambda,\alpha)\mathcal{J} \in \mathbb{C}^{n\times 2n}, \tag{4.22}$$

where $\mathcal{K}(b)$ denotes $\mathcal{K}(b,\lambda,\alpha,M)$, we have

$$\beta\mathcal{K}(b) = \mathcal{K}^*(b)\mathcal{J}\mathcal{K}(b) = \left[\frac{1}{(i\delta(\lambda))}\right]\mathcal{E}(M) = 0. \tag{4.23}$$

Moreover, $\operatorname{rank}\beta = n$, because the matrices $\Psi(b,\lambda,\alpha)$ and $\mathcal{J}$ are invertible and $\operatorname{rank}(I \quad M^*) = n$. In addition, the identity $\mathcal{J}^* = \mathcal{J}^{-1}$ yields

$$\beta\mathcal{J}\beta^* = \mathcal{K}^*(b)\mathcal{J}\mathcal{K}(b) \overset{(4.23)}{=} 0. \tag{4.24}$$

Now, if the condition $\beta\beta^* = I$ is not satisfied, then we replace β by $\widetilde{\beta} := (\beta\beta^*)^{-1/2}\beta$ (note that $\beta\beta^* > 0$, so that $(\beta\beta^*)^{-1/2}$ is well defined), and in this case

$$\begin{aligned} \widetilde{\beta}\mathcal{K}(b) &= (\beta\beta^*)^{-1/2}\beta\mathcal{K}(b) \overset{(4.23)}{=} 0, \\ \widetilde{\beta}\mathcal{J}\widetilde{\beta}^* &= (\beta\beta^*)^{-1/2}\beta\mathcal{J}\beta^*(\beta\beta^*)^{-1/2} \overset{(4.24)}{=} 0, \\ \widetilde{\beta}\widetilde{\beta}^* &= (\beta\beta^*)^{-1/2}\beta\beta^*(\beta\beta^*)^{-1/2} = I. \end{aligned} \tag{4.25}$$

Conversely, suppose that for a given $M \in \mathbb{C}^{n\times n}$ there exists $\beta \in \Gamma$ such that $\beta\mathcal{K}(b) = 0$. Then from (4.3) it follows that $\mathcal{K}(b) = \mathcal{J}\beta^* P$ for the matrix $P := -\beta\mathcal{J}\mathcal{K}(b) \in \mathbb{C}^{n\times n}$. Hence,

$$\mathcal{E}(M) = i\delta(\lambda)P^*\beta\mathcal{J}^*\mathcal{J}\mathcal{J}\beta^* P = i\delta(\lambda)P^*\beta\mathcal{J}\beta^* P = 0, \tag{4.26}$$

that is, $M \in C(\lambda)$. Finally, since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then by Proposition 4.3(iii) the number λ is not an eigenvalue of (4.4), which by Lemma 4.4 shows that the matrix $\beta\widetilde{Z}(b,\lambda,\alpha)$ is invertible. The definition of the Weyl solution in (4.16) then yields

$$\beta Z(b,\lambda,\alpha) + \beta\widetilde{Z}(b,\lambda,\alpha)M = \beta\mathcal{K}(b,\lambda,\alpha,M) = 0, \tag{4.27}$$

which implies that $M = -[\beta\widetilde{Z}(b,\lambda,\alpha)]^{-1}\beta Z(b,\lambda,\alpha) = M(\lambda)$. □

Remark 4.11. The matrix $P := -\beta\mathcal{J}\mathcal{K}(b,\lambda,\alpha,M) \in \mathbb{C}^{n\times n}$ from the proof of Theorem 4.10 is invertible. This fact was not needed in that proof. However, we show that P is invertible because this argument will be used in the proof of Lemma 4.14. First we prove that $\operatorname{Ker} P = \operatorname{Ker}\mathcal{K}(b,\lambda,\alpha,M)$. For if $Pd = 0$ for some $d \in \mathbb{C}^n$, then from identity (4.2) we get $\mathcal{K}(b,\lambda,\alpha,M)d = (I - \beta^*\beta)\mathcal{K}(b,\lambda,\alpha,M)d = \mathcal{J}\beta^* Pd = 0$. Therefore, $\operatorname{Ker} P \subseteq \operatorname{Ker}\mathcal{K}(b,\lambda,\alpha,M)$. The opposite inclusion follows by the definition of P. And since, by (4.16), $\operatorname{rank}\mathcal{K}(b,\lambda,\alpha,M) = \operatorname{rank}(I \quad M^*)^* = n$, it follows that $\operatorname{Ker}\mathcal{K}(b,\lambda,\alpha,M) = \{0\}$. Hence, $\operatorname{Ker} P = \{0\}$ as well; that is, the matrix P is invertible.

The next result contains a characterization of the matrices $M \in \mathbb{C}^{n\times n}$ which lie "inside" the Weyl disk $D(\lambda)$. In the previous result (Theorem 4.10) we have characterized the elements of the boundary of the Weyl disk $D(\lambda)$, that is, the elements of the Weyl circle $C(\lambda)$, in terms of the matrices $\beta \in \Gamma$. For such β we have $\beta\mathcal{J}\beta^* = 0$, which yields $i\delta(\lambda)\beta\mathcal{J}\beta^* = 0$. Comparing with that statement we now utilize the matrices $\beta \in \mathbb{C}^{n\times 2n}$ which satisfy $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$. In the special cases of the continuous and discrete time, this result can be found in [8, Lemma 2.13], [9, Lemma 2.18], and [2, Theorem 3.13].

Theorem 4.12. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M satisfies $\mathcal{E}(M) < 0$ if and only if there exists $\beta \in \mathbb{C}^{n\times 2n}$ such that $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ and $\beta\mathcal{X}(b,\lambda,\alpha,M) = 0$. In this case and under Hypothesis 4.2, we have with such a matrix β that $M = M(\lambda)$ as defined in* (4.13) *and β may be chosen so that $\beta\beta^* = I$.*

Proof. For $M \in \mathbb{C}^{n\times n}$ consider on $[a,b]_{\mathbb{T}}$ the Weyl solution

$$\mathcal{X}(\cdot) := \mathcal{X}(\cdot,\lambda,\alpha,M) = \begin{pmatrix} \mathcal{X}_1(\cdot) \\ \mathcal{X}_2(\cdot) \end{pmatrix}, \quad \text{with } n\times n \text{ blocks } \mathcal{X}_1(\cdot) \text{ and } \mathcal{X}_2(\cdot). \tag{4.28}$$

Suppose first that $\mathcal{E}(M) < 0$. Then the matrices $\mathcal{X}_j(b)$, $j \in \{1,2\}$, are invertible. Indeed, if one of them is singular, then there exists a nonzero vector $v \in \mathbb{C}^n$ such that $\mathcal{X}_1(b)v = 0$ or $\mathcal{X}_2(b)v = 0$. Then

$$v^*\mathcal{E}(M)v = i\delta(\lambda)v^*\mathcal{X}^*(b)\mathcal{J}\mathcal{X}(b)v = i\delta(\lambda)v^*\left[\mathcal{X}_1^*(b)\mathcal{X}_2(b) - \mathcal{X}_2^*(b)\mathcal{X}_1(b)\right]v = 0, \tag{4.29}$$

which contradicts $\mathcal{E}(M) < 0$. Now we set $\beta_1 := I$, $\beta_2 := -\mathcal{X}_1(b)\mathcal{X}_2^{-1}(b)$, and $\beta := (\beta_1 \;\; \beta_2)$. Then for this $2n\times n$ matrix β we have $\beta\mathcal{X}(b) = 0$ and, by a similar calculation as in (4.29),

$$\begin{aligned} \mathcal{E}(M) &= i\delta(\lambda)\mathcal{X}^*(b)\mathcal{J}\mathcal{X}(b) = i\delta(\lambda)\mathcal{X}_2^*(b)\left(\beta_2\beta_1^* - \beta_1\beta_2^*\right)\mathcal{X}_2(b) \\ &= 2\delta(\lambda)\mathcal{X}_2^*(b)\operatorname{Im}\left(\beta_1\beta_2^*\right)\mathcal{X}_2(b) = -i\delta(\lambda)\mathcal{X}_2^*(b)\beta\mathcal{J}\beta^*\mathcal{X}_2(b), \end{aligned} \tag{4.30}$$

where we used the equality $\beta\mathcal{J}\beta^* = 2i\operatorname{Im}(\beta_1\beta_2^*)$. Since $\mathcal{E}(M) < 0$ and $\mathcal{X}_2(b)$ is invertible, it follows that $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$. Conversely, assume that for a given matrix $M \in \mathbb{C}^{n\times n}$ there is $\beta = (\beta_1 \;\; \beta_2) \in \mathbb{C}^{n\times 2n}$ satisfying $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ and $\beta\mathcal{X}(b) = 0$. Condition $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ is equivalent to $\operatorname{Im}(\beta_1\beta_2^*) < 0$ when $\operatorname{Im}(\lambda) > 0$ and to $\operatorname{Im}(\beta_1\beta_2^*) > 0$ when $\operatorname{Im}(\lambda) < 0$. The positive or negative definiteness of $\operatorname{Im}(\beta_1\beta_2^*)$ implies the invertibility of β_1 and β_2; see Remark 2.2. Therefore, from the equality $\beta_1\mathcal{X}_1(b) + \beta_2\mathcal{X}_2(b) = \beta\mathcal{X}(b) = 0$, we obtain $\mathcal{X}_1(b) = -\beta_1^{-1}\beta_2\mathcal{X}_2(b)$, and so

$$\begin{aligned} \mathcal{E}(M) &= i\delta(\lambda)\left[\mathcal{X}_1^*(b)\mathcal{X}_2(b) - \mathcal{X}_2^*(b)\mathcal{X}_1(b)\right] \\ &= i\delta(\lambda)\mathcal{X}_2^*(b)\beta_1^{-1}\left(\beta_2\beta_1^* - \beta_1\beta_2^*\right)\beta_1^{*-1}\mathcal{X}_2(b) \\ &= -i\delta(\lambda)\mathcal{X}_2^*(b)\beta_1^{-1}\beta\mathcal{J}\beta^*\beta_1^{*-1}\mathcal{X}_2(b). \end{aligned} \tag{4.31}$$

The matrix $\mathcal{X}_2(b)$ is invertible, because if $\mathcal{X}_2(b)d = 0$ for some nonzero vector $d \in \mathbb{C}^n$, then $\mathcal{X}_1(b)d = -\beta_1^{-1}\beta_2\mathcal{X}_2(b)d = 0$, showing that $\operatorname{rank}\mathcal{X}(b) < n$. This however contradicts $\operatorname{rank}\mathcal{X}(b) = n$ which we have from the definition of the Weyl solution $\mathcal{X}(\cdot)$ in (4.16). Consequently, (4.31) yields through $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ that $\mathcal{E}(M) < 0$.

If the matrix β does not satisfy $\beta\beta^* = I$, then we modify it according to the procedure described in the proof of Theorem 4.10. Finally, since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we get from Proposition 4.3(iii) and Lemma 4.4 that the matrix $\beta\widetilde{Z}(b,\lambda,\alpha)$ is invertible which in turn implies through the calculation in (4.27) that $M = -[\beta\widetilde{Z}(b,\lambda,\alpha)]^{-1}\beta Z(b,\lambda,\alpha) = M(\lambda)$. □

In the following lemma we derive some additional properties of the Weyl disk and the $M(\lambda)$-function. Special cases of this statement can be found in [8, Lemma 2.9], [33, Theorem 3.1], [9, Lemma 2.14], [14, Lemma 3.2(ii)], [1, Theorem 3.7], [2, Lemma 3.7], and [3, Theorem 4.13].

Theorem 4.13. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. For any matrix $M \in D(\lambda)$ we have*

$$\delta(\lambda)\operatorname{Im}(M) \geq |\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t \geq 0. \tag{4.32}$$

In addition, under Hypothesis 4.2, we have $\delta(\lambda)\operatorname{Im}(M) > 0$.

Proof. By identity (4.18), for any matrix $M \in D(\lambda)$, we have

$$\begin{aligned} 2\delta(\lambda)\operatorname{Im}(M) &= -\mathcal{E}(M) + 2|\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t \\ &\geq 2|\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t, \end{aligned} \tag{4.33}$$

which yields together with $\widetilde{\mathcal{W}}(t) \geq 0$ on $[a,\rho(b)]_{\mathbb{T}}$ the inequalities in (4.32). The last assertion in Theorem 4.13 is a simple consequence of Hypothesis 4.2. □

In the last part of this section we wish to study the effect of changing α, which is one of the parameters of the $M(\lambda)$-function and the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$, when α varies within the set Γ. For this purpose we will use the $M(\lambda)$-function with all its arguments in the following two statements.

Lemma 4.14. *Let $\alpha,\beta,\gamma \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$M(\lambda,b,\alpha,\beta) = \left[\alpha\mathcal{J}\gamma^* + \alpha\gamma^* M(\lambda,b,\gamma,\beta)\right]\left[\alpha\gamma^* - \alpha\mathcal{J}\gamma^* M(\lambda,b,\gamma,\beta)\right]^{-1}. \tag{4.34}$$

Proof. Let $M(b,\lambda,\alpha,\beta)$ and $M(b,\lambda,\gamma,\beta)$ be given via (4.13), and consider the Weyl solutions $\mathcal{X}_\alpha(\cdot) := \mathcal{X}(\cdot,\lambda,\alpha,M(b,\lambda,\alpha,\beta))$ and $\mathcal{X}_\gamma(\cdot) := \mathcal{X}(\cdot,\lambda,\gamma,M(b,\lambda,\gamma,\beta))$ defined by (4.16) with $M = M(b,\lambda,\alpha,\beta)$ and $M = M(b,\lambda,\gamma,\beta)$, respectively. First we prove that the two Weyl solutions $\mathcal{X}_\alpha(\cdot)$ and $\mathcal{X}_\gamma(\cdot)$ differ by a constant nonsingular multiple. By definition, $\beta\mathcal{X}_\alpha(b) = 0$ and $\beta\mathcal{X}_\gamma(b) = 0$, which implies through (4.3) that $\mathcal{X}_\alpha(b) = \mathcal{J}\beta^* P_\alpha$ and $\mathcal{X}_\gamma(b) = \mathcal{J}\beta^* P_\gamma$

for some matrices $P_\alpha, P_\gamma \in \mathbb{C}^{n\times n}$, which are invertible by Remark 4.11. This implies that $\mathcal{X}_\alpha(b)P_\alpha^{-1} = \mathcal{J}\beta^* = \mathcal{X}_\gamma(b)P_\gamma^{-1}$. Consequently, $\mathcal{X}_\alpha(b) = \mathcal{X}_\gamma(b)P$, where $P := P_\gamma^{-1}P_\alpha$. By the uniqueness of solutions of system $(\mathcal{S}_\lambda)$, see Theorem 3.4, we obtain that $\mathcal{X}_\alpha(\cdot) = \mathcal{X}_\gamma(\cdot)P$ on $[a,b]_{\mathbb{T}}$. Upon the evaluation at $t = a$ we get

$$\Psi(a,\lambda,\alpha)\begin{pmatrix} I \\ M(\lambda,b,\alpha,\beta)\end{pmatrix} = \Psi(a,\lambda,\gamma)\begin{pmatrix} I \\ M(\lambda,b,\gamma,\beta)\end{pmatrix}P. \tag{4.35}$$

Since the matrices $\Psi(a,\lambda,\alpha) = (\alpha^* \quad -\mathcal{J}\alpha^*)$ and $\Psi(a,\lambda,\gamma) = (\gamma^* \quad -\mathcal{J}\gamma^*)$ are unitary, it follows from (4.35) that

$$\begin{aligned}\begin{pmatrix} I \\ M(\lambda,b,\alpha,\beta)\end{pmatrix} &= \begin{pmatrix} \alpha \\ \alpha\mathcal{J}\end{pmatrix}(\gamma^* \quad -\mathcal{J}\gamma^*)\begin{pmatrix} I \\ M(\lambda,b,\gamma,\beta)\end{pmatrix}P \\ &= \begin{pmatrix} \alpha\gamma^* - \alpha\mathcal{J}\gamma^*M(\lambda,b,\gamma,\beta) \\ \alpha\mathcal{J}\gamma^* + \alpha\gamma^*M(\lambda,b,\gamma,\beta)\end{pmatrix}P.\end{aligned} \tag{4.36}$$

The first row above yields that $P = [\alpha\gamma^* - \alpha\mathcal{J}\gamma^*M(\lambda,b,\gamma,\beta)]^{-1}$, while the second row is then written as identity (4.34). □

Corollary 4.15. *Let $\alpha,\beta,\gamma \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. With notation* (4.16) *and* (4.13) *we have*

$$\mathcal{X}(\cdot,\lambda,\alpha,M(\lambda,b,\alpha,\beta)) = \mathcal{X}(\cdot,\lambda,\gamma,M(\lambda,b,\gamma,\beta))\left[\alpha\gamma^* - \alpha\mathcal{J}\gamma^*M(\lambda,b,\gamma,\beta)\right]^{-1}. \tag{4.37}$$

Proof. The above identity follows from (4.35) and the formula for the matrix P from the end of the proof of Lemma 4.14. □

5. Geometric Properties of Weyl Disks

In this section we study the geometric properties of the Weyl disks as the point b moves through the interval $[a,\infty)_{\mathbb{T}}$. Our first result shows that the Weyl disks $D(\lambda,b)$ are nested. This statement generalizes the results in [11, Theorem 4.5], [66, Section 3.2.1], [9, equation (2.70)], [14, Theorem 3.1], [3, Theorem 4.4], and [5, Theorem 3.3(i)].

Theorem 5.1 (nesting property of Weyl disks). *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then*

$$D(\lambda,b_2) \subseteq D(\lambda,b_1), \quad \text{for every } b_1,b_2 \in [a,\infty)_{\mathbb{T}},\ b_1 < b_2. \tag{5.1}$$

Proof. Let $b_1, b_2 \in [a,\infty)_{\mathbb{T}}$ with $b_1 < b_2$, and take $M \in D(\lambda, b_2)$, that is, $\mathcal{E}(M, b_2) \leq 0$. From identity (4.18) with $b = b_1$ and later with $b = b_2$ and by using $\widetilde{\mathcal{W}}(\cdot) \geq 0$, we have

$$\begin{aligned}
\mathcal{E}(M, b_1) &\overset{(4.18)}{=} -2\delta(\lambda)\operatorname{Im}(M) + 2|\operatorname{Im}(\lambda)| \int_a^{b_1} \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t \\
&\leq -2\delta(\lambda)\operatorname{Im}(M) + 2|\operatorname{Im}(\lambda)| \int_a^{b_2} \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t \\
&\overset{(4.18)}{=} \mathcal{E}(M, b_2) \leq 0.
\end{aligned} \tag{5.2}$$

Therefore, by Definition 4.9, the matrix M belongs to $D(\lambda, b_1)$, which shows the result. □

Similarly for the regular case (Hypothesis 4.2) we now introduce the following assumption.

Hypothesis 5.2. There exists $b_0 \in (a,\infty)_{\mathbb{T}}$ such that Hypothesis 4.2 is satisfied with $b = b_0$; that is, inequality (4.8) holds with $b = b_0$ for every $\lambda \in \mathbb{C}$.

From Hypothesis 5.2 it follows by $\widetilde{\mathcal{W}}(\cdot) \geq 0$ that inequality (4.8) holds for every $b \in [b_0,\infty)_{\mathbb{T}}$.

For the study of the geometric properties of Weyl disks we will use the following representation:

$$\mathcal{E}(M,b) = i\delta(\lambda)\mathcal{X}^*(b,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(b,\lambda,\alpha,M) = \begin{pmatrix} I & M^* \end{pmatrix} \begin{pmatrix} \mathcal{F}(b,\lambda,\alpha) & \mathcal{G}^*(b,\lambda,\alpha) \\ \mathcal{G}(b,\lambda,\alpha) & \mathcal{H}(b,\lambda,\alpha) \end{pmatrix} \begin{pmatrix} I \\ M \end{pmatrix}, \tag{5.3}$$

of the matrix $\mathcal{E}(M,b)$, where we define on $[a,\infty)_{\mathbb{T}}$ the $n \times n$ matrices

$$\begin{aligned}
\mathcal{F}(\cdot,\lambda,\alpha) &:= i\delta(\lambda)Z^*(\cdot,\lambda,\alpha)\mathcal{J}Z(\cdot,\lambda,\alpha), \\
\mathcal{G}(\cdot,\lambda,\alpha) &:= i\delta(\lambda)\widetilde{Z}^*(\cdot,\lambda,\alpha)\mathcal{J}Z(\cdot,\lambda,\alpha), \\
\mathcal{H}(\cdot,\lambda,\alpha) &:= i\delta(\lambda)\widetilde{Z}^*(\cdot,\lambda,\alpha)\mathcal{J}\widetilde{Z}(\cdot,\lambda,\alpha).
\end{aligned} \tag{5.4}$$

Since $\mathcal{E}(M,b)$ is Hermitian, it follows that $\mathcal{F}(\cdot,\lambda,\alpha)$ and $\mathcal{H}(\cdot,\lambda,\alpha)$ are also Hermitian. Moreover, by (4.7), we have $\mathcal{H}(a,\lambda,\alpha) = 0$. In addition, if $b \in [b_0,\infty)_{\mathbb{T}}$, then Corollary 3.7 and Hypothesis 5.2 yield for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\mathcal{H}(b,\lambda,\alpha) = 2|\operatorname{Im}(\lambda)| \int_a^b \widetilde{Z}^{\sigma *}(t,\lambda,\alpha)\widetilde{\mathcal{W}}(t)\widetilde{Z}^{\sigma}(t,\lambda,\alpha)\Delta t > 0. \tag{5.5}$$

Therefore, $\mathcal{H}(b,\lambda,\alpha)$ is invertible (positive definite) for all $b \in [b_0,\infty)_{\mathbb{T}}$ and monotone nondecreasing as $b \to \infty$, with a consequence that $\mathcal{H}^{-1}(b,\lambda,\alpha)$ is monotone nonincreasing as $b \to \infty$. The following factorization of $\mathcal{E}(M,b)$ holds true; see also [2, equation (4.11)].

Lemma 5.3. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. With the notation* (5.4), *for any $M \in \mathbb{C}^{n\times n}$ and $b \in [a,\infty)_{\mathbb{T}}$ we have*

$$\begin{aligned}\mathcal{E}(M,b) &= \mathcal{F}(b,\lambda,\alpha) - \mathcal{G}^*(b,\lambda,\alpha)\mathcal{H}^{-1}(b,\lambda,\alpha)\mathcal{G}(b,\lambda,\alpha)\\ &\quad + \left[\mathcal{G}^*(b,\lambda,\alpha)\mathcal{H}^{-1}(b,\lambda,\alpha) + M^*\right]\mathcal{H}(b,\lambda,\alpha)\left[\mathcal{H}^{-1}(b,\lambda,\alpha)\mathcal{G}(b,\lambda,\alpha) + M\right],\end{aligned} \tag{5.6}$$

whenever the matrix $\mathcal{H}(b,\lambda,\alpha)$ is invertible.

Proof. The result is shown by a direct calculation. □

The following identity is a generalization of its corresponding versions in [11, Lemma 4.3], [1, Lemma 3.3], [14, Proposition 3.2], [2, Lemma 4.2], [3, Lemma 4.6], and [5, Theorem 5.6].

Lemma 5.4. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. With the notation* (5.4), *for any $b \in [a,\infty)_{\mathbb{T}}$, we have*

$$\mathcal{G}^*(b,\lambda,\alpha)\mathcal{H}^{-1}(b,\lambda,\alpha)\mathcal{G}(b,\lambda,\alpha) - \mathcal{F}(b,\lambda,\alpha) = \mathcal{H}^{-1}\left(b,\overline{\lambda},\alpha\right), \tag{5.7}$$

whenever the matrices $\mathcal{H}(b,\lambda,\alpha)$ and $\mathcal{H}(b,\overline{\lambda},\alpha)$ are invertible.

Proof. In order to simplify and abbreviate the notation we introduce the matrices

$$\begin{aligned}&\mathcal{F} := \mathcal{F}(b,\lambda,\alpha), \qquad \mathcal{G} := \mathcal{G}(b,\lambda,\alpha), \qquad \mathcal{H} := \mathcal{H}(b,\lambda,\alpha),\\ &\tilde{\mathcal{F}} := \mathcal{F}\left(b,\overline{\lambda},\alpha\right), \qquad \tilde{\mathcal{G}} := \mathcal{G}\left(b,\overline{\lambda},\alpha\right), \qquad \widetilde{\mathcal{H}} := \mathcal{H}\left(b,\overline{\lambda},\alpha\right),\end{aligned} \tag{5.8}$$

and use the notation $Z(\lambda)$ and $\tilde{Z}(\lambda)$ for $Z(b,\lambda,\alpha)$ and $\tilde{Z}(b,\lambda,\alpha)$, respectively. Then, since $\mathcal{F}^* = \mathcal{F}$ and $\delta(\lambda)\delta(\overline{\lambda}) = -1$, we get the identities

$$\mathcal{G}^*\tilde{\mathcal{F}} - \mathcal{F}^*\tilde{\mathcal{G}} = Z^*(\lambda)\mathcal{J}\left[\tilde{Z}(\lambda)Z^*\left(\overline{\lambda}\right) - Z(\lambda)\tilde{Z}^*\left(\overline{\lambda}\right)\right]\mathcal{J}Z\left(\overline{\lambda}\right) \overset{(3.21)}{=} Z^*(\lambda)\mathcal{J}Z\left(\overline{\lambda}\right) \overset{(3.20)}{=} 0, \tag{5.9}$$

$$\mathcal{H}\tilde{\mathcal{G}}^* - \mathcal{G}\mathcal{H}^* = \tilde{Z}^*(\lambda)\mathcal{J}\left[\tilde{Z}(\lambda)Z^*\left(\overline{\lambda}\right) - Z(\lambda)\tilde{Z}^*\left(\overline{\lambda}\right)\right]\mathcal{J}\tilde{Z}\left(\overline{\lambda}\right) \overset{(3.21)}{=} \tilde{Z}^*(\lambda)\mathcal{J}\tilde{Z}\left(\overline{\lambda}\right) \overset{(3.20)}{=} 0, \tag{5.10}$$

$$\mathcal{G}\tilde{\mathcal{G}} - \mathcal{H}\tilde{\mathcal{F}} = \tilde{Z}^*(\lambda)\mathcal{J}\left[Z(\lambda)\tilde{Z}^*\left(\overline{\lambda}\right) - \tilde{Z}(\lambda)Z^*\left(\overline{\lambda}\right)\right]\mathcal{J}Z\left(\overline{\lambda}\right) \overset{(3.21)}{=} -\tilde{Z}^*(\lambda)\mathcal{J}Z\left(\overline{\lambda}\right) \overset{(3.20)}{=} I, \tag{5.11}$$

$$\mathcal{G}^*\tilde{\mathcal{G}}^* - \mathcal{F}\widetilde{\mathcal{H}} = Z^*(\lambda)\mathcal{J}\left[\tilde{Z}(\lambda)Z^*\left(\overline{\lambda}\right) - Z(\lambda)\tilde{Z}^*\left(\overline{\lambda}\right)\right]\mathcal{J}\tilde{Z}\left(\overline{\lambda}\right) \overset{(3.21)}{=} Z^*(\lambda)\mathcal{J}\tilde{Z}\left(\overline{\lambda}\right) \overset{(3.20)}{=} I. \tag{5.12}$$

Hence, by using that $\widetilde{\mathcal{H}}$ is Hermitian, we see that

$$\widetilde{\mathcal{H}}^{-1} \overset{(5.12)}{=} \mathcal{G}^*\tilde{\mathcal{G}}^*\widetilde{\mathcal{H}}^{-1} - \mathcal{F} = \mathcal{G}^*\tilde{\mathcal{G}}^*\widetilde{\mathcal{H}}^{*-1} - \mathcal{F} \overset{(5.10)}{=} \mathcal{G}^*\mathcal{H}^{-1}\mathcal{G} - \mathcal{F}. \tag{5.13}$$

Identity (5.7) is now proven. □

Corollary 5.5. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under Hypothesis 5.2, the matrix $\mathscr{H}(b,\lambda,\alpha)$ is invertible for every $b \in [b_0,\infty)_{\mathbb{T}}$, and for these values of b we have*

$$\mathcal{G}^*(b,\lambda,\alpha)\mathscr{H}^{-1}(b,\lambda,\alpha)\mathcal{G}(b,\lambda,\alpha) - \mathcal{F}(b,\lambda,\alpha) > 0. \tag{5.14}$$

Proof. Since $b \in [b_0,\infty)_{\mathbb{T}}$, then identity (5.5) yields that $\mathscr{H}(b,\lambda,\alpha) > 0$ and $\mathscr{H}(b,\overline{\lambda},\alpha) > 0$. Consequently, inequality (5.14) follows from (5.7) of Lemma 5.4. □

In the next result we justify the terminology for the sets $D(\lambda,b)$ and $C(\lambda,b)$ in Definition 4.9 to be called a "disk" and a "circle." It is a generalization of [14, Theorem 3.1], [2, Theorem 5.4], [5, Theorem 3.3(iii)]; see also [66, Theorem 3.5], [26, pages 70-71], [8, page 3485], [14, Proposition 3.3], [1, Theorem 3.3], [3, Theorem 4.8]. Consider the sets $\mathcal{V}$ and $\mathcal{U}$ of contractive and unitary matrices in $\mathbb{C}^{n\times n}$, respectively, that is,

$$\mathcal{V} := \left\{V \in \mathbb{C}^{n\times n},\ V^*V \le I\right\}, \qquad \mathcal{U} := \partial\mathcal{V} = \left\{U \in \mathbb{C}^{n\times n},\ U^*U = I\right\}. \tag{5.15}$$

The set $\mathcal{V}$ is known to be closed (in fact compact, since $\mathcal{V}$ is bounded) and convex.

Theorem 5.6. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Under Hypothesis 5.2, for every $b \in [b_0,\infty)_{\mathbb{T}}$, the Weyl disk and Weyl circle have the representations*

$$D(\lambda,b) = \left\{P(\lambda,b) + R(\lambda,b)VR\left(\overline{\lambda},b\right),\ V \in \mathcal{V}\right\}, \tag{5.16}$$

$$C(\lambda,b) = \left\{P(\lambda,b) + R(\lambda,b)UR\left(\overline{\lambda},b\right),\ U \in \mathcal{U}\right\}, \tag{5.17}$$

where, with the notation (5.4),

$$P(\lambda,b) := -\mathscr{H}^{-1}(\lambda,b,\alpha)\mathcal{G}(\lambda,b,\alpha), \qquad R(\lambda,b) := \mathscr{H}^{-1/2}(\lambda,b,\alpha). \tag{5.18}$$

Consequently, for every $b \in [b_0,\infty)_{\mathbb{T}}$, the sets $D(\lambda,b)$ are closed and convex.

The representations of $D(\lambda,b)$ and $C(\lambda,b)$ in (5.16) and (5.17) can be written as $D(\lambda,b) = P(\lambda,b) + R(\lambda,b)\mathcal{V}R(\overline{\lambda},b)$ and $C(\lambda,b) = P(\lambda,b) + R(\lambda,b)\mathcal{U}R(\overline{\lambda},b)$. The importance of the matrices $P(\lambda,b)$ and $R(\lambda,b)$ is justified in the following.

Definition 5.7. For $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and $b \in [a,\infty)_{\mathbb{T}}$ such that $\mathscr{H}(\lambda,b,\alpha)$ and $\mathscr{H}(\overline{\lambda},b,\alpha)$ are positive definite, the matrix $P(\lambda,b)$ is called the center of the Weyl disk or the Weyl circle. The matrices $R(\lambda,b)$ and $R(\overline{\lambda},b)$ are called the *matrix radii* of the Weyl disk or the Weyl circle.

Proof of Theorem 5.6. By (5.5) and for any $b \in [b_0,\infty)_{\mathbb{T}}$, the matrices $\mathscr{H} := \mathscr{H}(\lambda,b,\alpha)$ and $\widetilde{\mathscr{H}} := \mathscr{H}(\overline{\lambda},b,\alpha)$ are positive definite, so that the matrices $P := P(\lambda,b)$, $R(\lambda) := R(\lambda,b)$, and

$R(\overline{\lambda}) := R(\overline{\lambda}, b)$ are well defined. By Definition 4.9, for $M \in D(\lambda, b)$, we have $\mathcal{E}(M, b) \leq 0$, which in turn with notation (5.8) implies by Lemmas 5.3 and 5.4 that

$$\begin{aligned} &-R^2\left(\overline{\lambda}\right) + (M^* - P^*)R^{-2}(\lambda)(M - P) \\ &\quad \overset{(5.7)}{=} \mathcal{F} - \mathcal{G}^*\mathcal{H}^{-1}\mathcal{G} + \left(\mathcal{H}^{-1}\mathcal{G} + M\right)^*\mathcal{H}\left(\mathcal{H}^{-1}\mathcal{G} + M\right) = \mathcal{E}(M, b) \leq 0. \end{aligned} \tag{5.19}$$

Therefore, the matrix

$$V := R^{-1}(\lambda)(M - P)R^{-1}\left(\overline{\lambda}\right), \tag{5.20}$$

satisfies $V^*V \leq I$. This relation between the matrices $M \in D(\lambda, b)$ and $V \in \mathcal{U}$ is bijective (more precisely, it is a homeomorphism), and the inverse to (5.20) is given by $M = P + R(\lambda)VR(\overline{\lambda})$. The latter formula proves that the Weyl disk $D(\lambda, b)$ has the representation in (5.16). Moreover, since by the definition $M \in C(\lambda, b)$ means that $\mathcal{E}(M, b) = 0$, it follows that the elements of the Weyl circle $C(\lambda, b)$ are in one-to-one correspondence with the matrices V defined in (5.20) which, similarly as in (5.19), now satisfy $V^*V = I$. Hence, the representation of $C(\lambda, b)$ in (5.17) follows. The fact that for $b \in [b_0, \infty)_{\mathbb{T}}$ the sets $D(\lambda, b)$ are closed and convex follows from the same properties of the set $\mathcal{U}$, being homeomorphic to $D(\lambda, b)$. □

6. Limiting Weyl Disk and Weyl Circle

In this section we study the limiting properties of the Weyl disk and Weyl circle and their center and matrix radii. Since under Hypothesis 5.2 the matrix function $\mathcal{H}(\cdot, \lambda, \alpha)$ is monotone nondecreasing as $b \to \infty$, it follows from the definition of $R(\lambda, b)$ and $R(\overline{\lambda}, b)$ in (5.18) that the two matrix functions $R(\lambda, \cdot)$ and $R(\overline{\lambda}, \cdot)$ are monotone nonincreasing for $b \to \infty$. Furthermore, since $R(\lambda, b)$ and $R(\overline{\lambda}, b)$ are Hermitian and positive definite for $b \in [b_0, \infty)_{\mathbb{T}}$, the limits

$$R_+(\lambda) := \lim_{b \to \infty} R(\lambda, b), \qquad R_+\left(\overline{\lambda}\right) := \lim_{b \to \infty} R\left(\overline{\lambda}, b\right), \tag{6.1}$$

exist and satisfy $R_+(\lambda) \geq 0$ and $R_+(\overline{\lambda}) \geq 0$. The index "+" in the above notation as well as in Definition 6.2 refers to the limiting disk at $+\infty$. In the following result we will see that the center $P(\lambda, b)$ also converges to a limiting matrix when $b \to \infty$. This is a generalization of [11, Theorem 4.7], [1, Theorem 3.5], [14, Proposition 3.5], [2, Theorem 4.5], and [3, Theorem 4.10].

Theorem 6.1. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under Hypothesis 5.2, the center $P(\lambda, b)$ converges as $b \to \infty$ to a limiting matrix $P_+(\lambda) \in \mathbb{C}^{n \times n}$, that is,*

$$P_+(\lambda) := \lim_{b \to \infty} P(\lambda, b). \tag{6.2}$$

Proof. We prove that the matrix function $P(\lambda, \cdot)$ satisfies the Cauchy convergence criterion. Let $b_1, b_2 \in [b_0, \infty)_{\mathbb{T}}$ be given with $b_1 < b_2$. By Theorem 5.1, we have that $D(\lambda, b_2) \subseteq D(\lambda, b_1)$.

Therefore, by (5.16) of Theorem 5.6, for a matrix $M \in D(\lambda, b_2)$, there are (unique) matrices $V_1, V_2 \in \mathcal{V}$ such that

$$M = P(\lambda, b_j) + R(\lambda, b_j) V_j R\left(\overline{\lambda}, b_j\right), \quad j \in \{1, 2\}. \tag{6.3}$$

Upon subtracting the two equations in (6.3), we get

$$P(\lambda, b_2) - P(\lambda, b_1) + R(\lambda, b_2) V_2 R\left(\overline{\lambda}, b_2\right) = R(\lambda, b_1) V_1 R\left(\overline{\lambda}, b_1\right). \tag{6.4}$$

This equation, when solved for V_1 in terms of V_2, has the form

$$V_1 = R^{-1}(\lambda, b_1)\left[P(\lambda, b_2) - P(\lambda, b_1) + R(\lambda, b_2) V_2 R\left(\overline{\lambda}, b_2\right)\right] R^{-1}\left(\overline{\lambda}, b_1\right) =: T(V_2), \tag{6.5}$$

which defines a continuous mapping $T : \mathcal{V} \to \mathcal{V}$, $T(V_2) = V_1$. Since $\mathcal{V}$ is compact, it follows that the mapping T has a fixed point in $\mathcal{V}$, that is, $T(V) = V$ for some matrix $V \in \mathcal{V}$. Equation $T(V) = V$ implies that

$$\begin{aligned} P(\lambda, b_2) - P(\lambda, b_1) &= R(\lambda, b_1) V R\left(\overline{\lambda}, b_1\right) - R(\lambda, b_2) V R\left(\overline{\lambda}, b_2\right) \\ &= [R(\lambda, b_1) - R(\lambda, b_2)] V R\left(\overline{\lambda}, b_1\right) - R(\lambda, b_2) V\left[R\left(\overline{\lambda}, b_1\right) - R\left(\overline{\lambda}, b_2\right)\right]. \end{aligned} \tag{6.6}$$

Hence, by $\|V\| \le 1$, we have

$$\|P(\lambda, b_2) - P(\lambda, b_1)\| \le \|R(\lambda, b_1) - R(\lambda, b_2)\| \left\|R\left(\overline{\lambda}, b_1\right)\right\| + \|R(\lambda, b_2)\| \left\|R\left(\overline{\lambda}, b_1\right) - R\left(\overline{\lambda}, b_2\right)\right\|. \tag{6.7}$$

Since the functions $R(\lambda, \cdot)$ and $R(\overline{\lambda}, \cdot)$ are monotone nonincreasing, they are bounded; that is, for some $K > 0$, we have $\|R(\lambda, b)\| \le K$ and $\|R(\overline{\lambda}, b)\| \le K$ for all $b \in [b_0, \infty)_{\mathbb{T}}$.

Let $\varepsilon > 0$ be arbitrary. The convergence of $R(\lambda, b)$ and $R(\overline{\lambda}, b)$ as $b \to \infty$ yields the existence of $b_3 \in [b_0, \infty)_{\mathbb{T}}$ such that for every $b_1, b_2 \in [b_3, \infty)_{\mathbb{T}}$ with $b_1 < b_2$ we have

$$\|R(\nu, b_1) - R(\nu, b_2)\| \le \frac{\varepsilon}{(2K)}, \quad \nu \in \left\{\lambda, \overline{\lambda}\right\}. \tag{6.8}$$

Using estimate (6.8) in inequality (6.7) we obtain for $b_2 > b_1 \ge b_3$

$$\|P(\lambda, b_2) - P(\lambda, b_1)\| < \frac{\varepsilon}{(2K)} \cdot K + \frac{\varepsilon}{(2K)} \cdot K = \varepsilon. \tag{6.9}$$

This means that the limit $P_+(\lambda) \in \mathbb{C}^{n \times n}$ in (6.2) exists, which completes the proof. □

By Theorems 5.1 and 5.6 we know that the Weyl disks $D(\lambda, b)$ are closed, convex, and nested as $b \to \infty$. Thereore the limit of $D(\lambda, b)$ as $b \to \infty$ is a closed, convex, and nonempty set. This motivates the following definition, which can be found in the special cases of system $(\mathcal{S}_\lambda)$ in [26, Theorem 3.3], [1, Theorem 3.6], [2, Definition 4.7], and [3, Theorem 4.12].

Definition 6.2 (limiting Weyl disk). Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the set

$$D_+(\lambda) := \bigcap_{b \in [a,\infty)_{\mathbb{T}}} D(\lambda, b), \tag{6.10}$$

is called the *limiting Weyl disk*. The matrix $P_+(\lambda)$ from Theorem 6.1 is called the *center* of $D_+(\lambda)$ and the matrices $R_+(\lambda)$ and $R_+(\overline{\lambda})$ from (6.1) its *matrix radii*.

As a consequence of Theorem 5.6, we obtain the following characterization of the limiting Weyl disk.

Corollary 6.3. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under Hypothesis 5.2, we have*

$$D_+(\lambda) = P_+(\lambda) + R_+(\lambda)\mathcal{U}R_+\left(\overline{\lambda}\right), \tag{6.11}$$

where $\mathcal{U}$ is the set of all contractive matrices defined in (5.15).

From now on we assume that Hypothesis 5.2 holds, so that the limiting center $P_+(\lambda)$ and the limiting matrix radii $R_+(\lambda)$ and $R_+(\overline{\lambda})$ of $D_+(\lambda)$ are well defined.

Remark 6.4. By means of the nesting property of the disks (Theorem 5.1) and Theorems 4.10 and 4.12, it follows that the elements of the limiting Weyl disk $D_+(\lambda)$ are of the form

$$M_+(\lambda) \in D_+(\lambda), \quad M_+(\lambda) = \lim_{b \to \infty} M(\lambda, b, \alpha, \beta(b)), \tag{6.12}$$

where $\beta(b) \in \mathbb{C}^{n \times 2n}$ satisfies $\beta(b)\beta^*(b) = I$ and $i\delta(\lambda)\beta(b)\mathcal{J}\beta^*(b) \geq 0$ for all $b \in [a, \infty)$. Moreover, from Lemma 4.6, we conclude that

$$M_+^*(\lambda) = M_+\left(\overline{\lambda}\right). \tag{6.13}$$

A matrix $M_+(\lambda)$ from (6.12) is called a *half-line Weyl-Titchmarsh $M(\lambda)$-function*. Also, as noted in [2, Section 4], see also [8, Theorem 2.18], the function $M_+(\lambda)$ is a Herglotz function with rank n and has a certain integral representation (which will not be needed in this paper).

Our next result shows another characterization of the elements of $D_+(\lambda)$ in terms of the Weyl solution $\mathcal{X}(\cdot, \alpha, \lambda, M)$ defined in (4.16). This is a generalization of [11, page 671], [26, equation (3.2)], [1, Theorem 3.8(i)], [2, Theorem 4.8], and [3, Theorem 4.15].

Theorem 6.5. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M belongs to the limiting Weyl disk $D_+(\lambda)$ if and only if*

$$\int_a^\infty \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t \le \frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}. \tag{6.14}$$

Proof. By Definition 6.2, we have $M \in D_+(\lambda)$ if and only if $M \in D(\lambda, b)$, that is, $\mathcal{E}(M,b) \le 0$, for all $b \in [a,\infty)_{\mathbb{T}}$. Therefore, by formula (4.18), we get

$$\int_a^b \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t = \frac{\mathcal{E}(M,b)}{2|\operatorname{Im}(\lambda)|} + \frac{\delta(\lambda)\operatorname{Im}(M)}{|\operatorname{Im}(\lambda)|} \le \frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}, \tag{6.15}$$

for every $b \in [a,\infty)_{\mathbb{T}}$, which is equivalent to inequality (6.14). □

Remark 6.6. In [1, Definition 3.4], the notion of a boundary of the limiting Weyl disk $D_+(\lambda)$ is discussed. This would be a "limiting Weyl circle" according to Definitions 4.9 and 6.2. The description of matrices $M \in \mathbb{C}^{n\times n}$ laying on this boundary follows from Theorems 6.5 and 4.10, giving for such matrices M the equality

$$\int_a^\infty \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t = \frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}. \tag{6.16}$$

Condition (6.16) is also equivalent to

$$\lim_{t\to\infty} \mathcal{X}^{*}(t,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(t,\lambda,\alpha,M) = 0. \tag{6.17}$$

This is because, by (4.19) and the Lagrange identity (Corollary 3.6),

$$\begin{aligned} &\mathcal{X}^{*}(t,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(t,\lambda,\alpha,M) \\ &\quad = 2i\operatorname{Im}(\lambda)\left[\frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)} - \int_a^t \mathcal{X}^{\sigma *}(s,\lambda,\alpha,M)\widetilde{\mathcal{W}}(s)\mathcal{X}^{\sigma}(s,\lambda,\alpha,M)\Delta s\right], \end{aligned} \tag{6.18}$$

for every $t \in [a,\infty)_{\mathbb{T}}$. From this we can see that the integral on the right-hand side above converges for $t \to \infty$ and (6.16) holds if and only if condition (6.17) is satisfied. Characterizations (6.16) and (6.17) of the matrices M on the boundary of the limiting Weyl disk $D_+(\lambda)$ generalize the corresponding results in [1, Theorems 3.8(ii) and 3.9]; see also [14, Theorem 6.3].

Consider the linear space of square integrable $\mathrm{C}^1_{\mathrm{prd}}$ functions

$$L^2_{\mathcal{W}} = L^2_{\mathcal{W}}[a,\infty)_{\mathbb{T}} := \left\{z : [a,\infty)_{\mathbb{T}} \longrightarrow \mathbb{C}^{2n},\ z \in \mathrm{C}^1_{\mathrm{prd}},\ \|z(\cdot)\|_{\mathcal{W}} < \infty\right\}, \tag{6.19}$$

where we define

$$\|z(\cdot)\|_{\mathcal{W}} := \sqrt{\langle z(\cdot), z(\cdot)\rangle_{\mathcal{W}}}, \qquad \langle z(\cdot), \tilde{z}(\cdot)\rangle_{\mathcal{W}} := \int_a^\infty z^{\sigma *}(t)\widetilde{\mathcal{W}}(t)\tilde{z}^{\sigma}(t)\,\Delta t. \tag{6.20}$$

In the following result we prove that the space $L^2_{\mathcal{W}}$ contains the columns of the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$ when M belongs to the limiting Weyl disk $D_+(\lambda)$. This implies that there are at least n linearly independent solutions of system $(\mathcal{S}_\lambda)$ in $L^2_{\mathcal{W}}$. This is a generalization of [11, Theorem 5.1], [14, Theorem 4.1], [2, Theorem 4.10], and [5, page 716].

Theorem 6.7. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in D_+(\lambda)$. The columns of $\mathcal{X}(\cdot,\lambda,\alpha,M)$ form a linearly independent system of solutions of system $(\mathcal{S}_\lambda)$, each of which belongs to $L^2_{\mathcal{W}}$.*

Proof. Let $z_j(\cdot) := \mathcal{X}(\cdot,\lambda,\alpha,M)e_j$ for $j \in \{1,\dots,n\}$ be the columns of the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$, where e_j is the jth unit vector. We prove that the functions $z_1(\cdot),\dots,z_n(\cdot)$ are linearly independent. Assume that $\sum_{j=1}^n c_j z_j(\cdot) = 0$ on $[a,\infty)_{\mathbb{T}}$ for some $c_1,\dots,c_n \in \mathbb{C}$. Then $\mathcal{X}(\cdot,\lambda,\alpha,M)c = 0$, where $c := (c_1^*,\dots,c_n^*)^* \in \mathbb{C}^n$. It follows by (4.19) that

$$2ic^* \operatorname{Im}(M)c = c^*\mathcal{X}^*(a,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(a,\lambda,\alpha,M)c = 0, \tag{6.21}$$

which implies the equality $c^*\delta(\lambda)\operatorname{Im}(M)c = 0$. Using that $M \in D_+(\lambda) \subseteq D(\lambda,b)$ for some $b \in [b_0,\infty)_{\mathbb{T}}$, we obtain from Theorem 4.13 that the matrix $\delta(\lambda)\operatorname{Im}(M)$ is positive definite. Hence, $c = 0$ so that the functions $z_1(\cdot),\dots,z_n(\cdot)$ are linearly independent. Finally, for every $j \in \{1,\dots,n\}$ we get from Theorem 6.5 the inequality

$$\|z_j(\cdot)\|^2_{\mathcal{W}} = \int_a^\infty z_j^{\sigma *}(t)\widetilde{\mathcal{W}}(t)z_j^{\sigma}(t)\,\Delta t \overset{(6.14)}{\leq} e_j^*\frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}e_j \leq \frac{\|\delta(\lambda)\operatorname{Im}(M)\|}{|\operatorname{Im}(\lambda)|} < \infty. \tag{6.22}$$

Thus, $z_j(\cdot) \in L^2_{\mathcal{W}}$ for every $j \in \{1,\dots,n\}$, and the proof is complete. □

Denote by $\mathcal{N}(\lambda)$ the linear space of all square integrable solutions of system $(\mathcal{S}_\lambda)$, that is,

$$\mathcal{N}(\lambda) := \left\{z(\cdot) \in L^2_{\mathcal{W}},\ z(\cdot) \text{ solves } (\mathcal{S}_\lambda)\right\}. \tag{6.23}$$

Then as a consequence of Theorem 6.7 we obtain the estimate

$$\dim \mathcal{N}(\lambda) \geq n, \quad \text{for each } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{6.24}$$

Next we discuss the situation when $\dim \mathcal{N}(\lambda) = n$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 6.8. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $\dim \mathcal{N}(\lambda) = n$. Then the matrix radii of the limiting Weyl disk $D_+(\lambda)$ satisfy $R_+(\lambda) = 0 = R_+(\overline{\lambda})$. Consequently, the set $D_+(\lambda)$ consists of the single matrix $M = P_+(\lambda)$, that is, the center of $D_+(\lambda)$, which is given by formula* (6.2) *of Theorem 6.1.*

Proof. With the matrix radii $R_+(\lambda)$ and $R_+(\overline{\lambda})$ of $D_+(\lambda)$ defined in (6.1) and with the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$ given by a matrix $M \in D_+(\lambda)$, we observe that the columns of $\mathcal{X}(\cdot,\lambda,\alpha,M)$ form a basis of the space $\mathcal{N}(\lambda)$. Since the columns of the fundamental matrix $\Psi(\cdot,\lambda,\alpha) = (Z(\cdot,\lambda,\alpha)\ \ \widetilde{Z}(\cdot,\lambda,\alpha))$ span all solutions of system $(\mathcal{S}_\lambda)$, the definition of $\mathcal{X}(\cdot,\lambda,\alpha,M) = Z(\cdot,\lambda,\alpha) + \widetilde{Z}(\cdot,\lambda,\alpha)M$ yields that the columns of $\widetilde{Z}(\cdot,\lambda,\alpha)$ together with the columns of $\mathcal{X}(\cdot,\lambda,\alpha,M)$ form a basis of all solutions of system $(\mathcal{S}_\lambda)$. Hence, from $\dim \mathcal{N}(\lambda) = n$ and Theorem 6.7, we get that the columns of $\widetilde{Z}(\cdot,\lambda,\alpha)$ do not belong to $L^2_{\mathcal{W}}$. Consequently, by formula (5.5), the Hermitian matrix functions $\mathcal{H}(\cdot,\lambda,\alpha)$ and $\mathcal{H}(\cdot,\overline{\lambda},\alpha)$ defined in (5.4) are monotone nondecreasing on $[a,\infty)_{\mathbb{T}}$ without any upper bound; that is, their eigenvalues—being real—tend to ∞. Therefore, the functions $R(\lambda,\cdot)$ and $R(\overline{\lambda},\cdot)$ as defined in (5.18) have limits at ∞ equal to zero; that is, $R_+(\lambda) = 0$ and $R_+(\overline{\lambda}) = 0$. The fact that the set $D_+(\lambda) = \{P_+(\lambda)\}$ then follows from the characterization of $D_+(\lambda)$ in Corollary 6.3. □

In the final result of this section, we establish another characterization of the matrices M from the limiting Weyl disk $D_+(\lambda)$. In comparison with Theorem 6.5, we now use a similar condition to the one in Theorem 4.12 for the regular spectral problem. However, a stronger assumption than Hypothesis 5.2 is now required for this result to hold; compare with [9, Lemma 2.21] and [2, Theorem 4.16].

Hypothesis 6.9. For every $a_0, b_0 \in (a,\infty)_{\mathbb{T}}$ with $a_0 < b_0$ and for every $\lambda \in \mathbb{C}$, we have

$$\int_{a_0}^{b_0} \Psi^{\sigma *}(t,\lambda,\alpha)\widetilde{\mathcal{W}}(t)\Psi^{\sigma}(t,\lambda,\alpha)\Delta t > 0. \tag{6.25}$$

Under Hypothesis 6.9, the Weyl disks $D(\lambda,b)$ converge to the limiting disk "monotonically" as $b \to \infty$; that is, the limiting Weyl disk $D_+(\lambda)$ is "open" in the sense that all of its elements lie inside $D_+(\lambda)$. This can be interpreted in view of Theorem 4.12 as $\mathcal{E}(M,t) < 0$ for all $t \in [a,\infty)_{\mathbb{T}}$.

Theorem 6.10. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. Under Hypothesis 6.9, the matrix $M \in D_+(\lambda)$ if and only if*

$$\mathcal{E}(M,t) < 0, \quad \forall t \in [a,\infty)_{\mathbb{T}}. \tag{6.26}$$

Proof. If condition (6.26) holds, then $M \in D_+(\lambda)$ follows from the definition of $D_+(\lambda)$. Conversely, suppose that $M \in D_+(\lambda)$, and let $t \in [a,\infty)_{\mathbb{T}}$ be given. Then for any $b \in (t,\infty)_{\mathbb{T}}$ we have by formula (4.18) that

$$\begin{aligned}\mathcal{E}(M,t) &= -2\delta(\lambda)\operatorname{Im}(M) + 2|\operatorname{Im}(\lambda)|\int_a^t \mathcal{X}^{\sigma *}(s,\lambda,\alpha,M)\widetilde{\mathcal{W}}(s)\mathcal{X}^{\sigma}(s,\lambda,\alpha,M)\Delta s \\ &= \mathcal{E}(M,b) - 2|\operatorname{Im}(\lambda)|\int_t^b \mathcal{X}^{\sigma *}(s,\lambda,\alpha,M)\widetilde{\mathcal{W}}(s)\mathcal{X}^{\sigma}(s,\lambda,\alpha,M)\Delta s,\end{aligned} \tag{6.27}$$

where we used the property $\int_a^t f(s)\Delta s = \int_a^b f(s)\Delta s - \int_t^b f(s)\Delta s$. Since $M \in D_+(\lambda)$ is assumed, we have $M \in D(\lambda, b)$, that is, $\mathcal{E}(M, b) \leq 0$, while Hypothesis 6.9 implies the positivity of the integral over $[t, b]_{\mathbb{T}}$ in (6.27). Consequently, (6.27) yields that $\mathcal{E}(M, t) < 0$. □

Remark 6.11. If we partition the Weyl solution $\mathcal{X}(\cdot, \lambda) := \mathcal{X}(\cdot, \lambda, \alpha, M)$ into two $n \times n$ blocks $\mathcal{X}_1(\cdot, \lambda)$ and $\mathcal{X}_2(\cdot, \lambda)$ as in (4.28), then condition (6.26) can be written as

$$\delta(\lambda)\operatorname{Im}\big(\mathcal{X}_1^*(t, \lambda)\mathcal{X}_2(t, \lambda)\big) > 0, \quad \forall t \in [a, \infty)_{\mathbb{T}}. \tag{6.28}$$

Therefore, by Remark 2.2, the matrices $\mathcal{X}_1(t, \lambda)$ and $\mathcal{X}_2(t, \lambda)$ are invertible for all $t \in [a, \infty)_{\mathbb{T}}$. A standard argument then yields that the quotient $Q(\cdot, \lambda) := \mathcal{X}_2(\cdot, \lambda)\mathcal{X}_1^{-1}(\cdot, \lambda)$ satisfies the *Riccati matrix equation* (suppressing the argument t in the coefficients)

$$Q^{\Delta} - (\mathcal{C} + \mathcal{D}Q) + Q^{\sigma}(\mathcal{A} + \mathcal{B}Q) + \lambda\mathcal{W}\big[I + \mu(\mathcal{A} + \mathcal{B}Q)\big] = 0, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{6.29}$$

see [57, Theorem 3], [48, Section 6], and [49].

7. Limit Point and Limit Circle Criteria

Throughout this section we assume that Hypothesis 5.2 is satisfied. The results from Theorem 6.7 and Lemma 6.8 motivate the following terminology; compare with [4, page 75], [43, Definition 1.2] in the time scales scalar case $n = 1$, with [8, page 3486], [36, page 1668], [30, page 274], [38, Definition 3.1], [37, Definition 1], [67, page 2826] in the continuous case, and with [14, Definition 5.1], [2, Definition 4.12] in the discrete case.

Definition 7.1 (limit point and limit circle case for system $(\mathcal{S}_\lambda)$). The system $(\mathcal{S}_\lambda)$ is said to be in the *limit point case* at ∞ (or of the *limit point type*) if

$$\dim \mathcal{N}(\lambda) = n, \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{7.1}$$

The system $(\mathcal{S}_\lambda)$ is said to be in the *limit circle case* at ∞ (or of the *limit circle type*) if

$$\dim \mathcal{N}(\lambda) = 2n, \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{7.2}$$

Remark 7.2. According to Remark 6.4 (in which $\beta(b) \equiv \beta$), the center $P_+(\lambda)$ of the limiting Weyl disk $D_+(\lambda)$ can be expressed in the limit point case as

$$P_+(\lambda) = M_+(\lambda) = \lim_{b \to \infty} M(\lambda, b, \alpha, \beta), \tag{7.3}$$

where $\beta \in \Gamma$ is arbitrary but fixed.

Next we establish the first main result of this section. Its continuous time version can be found in [30, Theorem 2.1], [11, Theorem 8.5] and the discrete time version in [9, Lemma 3.2], [2, Theorem 4.13].

Theorem 7.3. *Let the system $(\mathcal{S}_\lambda)$ be in the limit point or limit circle case, fix $\alpha \in \Gamma$, and let $\lambda, \nu \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\lim_{t\to\infty} \mathcal{X}_+^*(t,\lambda,\alpha,M_+(\lambda))\mathcal{J}\mathcal{X}_+(t,\nu,\alpha,M_+(\nu)) = 0, \tag{7.4}$$

where $\mathcal{X}_+(\cdot,\lambda,\alpha,M_+(\lambda))$ and $\mathcal{X}_+(\cdot,\nu,\alpha,M_+(\nu))$ are the Weyl solutions of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, respectively, defined by (4.16) *through the matrices $M_+(\lambda)$ and $M_+(\nu)$, which are determined by the limit in* (6.12).

Proof. For every $t \in [a,\infty)_{\mathbb{T}}$ and matrices $\beta(t) \in \mathbb{C}^{n\times 2n}$ such that $\beta(t)\beta^*(t) = I$ and $i\delta(\lambda)\beta(t)\mathcal{J}\beta^*(t) \geq 0$ and for $\kappa \in \{\lambda,\nu\}$, we define the matrix (compare with Definition 4.5)

$$M(\kappa,t,\alpha,\beta(t)) := -\left[\beta(t)\widetilde{Z}(t,\kappa,\alpha)\right]^{-1}\beta(t)Z(t,\kappa,\alpha). \tag{7.5}$$

Then, by Theorems 4.10 and 4.12, we have $M(\kappa,t,\alpha,\beta(t)) \in D(\kappa,t)$. Following the notation in (4.16), we consider the Weyl solutions $\mathcal{X}(\cdot,\kappa) := \mathcal{X}(\cdot,\kappa,\alpha,M(\kappa,t,\alpha,\beta(\cdot)))$. Similarly, let $\mathcal{X}_+(\cdot,\kappa) := \mathcal{X}(\cdot,\kappa,\alpha,M_+(\kappa))$ be the Weyl solutions corresponding to the matrices $M_+(\kappa) \in D_+(\kappa)$ from the statement of this theorem.

First assume that the system $(\mathcal{S}_\lambda)$ is of the limit point type. In this case, by Remark 7.2, we may take $\beta(t) \in \Gamma$ for all $t \in [a,\infty)_{\mathbb{T}}$. Hence, from Theorem 4.10, we get that $\beta(\cdot)\mathcal{X}(\cdot,\kappa) = 0$ on $[a,\infty)_{\mathbb{T}}$. By (4.3), for each $t \in [a,\infty)_{\mathbb{T}}$ and $\kappa \in \{\lambda,\nu\}$, there is a matrix $Q_\kappa(t) \in \mathbb{C}^{n\times n}$ such that $\mathcal{X}(\cdot,\kappa) = \mathcal{J}\beta^*(\cdot)Q_\kappa(\cdot)$ on $[a,\infty)_{\mathbb{T}}$. Hence, we have on $[a,\infty)_{\mathbb{T}}$

$$\begin{aligned}
&\mathcal{X}_+^*(t,\lambda)\mathcal{J}\mathcal{X}_+(t,\nu) + F(t,\lambda,\nu,\beta(t)) + G(t,\lambda,\nu,\beta(t)) \\
&\quad = \mathcal{X}^*(t,\lambda)\mathcal{J}\mathcal{X}(t,\nu) = Q_\lambda^*(t)\beta(t)\mathcal{J}\beta^*(t)Q_\nu(t) = 0,
\end{aligned} \tag{7.6}$$

where we define

$$\begin{aligned}
F(t,\lambda,\nu,\beta(t)) &:= \mathcal{X}_+^*(t,\lambda)\mathcal{J}\widetilde{Z}(t,\nu,\alpha)\left[M(\nu,t,\alpha,\beta(t)) - M_+(\nu)\right], \\
G(t,\lambda,\nu,\beta(t)) &:= \left[M^*(\lambda,t,\alpha,\beta(t)) - M_+^*(\lambda)\right]\widetilde{Z}^*(t,\lambda,\alpha)\mathcal{J}\mathcal{X}(t,\nu).
\end{aligned} \tag{7.7}$$

If we show that

$$\lim_{t\to\infty} F(t,\lambda,\nu,\beta(t)) = 0, \qquad \lim_{t\to\infty} G(t,\lambda,\nu,\beta(t)) = 0, \tag{7.8}$$

then (7.6) implies the result claimed in (7.4). First we prove the second limit in (7.8). Pick any $t \in [b_0,\infty)_{\mathbb{T}}$. By Theorem 5.6, Corollary 6.3, and $D_+(\lambda) \subseteq D(\lambda,t)$, we have

$$M(\lambda,t,\alpha,\beta(t)) = P(\lambda,t) + R(\lambda,t)U(t)R\left(\overline{\lambda},t\right), \qquad M_+(\lambda) = P(\lambda,t) + R(\lambda,t)V(t)R\left(\overline{\lambda},t\right), \tag{7.9}$$

where $U(t) \in \mathcal{U}$ and $V(t) \in \mathcal{V}$. Therefore,

$$M(\lambda, t, \alpha, \beta(t)) - M_+(\lambda) = R(\lambda, t)[U(t) - V(t)]R\left(\overline{\lambda}, t\right). \tag{7.10}$$

Since $\widetilde{Z}(\cdot, \lambda, \alpha)$ and $\mathcal{X}(\cdot, \nu)$ are, respectively, solutions of systems $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$ which satisfy $\widetilde{Z}^*(a, \lambda, \alpha)\mathcal{J}\mathcal{X}(a, \nu) = -I$, it follows from Corollary 3.6 that

$$\widetilde{Z}^*(t, \lambda, \alpha)\mathcal{J}\mathcal{X}(t, \nu) = -I + \left(\overline{\lambda} - \nu\right)\int_a^t \widetilde{Z}^{\sigma *}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\mathcal{X}^\sigma(s, \nu)\Delta s. \tag{7.11}$$

Hence, we can write

$$G(t, \lambda, \nu, \beta(t)) = R\left(\overline{\lambda}, t\right)[U^*(t) - V^*(t)]R(\lambda, t)\left[\left(\overline{\lambda} - \nu\right)\int_a^t \widetilde{Z}^{\sigma *}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\mathcal{X}^\sigma(s, \nu)\Delta s - I\right], \tag{7.12}$$

where we used the Hermitian property of $R(\lambda, t)$ and $R(\overline{\lambda}, t)$. Since we now assume that system $(\mathcal{S}_\lambda)$ is in the limit point case, we know from Lemma 6.8 that $\lim_{t\to\infty} R(\lambda, t) = 0$ and $\lim_{t\to\infty} R(\overline{\lambda}, t) = 0$. Therefore, in order to establish (7.8)(ii), it is sufficient to show that

$$R(\lambda, t)\int_a^t \widetilde{Z}^{\sigma *}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\mathcal{X}^\sigma(s, \nu)\Delta s, \tag{7.13}$$

is bounded for $t \in [b_0, \infty)_{\mathbb{T}}$. Let $\eta \in \mathbb{C}^n$ be a unit vector, and denote by $\mathcal{X}_j(\cdot, \nu) := \mathcal{X}(\cdot, \nu)e_j$ the jth column of $\mathcal{X}(\cdot, \nu)$ for $j \in \{1, \ldots, n\}$. With the definition of $R(\lambda, \cdot)$ in (5.18) we have

$$\begin{aligned}
&\left|\int_a^t \eta^* R(\lambda, s)\widetilde{Z}^{\sigma *}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\mathcal{X}_j^\sigma(s, \nu)\Delta s\right| \\
&\quad \le \int_a^t \left|\widetilde{\mathcal{W}}^{1/2}(s)\widetilde{Z}^{\sigma *}(s, \lambda, \alpha)R(\lambda, s)\eta\right|\left|\widetilde{\mathcal{W}}^{1/2}(s)\mathcal{X}_j^\sigma(s, \nu)\right|\Delta s \\
&\quad \overset{\text{C-S}}{\le} \left(\int_a^t \eta^* R(\lambda, s)\widetilde{Z}^{\sigma *}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\widetilde{Z}^\sigma(s, \lambda, \alpha)R(\lambda, s)\eta\Delta s\right)^{1/2} \\
&\qquad \times \left(\int_a^t \mathcal{X}_j^{\sigma *}(s, \nu)\widetilde{\mathcal{W}}(s)\mathcal{X}_j^\sigma(s, \nu)\Delta s\right)^{1/2},
\end{aligned} \tag{7.14}$$

where the last step follows from the Cauchy-Schwarz inequality (C-S) on time scales. From (5.5) we obtain

$$\mathcal{H}^{-1/2}(t, \lambda, \alpha)\int_a^t \widetilde{Z}^{\sigma *}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\widetilde{Z}^\sigma(s, \lambda, \alpha)\Delta s\mathcal{H}^{-1/2}(t, \lambda, \alpha) = \frac{1}{2|\mathrm{Im}(\lambda)|}I, \tag{7.15}$$

so that the first term in the product in (7.14) is bounded by $1/\sqrt{2|\operatorname{Im}(\lambda)|}$. Moreover, from formula (4.18) we get that the second term in the product in (7.14) is bounded by the number $[e_j^* \operatorname{Im}(M(\nu,t,\alpha,\beta(t)))e_j]/\operatorname{Im}(\nu)$. Hence, upon recalling the limit in (6.12), we conclude that the product in (7.14) is bounded by

$$\frac{1}{2|\operatorname{Im}(\lambda)|}\cdot\frac{e_j^*\operatorname{Im}(M_+(\nu))e_j}{\operatorname{Im}(\nu)}, \tag{7.16}$$

which is independent of t. Consequently, the second limit in (7.8) is established. The first limit in (7.8) is then proven in a similar manner. The proof for the limit point case is finished.

If the system $(\mathcal{S}_\lambda)$ is in the limit circle case, then for $\kappa\in\{\lambda,\nu\}$ the columns of $\widetilde{Z}(\cdot,\kappa,\alpha)$ and $\mathcal{X}_+(\cdot,\kappa)$ belong to $L^2_{\mathcal{W}}$; hence, they are bounded in the $L^2_{\mathcal{W}}$ norm. In this case the limits in (7.8) easily follow from the limit (6.12) for $M_+(\kappa)$, $\kappa\in\{\lambda,\nu\}$. □

In the next result we provide a characterization of the system $(\mathcal{S}_\lambda)$ being of the limit point type. Special cases of this statement can be found, for example, in [14, Theorem 6.12] and [2, Theorem 4.14].

Theorem 7.4. *Let $\alpha\in\Gamma$. The system $(\mathcal{S}_\lambda)$ is in the limit point case if and only if, for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$ and every square integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\overline{\lambda})$ of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_{\overline{\lambda}})$, respectively, we have*

$$z_1^*(t,\lambda)\mathcal{J}z_2\left(t,\overline{\lambda}\right)=0,\quad \forall t\in[b_0,\infty)_{\mathbb{T}}. \tag{7.17}$$

Proof. Let $(\mathcal{S}_\lambda)$ be in the limit point case. Fix any $\lambda\in\mathbb{C}\setminus\mathbb{R}$, and suppose that $z_1(\cdot,\lambda)$ and $z_2(\cdot,\overline{\lambda})$ are solutions of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_{\overline{\lambda}})$, respectively. Then, by Theorem 6.7 and Remark 6.4, there are vectors $\xi_1,\xi_2\in\mathbb{C}^n$ such that $z_1(\cdot,\lambda)=\mathcal{X}_+(\cdot,\lambda)\xi_1$ and $z_2(\cdot,\overline{\lambda})=\mathcal{X}_+(\cdot,\overline{\lambda})\xi_2$ on $[a,\infty)_{\mathbb{T}}$, where $\mathcal{X}_+(\cdot,\kappa):=\mathcal{X}_+(\cdot,\kappa,\alpha,M_+(\kappa))$ are the Weyl solutions corresponding to some matrices $M_+(\kappa)\in D_+(\kappa)$ for $\kappa\in\{\lambda,\overline{\lambda}\}$. In fact, by Lemma 6.8, the matrix $M_+(\kappa)$ is equal to the center of the disk $D_+(\kappa)$. It follows that for any $t\in[b_0,\infty)_{\mathbb{T}}$ equality

$$\begin{aligned}&\mathcal{X}_+^*(t,\lambda)\mathcal{J}\mathcal{X}_+\left(t,\overline{\lambda}\right)\\&\quad\overset{(4.16)}{=}\left(I\quad M_+^*(\lambda)\right)\Psi^*(t,\lambda,\alpha)\mathcal{J}\Psi\left(t,\overline{\lambda},\alpha\right)\left(I\quad M_+^*(\overline{\lambda})\right)^*\overset{(3.19)(\mathrm{i})}{=}M_+^*\left(\overline{\lambda}\right)-M_+^*(\lambda)\overset{(6.13)}{=}0,\end{aligned} \tag{7.18}$$

holds, so that (7.17) is established. Conversely, let $\nu\in\mathbb{C}\setminus\mathbb{R}$ be arbitrary but fixed, set $\lambda:=\overline{\nu}$, and suppose that, for every square integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\nu)$ of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, condition (7.17) is satisfied. From Theorem 6.7 we know that for $M_+(\kappa)\in D_+(\kappa)$ the columns $\mathcal{X}_+^{[j]}(\cdot,\kappa)$, $j\in\{1,\dots,n\}$, of the Weyl solution $\mathcal{X}_+(\cdot,\kappa)$ are linearly independent square integrable solutions of $(\mathcal{S}_\kappa)$, $\kappa\in\{\lambda,\nu\}$. Therefore, $\dim\mathcal{N}(\lambda)\geq n$, and $\dim\mathcal{N}(\nu)\geq n$. Moreover, by identity (3.19)(i), we have

$$\mathcal{X}_+^*(t,\lambda)\mathcal{J}\mathcal{X}_+^{[j]}(t,\nu)=0,\quad\forall t\in[b_0,\infty)_{\mathbb{T}},\ j\in\{1,\dots,n\}. \tag{7.19}$$

Let $z(\cdot,\nu)$ be any square integrable solution of system $(\mathcal{S}_\nu)$. Then, by our assumption (7.17),

$$\mathcal{X}_+^*(t,\lambda)\mathcal{J}z(t,\nu)=0,\quad \forall t\in[b_0,\infty)_{\mathbb{T}}. \tag{7.20}$$

From (7.19) and (7.20) it follows that the vectors $\mathcal{X}_+^{[j]}(a,\nu)$, $j\in\{1,\dots,n\}$, and $z(a,\nu)$ are solutions of the linear homogeneous system

$$\mathcal{X}_+^*(a,\lambda)\mathcal{J}\eta=0. \tag{7.21}$$

Since, by Theorem 6.7, the vectors $\mathcal{X}_+^{[j]}(a,\nu)$ for $j\in\{1,\dots,n\}$ represent a basis of the solution space of system (7.21), there exists a vector $\xi\in\mathbb{C}^n$ such that $z(a,\nu)=\mathcal{X}_+(a,\nu)\xi$. By the uniqueness of solutions of system $(\mathcal{S}_\nu)$ we then get $z(\cdot,\nu)=\mathcal{X}_+(\cdot,\nu)\xi$ on $[a,\infty)_{\mathbb{T}}$. Hence, the solution $z(\cdot,\nu)$ is square integrable and $\dim\mathcal{N}(\nu)=n$. Since $\nu\in\mathbb{C}\setminus\mathbb{R}$ was arbitrary, it follows that the system $(\mathcal{S}_\lambda)$ is in the limit point case. □

As a consequence of the above result, we obtain a characterization of the limit point case in terms of a condition similar to (7.17), but using a limit. This statement is a generalization of [30, Corollary 2.3], [9, Corollary 3.3], [14, Theorem 6.14], [2, Corollary 4.15], [1, Theorem 3.9], [3, Theorem 4.16].

Corollary 7.5. *Let $\alpha\in\Gamma$. The system $(\mathcal{S}_\lambda)$ is in the limit point case if and only if, for every $\lambda,\nu\in\mathbb{C}\setminus\mathbb{R}$ and every square integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\nu)$ of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, respectively, we have*

$$\lim_{t\to\infty} z_1^*(t,\lambda)\mathcal{J}z_2(t,\nu)=0. \tag{7.22}$$

Proof. The necessity follows directly from Theorem 7.3. Conversely, assume that condition (7.22) holds for every $\lambda,\nu\in\mathbb{C}\setminus\mathbb{R}$ and every square integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\nu)$ of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$. Fix $\lambda\in\mathbb{C}\setminus\mathbb{R}$, and set $\nu:=\overline{\lambda}$. By Corollary 3.7 we know that $z_1^*(\cdot,\lambda)\mathcal{J}z_2(\cdot,\nu)$ is constant on $[a,\infty)_{\mathbb{T}}$. Therefore, by using condition (7.22), we can see that identity (7.17) must be satisfied, which yields by Theorem 7.4 that the system $(\mathcal{S}_\lambda)$ is of the limit point type. □

8. Nonhomogeneous Time Scale Symplectic Systems

In this section we consider the nonhomogeneous time scale symplectic system

$$z^\Delta(t,\lambda)=\mathcal{S}(t,\lambda)z(t,\lambda)-\mathcal{J}\widetilde{\mathcal{W}}(t)f^\sigma(t),\quad t\in[a,\infty)_{\mathbb{T}}, \tag{8.1}$$

where the matrix function $\mathcal{S}(\cdot,\lambda)$ and $\widetilde{\mathcal{W}}(\cdot)$ are defined in (3.3) and (3.1), $f\in L^2_{\mathcal{W}}$, and where the associated homogeneous system $(\mathcal{S}_\lambda)$ is either of the limit point or limit circle type at ∞. Together with system (8.1) we consider a second system of the same form but with a different spectral parameter and a different nonhomogeneous term

$$y^\Delta(t,\nu)=\mathcal{S}(t,\nu)y(t,\nu)-\mathcal{J}\widetilde{\mathcal{W}}(t)g^\sigma(t),\quad t\in[a,\infty)_{\mathbb{T}}, \tag{8.2}$$

with $g\in L^2_{\mathcal{W}}$. The following is a generalization of Theorem 3.5 to nonhomogeneous systems.

Theorem 8.1 (Lagrange identity). *Let $\lambda, \nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If $z(\cdot,\lambda)$ and $y(\cdot,\nu)$ are $2n \times m$ solutions of systems* (8.1) *and* (8.2)*, respectively, then*

$$\begin{aligned}&\left[z^*(t,\lambda)\mathcal{J}y(t,\nu)\right]^{\Delta}\\&\quad=\left(\overline{\lambda}-\nu\right)z^{\sigma*}(t,\lambda)\widetilde{\mathcal{W}}(t)y^{\sigma}(t,\nu)-f^{\sigma*}(t)\widetilde{\mathcal{W}}(t)y^{\sigma}(t,\nu)+z^{\sigma*}(t,\lambda)\widetilde{\mathcal{W}}(t)g^{\sigma}(t),\quad t\in[a,\infty)_{\mathbb{T}}.\end{aligned}\tag{8.3}$$

Proof. Formula (8.3) follows by the product rule (2.1) with the aid of the relation

$$z^{\sigma}(t,\lambda)=\left[I+\mu(t)\mathcal{S}(t,\lambda)\right]z(t,\lambda)+\mu(t)\widetilde{\mathcal{W}}(t)f^{\sigma}(t),\tag{8.4}$$

and identity (3.6). □

For $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $t, s \in [a,\infty)_{\mathbb{T}}$, we define the function

$$G(t,s,\lambda,\alpha):=\begin{cases}\widetilde{Z}(t,\lambda,\alpha)\mathcal{K}_{+}^{*}\left(s,\overline{\lambda},\alpha\right), & \text{for } t\in[a,s)_{\mathbb{T}},\\ \mathcal{K}_{+}(t,\lambda,\alpha)\widetilde{Z}^{*}\left(s,\overline{\lambda},\alpha\right), & \text{for } t\in[s,\infty)_{\mathbb{T}},\end{cases}\tag{8.5}$$

where $\widetilde{Z}(\cdot,\lambda,\alpha)$ is the solution of system $(\mathcal{S}_{\lambda})$ given in (4.10), that is, $\widetilde{Z}(a,\lambda,\alpha)=-\mathcal{J}\alpha^{*}$, and $\mathcal{K}_{+}(\cdot,\lambda,\alpha):=\mathcal{K}(\cdot,\lambda,\alpha,M_{+}(\lambda))$ is the Weyl solution of $(\mathcal{S}_{\lambda})$ as in (4.16) determined by a matrix $M_{+}(\lambda)\in D_{+}(\lambda)$. This matrix $M_{+}(\lambda)\in D_{+}(\lambda)$ is arbitrary but fixed throughout this section. By interchanging the order of the arguments t and s, we have

$$G(t,s,\lambda,\alpha)=\begin{cases}\mathcal{K}_{+}(t,\lambda,\alpha)\widetilde{Z}^{*}\left(s,\overline{\lambda},\alpha\right), & \text{for } s\in[a,t]_{\mathbb{T}},\\ \widetilde{Z}(t,\lambda,\alpha)\mathcal{K}_{+}^{*}\left(s,\overline{\lambda},\alpha\right), & \text{for } s\in(t,\infty)_{\mathbb{T}}.\end{cases}\tag{8.6}$$

In the literature the function $G(\cdot,\cdot,\lambda,\alpha)$ is called a resolvent kernel, compare with [30, page 283], [32, page 15], [2, equation (5.4)], and in this section it will play a role of the Green function.

Lemma 8.2. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\mathcal{K}_{+}(t,\lambda,\alpha)\widetilde{Z}^{*}\left(t,\overline{\lambda},\alpha\right)-\widetilde{Z}(t,\lambda,\alpha)\mathcal{K}_{+}^{*}\left(t,\overline{\lambda},\alpha\right)=\mathcal{J},\quad\forall t\in[a,\infty)_{\mathbb{T}}.\tag{8.7}$$

Proof. Identity (8.7) follows by a direct calculation from the definition of $\mathcal{K}_{+}(\cdot,\lambda,\alpha)$ via (4.16) with a matrix $M_{+}(\lambda)\in D_{+}(\lambda)$ by using formulas (3.21) and (6.13). □

In the next lemma we summarize the properties of the function $G(\cdot,\cdot,\lambda,\alpha)$, which together with Proposition 8.4 and Theorem 8.5 justifies the terminology "Green function" of the system (8.1); compare with [68, Section 4]. A discrete version of the following result can be found in [2, Lemma 5.1].

Lemma 8.3. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The function $G(\cdot, \cdot, \lambda, \alpha)$ has the following properties:*

(i) $G^*(t, s, \lambda, \alpha) = G(s, t, \overline{\lambda}, \alpha)$ *for every* $t, s \in [a, \infty)_{\mathbb{T}}$, $t \neq s$,

(ii) $G^*(t, t, \lambda, \alpha) = G(t, t, \overline{\lambda}, \alpha) - \mathcal{J}$ *for every* $t \in [a, \infty)_{\mathbb{T}}$,

(iii) $G(\sigma(t), \sigma(t), \lambda, \alpha) = [I + \mu(t)\mathcal{S}(t, \lambda)]G(t, \sigma(t), \lambda, \alpha) + \mathcal{J}$ *for every right-scattered point* $t \in [a, \infty)_{\mathbb{T}}$,

(iv) *for every* $t, s \in [a, \infty)_{\mathbb{T}}$ *such that* $t \notin \mathcal{T}(s)$, *the function* $G(\cdot, s, \lambda, \alpha)$ *solves the homogeneous system* $(\mathcal{S}_\lambda)$ *on the set* $\mathcal{T}(s)$, *where*

$$\mathcal{T}(s) := \{\tau \in [a, \infty)_{\mathbb{T}},\ \tau \neq \rho(s) \text{ if } s \text{ is left-scattered}\}, \tag{8.8}$$

(v) *the columns of* $G(\cdot, s, \lambda, \alpha)$ *belong to* $L^2_{\mathcal{W}}$ *for every* $s \in [a, \infty)_{\mathbb{T}}$, *and the columns of* $G(t, \cdot, \lambda, \alpha)$ *belong to* $L^2_{\mathcal{W}}$ *for every* $t \in [a, \infty)_{\mathbb{T}}$.

Proof. Condition (i) follows from the definition of $G(\cdot, s, \lambda, \alpha)$ in (8.5). Condition (ii) is a consequence of Lemma 8.2. Condition (iii) is proven from the definition of $G(\sigma(t), \sigma(t), \lambda, \alpha)$ in (8.5) by using Lemma 8.2 and $\widetilde{Z}(t, \lambda, \alpha) = \widetilde{Z}^\sigma(t, \lambda, \alpha) - \mu(t)\mathcal{S}(t, \lambda)\widetilde{Z}(t, \lambda, \alpha)$. Concerning condition (iv), the function $G(\cdot, s, \lambda, \alpha)$ solves the system $(\mathcal{S}_\lambda)$ on $[s, \infty)_{\mathbb{T}}$ because $\mathcal{X}_+(\cdot, \lambda, \alpha)$ solves this system on $[s, \infty)_{\mathbb{T}}$. If $s \in (a, \infty)_{\mathbb{T}}$ is left-dense, then $G(\cdot, s, \lambda, \alpha)$ solves $(\mathcal{S}_\lambda)$ on $[a, s)_{\mathbb{T}}$, since $\widetilde{Z}(\cdot, \lambda, \alpha)$ solves this system on $[a, s)_{\mathbb{T}}$. For the same reason $G(\cdot, s, \lambda, \alpha)$ solves $(\mathcal{S}_\lambda)$ on $[a, \rho(s))_{\mathbb{T}}$ if $s \in (a, \infty)_{\mathbb{T}}$ is left-scattered. Condition (v) follows from the definition of $G(\cdot, s, \lambda, \alpha)$ in (8.5) used with $t \geq s$ and from the fact that the columns of $\mathcal{X}_+(\cdot, \lambda, \alpha)$ belong to $L^2_{\mathcal{W}}$, by Theorem 6.7. The columns of $G(t, \cdot, \lambda, \alpha)$ then belong to $L^2_{\mathcal{W}}$ by part (i) of this lemma. □

Since by Lemma 8.3(v) the columns of $G(t, \cdot, \lambda, \alpha)$ belong to $L^2_{\mathcal{W}}$, the function

$$\widehat{z}(t, \lambda, \alpha) := -\int_a^\infty G(t, \sigma(s), \lambda, \alpha)\widetilde{\mathcal{W}}(s)f^\sigma(s)\Delta s, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{8.9}$$

is well defined whenever $f \in L^2_{\mathcal{W}}$. Moreover, by using (8.6), we can write $\widehat{z}(t, \lambda, \alpha)$ as

$$\begin{aligned} \widehat{z}(t, \lambda, \alpha) = &-\mathcal{X}_+(t, \lambda, \alpha)\int_a^t \widetilde{Z}^{\sigma *}\left(s, \overline{\lambda}, \alpha\right)\widetilde{\mathcal{W}}(s)f^\sigma(s)\Delta s \\ &- \widetilde{Z}(t, \lambda, \alpha)\int_t^\infty \mathcal{X}_+^{\sigma *}\left(s, \overline{\lambda}, \alpha\right)\widetilde{\mathcal{W}}(s)f^\sigma(s)\Delta s, \quad t \in [a, \infty)_{\mathbb{T}}. \end{aligned} \tag{8.10}$$

Proposition 8.4. *For $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $f \in L^2_{\mathcal{W}}$, the function $\widehat{z}(\cdot, \lambda, \alpha)$ defined in* (8.9) *solves the nonhomogeneous system* (8.1) *with the initial condition $\alpha\widehat{z}(a, \lambda, \alpha) = 0$.*

Proof. By the time scales product rule (2.1) when we Δ-differentiate expression (8.10), we have for every $t \in [a,\infty)_{\mathbb{T}}$ (suppressing the dependence on α in the the following calculation)

$$\begin{aligned}
\widehat{z}^{\Delta}(t,\lambda) &= -\mathcal{X}_{+}^{\Delta}(t,\lambda)\int_a^t \widetilde{Z}^{\sigma *}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s - \mathcal{X}_{+}^{\sigma}(t,\lambda)\widetilde{Z}^{\sigma *}\left(t,\overline{\lambda}\right)\widetilde{\mathcal{W}}(t)f^{\sigma}(t) \\
&\quad - \widetilde{Z}^{\Delta}(t,\lambda)\int_t^{\infty} \mathcal{X}_{+}^{\sigma *}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s + \widetilde{Z}^{\sigma}(t,\lambda)\mathcal{X}_{+}^{\sigma *}\left(t,\overline{\lambda}\right)\widetilde{\mathcal{W}}(t)f^{\sigma}(t) \\
&= \mathcal{S}(t,\lambda)\widehat{z}(t,\lambda) - \left[\mathcal{X}_{+}^{\sigma}(t,\lambda)\widetilde{Z}^{\sigma *}\left(t,\overline{\lambda}\right) - \widetilde{Z}^{\sigma}(t,\lambda)\mathcal{X}_{+}^{\sigma *}\left(t,\overline{\lambda}\right)\right]\widetilde{\mathcal{W}}(t)f^{\sigma}(t) \\
&\overset{(8.7)}{=} \mathcal{S}(t,\lambda)\widehat{z}(t,\lambda) - \mathcal{J}\widetilde{\mathcal{W}}(t)f^{\sigma}(t).
\end{aligned} \tag{8.11}$$

This shows that $\widehat{z}(\cdot,\lambda,\alpha)$ is a solution of system (8.1). From (8.10) with $t = a$, we get

$$\alpha\widehat{z}(a,\lambda,\alpha) = -\alpha\widetilde{Z}(a,\lambda,\alpha)\int_a^{\infty} \mathcal{X}_{+}^{\sigma *}\left(s,\overline{\lambda},\alpha\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s = 0, \tag{8.12}$$

where we used the initial condition $\widetilde{Z}(a,\lambda,\alpha) = -\mathcal{J}\alpha^{*}$ and $\alpha\mathcal{J}\alpha^{*} = 0$ coming from $\alpha \in \Gamma$. □

The following theorem provides further properties of the solution $\widehat{z}(\cdot,\lambda,\alpha)$ of system (8.1). It is a generalization of [10, Lemma 4.2], [11, Theorem 7.5], [2, Theorem 5.2] to time scales.

Theorem 8.5. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $f \in L^2_{\mathcal{W}}$. Suppose that system $(\mathcal{S}_\lambda)$ is in the limit point or limit circle case. Then the solution $\widehat{z}(\cdot,\lambda,\alpha)$ of system* (8.1) *defined in* (8.9) *belongs to $L^2_{\mathcal{W}}$ and satisfies*

$$\|\widehat{z}(\cdot,\lambda,\alpha)\|_{\mathcal{W}} \le \frac{1}{|\mathrm{Im}(\lambda)|}\|f\|_{\mathcal{W}}, \tag{8.13}$$

$$\lim_{t\to\infty} \mathcal{X}_{+}^{*}(t,\nu,\alpha)\mathcal{J}\widehat{z}(t,\lambda,\alpha) = 0, \quad \textit{for every } \nu \in \mathbb{C} \setminus \mathbb{R}. \tag{8.14}$$

Proof. To shorten the notation we suppress the dependence on α in all quantities appearing in this proof. Assume first that system $(\mathcal{S}_\lambda)$ is in the limit point case. For every $r \in [a,\infty)_{\mathbb{T}}$ we define the function $f_r(\cdot) := f(\cdot)$ on $[a,r]_{\mathbb{T}}$ and $f_r(\cdot) := 0$ on $(r,\infty)_{\mathbb{T}}$ and the function

$$\widehat{z}_r(t,\lambda) := -\int_a^{\infty} G(t,\sigma(s),\lambda)\widetilde{\mathcal{W}}(s)f_r^{\sigma}(s)\Delta s = -\int_a^{r} G(t,\sigma(s),\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.15}$$

For every $t \in [r,\infty)_{\mathbb{T}}$ we have as in (8.10) that

$$\widehat{z}_r(t,\lambda) = -\mathcal{X}_{+}(t,\lambda)g(r,\lambda), \quad g(r,\lambda) := \int_a^{r} \widetilde{Z}^{\sigma *}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.16}$$

Since by Theorem 6.7 the solution $\mathcal{X}_+(\cdot,\lambda) \in L^2_{\mathcal{W}}$, (8.16) shows that $\widehat{z}_r(\cdot,\lambda)$, being a multiple of $\mathcal{X}_+(\cdot,\lambda)$, also belongs to $L^2_{\mathcal{W}}$. Moreover, by Theorem 7.3,

$$\lim_{t\to\infty} \widehat{z}_r^*(t,\lambda)\mathcal{J}\widehat{z}_r(t,\lambda) \overset{(8.16)}{=} g^*(r,\lambda)\lim_{t\to\infty}\mathcal{X}_+^*(t,\lambda)\mathcal{J}\mathcal{X}_+(t,\lambda)g(r,\lambda) \overset{(7.4)}{=} 0. \tag{8.17}$$

On the other hand, $\widehat{z}_r^*(a,\lambda)\mathcal{J}\widehat{z}_r(a,\lambda) = 0$, and for any $t \in [a,\infty)_{\mathbb{T}}$ identity (8.3) implies

$$\begin{aligned}
&\widehat{z}_r^*(t,\lambda)\mathcal{J}\widehat{z}_r(t,\lambda)\\
&\quad = -2i\,\mathrm{Im}(\lambda)\int_a^t \widehat{z}_r^{\sigma *}(s,\lambda)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s + 2i\,\mathrm{Im}\left(\int_a^t \widehat{z}_r^{\sigma *}(s,\lambda)\widetilde{\mathcal{W}}(s)f_r^{\sigma}(s)\Delta s\right).
\end{aligned} \tag{8.18}$$

Combining (8.18), where $t \to \infty$, formula (8.17), and the definition on $f_r(\cdot)$ yields

$$\|\widehat{z}_r(\cdot,\lambda)\|^2_{\mathcal{W}} = \int_a^\infty \widehat{z}_r^{\sigma *}(s,\lambda)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s = \frac{1}{\mathrm{Im}(\lambda)}\,\mathrm{Im}\left(\int_a^r \widehat{z}_r^{\sigma *}(s,\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s\right). \tag{8.19}$$

By using the Cauchy-Schwarz inequality (C-S) on time scales and $\widetilde{\mathcal{W}}(\cdot) \geq 0$, we then have

$$\begin{aligned}
\|\widehat{z}_r(\cdot,\lambda)\|^2_{\mathcal{W}} &= \frac{1}{2i\,\mathrm{Im}(\lambda)}\left[\int_a^r \widehat{z}_r^{\sigma *}(s,\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s - \int_a^r f^{\sigma *}(s)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s\right]\\
&\leq \frac{1}{|\mathrm{Im}(\lambda)|}\left|\int_a^r \widehat{z}_r^{\sigma *}(s,\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s\right|\\
&\overset{\text{C-S}}{\leq} \frac{1}{|\mathrm{Im}(\lambda)|}\left(\int_a^r \widehat{z}_r^{\sigma *}(s,\lambda)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s\right)^{1/2}\left(\int_a^r f^{\sigma *}(s)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s\right)^{1/2}\\
&\leq \frac{1}{|\mathrm{Im}(\lambda)|}\|\widehat{z}_r(\cdot,\lambda)\|_{\mathcal{W}}\|f\|_{\mathcal{W}}.
\end{aligned} \tag{8.20}$$

Since $\|\widehat{z}_r(\cdot,\lambda)\|_{\mathcal{W}}$ is finite by $\widehat{z}_r(\cdot,\lambda) \in L^2_{\mathcal{W}}$, we get from the above calculation that

$$\|\widehat{z}_r(\cdot,\lambda)\|_{\mathcal{W}} \leq \frac{1}{|\mathrm{Im}(\lambda)|}\|f\|_{\mathcal{W}}. \tag{8.21}$$

We will prove that (8.21) implies estimate (8.13) by the convergence argument. For any $t, r \in [a,\infty)_{\mathbb{T}}$ we observe that

$$\widehat{z}(t,\lambda) - \widehat{z}_r(t,\lambda) = -\int_r^\infty G(t,\sigma(s),\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.22}$$

Now we fix $q \in [a,r)_{\mathbb{T}}$. By the definition of $G(\cdot,\cdot,\lambda)$ in (8.5) we have for every $t \in [a,q]_{\mathbb{T}}$

$$\widehat{z}(t,\lambda) - \widehat{z}_r(t,\lambda) = -\widetilde{Z}(t,\lambda)\int_r^\infty \mathcal{X}_+^*\left(\sigma(s),\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.23}$$

Since the functions $\mathcal{X}_+(\cdot,\overline{\lambda})$ and $f(\cdot)$ belong to $L^2_{\mathcal{W}}$, it follows that the right-hand side of (8.23) converges to zero as $r \to \infty$ for every $t \in [a,q]_{\mathbb{T}}$. Hence, $\widehat{z}_r(\cdot,\lambda)$ converges to the function $\widehat{z}(\cdot,\lambda)$ uniformly on $[a,q]_{\mathbb{T}}$. Since $\widehat{z}(\cdot,\lambda) = \widehat{z}_r(\cdot,\lambda)$ on $[a,q]_{\mathbb{T}}$, we have by $\widetilde{\mathcal{W}}(\cdot) \geq 0$ and (8.21) that

$$\int_a^q \widehat{z}^{\sigma *}(s,\lambda)\widetilde{\mathcal{W}}(s)\widehat{z}^{\sigma}(s,\lambda)\Delta s \leq \|\widehat{z}_r(\cdot,\lambda)\|^2_{\mathcal{W}} \overset{(8.21)}{\leq} \frac{1}{|\mathrm{Im}(\lambda)|^2}\|f\|^2_{\mathcal{W}}. \tag{8.24}$$

Since $q \in [a,\infty)_{\mathbb{T}}$ was arbitrary, inequality (8.24) implies the result in (8.13). In the limit circle case inequality (8.13) follows by the same argument by using the fact that all solutions of system $(\mathcal{S}_\lambda)$ belong to $L^2_{\mathcal{W}}$.

Now we prove the existence of the limit (8.14). Assume that the system $(\mathcal{S}_\lambda)$ is in the limit point case, and let $\nu \in \mathbb{C} \setminus \mathbb{R}$ be arbitrary. Following the argument in the proof of [30, Lemma 4.1] and [2, Theorem 5.2], we have from identity (8.3) that for any $r,t \in [a,\infty)_{\mathbb{T}}$

$$\begin{aligned}\mathcal{X}^*_+(t,\nu)\mathcal{J}\widehat{z}_r(t,\lambda) &= \mathcal{X}^*_+(a,\nu)\mathcal{J}\widehat{z}_r(a,\lambda) + (\overline{\nu}-\lambda)\int_a^t \mathcal{X}^{\sigma *}_+(s,\nu)\widetilde{\mathcal{W}}(s)\widehat{z}^{\sigma}_r(s,\lambda)\Delta s \\ &\quad + \int_a^t \mathcal{X}^{\sigma *}_+(s,\nu)\widetilde{\mathcal{W}}(s)f^{\sigma}_r(s)\Delta s.\end{aligned} \tag{8.25}$$

Since for $t \in [r,\infty)_{\mathbb{T}}$ equality (8.16) holds, it follows that

$$\lim_{t\to\infty}\mathcal{X}^*_+(t,\nu)\mathcal{J}\widehat{z}_r(t,\lambda) = -\lim_{t\to\infty}\mathcal{X}^*_+(t,\nu)\mathcal{J}\mathcal{X}_+(t,\lambda)g(r,\lambda) \overset{(7.4)}{=} 0. \tag{8.26}$$

Hence, by (8.25),

$$\mathcal{X}^*_+(a,\nu)\mathcal{J}\widehat{z}_r(a,\lambda) = (\lambda-\overline{\nu})\int_a^\infty \mathcal{X}^{\sigma *}_+(s,\nu)\widetilde{\mathcal{W}}(s)\widehat{z}^{\sigma}_r(s,\lambda)\Delta s - \int_a^r \mathcal{X}^{\sigma *}_+(s,\nu)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.27}$$

By the uniform convergence of $\widehat{z}_r(\cdot,\lambda)$ to $\widehat{z}(\cdot,\lambda)$ on compact intervals, we get from (8.27) with $r \to \infty$ the equality

$$\mathcal{X}^*_+(a,\nu)\mathcal{J}\widehat{z}(a,\lambda) = (\lambda-\overline{\nu})\int_a^\infty \mathcal{X}^{\sigma *}_+(s,\nu)\widetilde{\mathcal{W}}(s)\widehat{z}^{\sigma}(s,\lambda)\Delta s - \int_a^\infty \mathcal{X}^{\sigma *}_+(s,\nu)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.28}$$

On the other hand, by (8.3), we obtain for every $t \in [a,\infty)_{\mathbb{T}}$

$$\begin{aligned}\mathcal{X}^*_+(t,\nu)\mathcal{J}\widehat{z}(t,\lambda) &= \mathcal{X}^*_+(a,\nu)\mathcal{J}\widehat{z}(a,\lambda) + (\overline{\nu}-\lambda)\int_a^t \mathcal{X}^{\sigma *}_+(s,\nu)\widetilde{\mathcal{W}}(s)\widehat{z}^{\sigma}(s,\lambda)\Delta s \\ &\quad + \int_a^t \mathcal{X}^{\sigma *}_+(s,\nu)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s.\end{aligned} \tag{8.29}$$

Upon taking the limit in (8.29) as $t \to \infty$ and using equality (8.28), we conclude that the limit in (8.14) holds true.

In the limit circle case, the limit in (8.14) can be proved similarly as above, because all the solutions of system $(\mathcal{S}_\lambda)$ now belong to $L^2_{\mathcal{W}}$. However, in this case, we can apply a direct argument to show that (8.14) holds. By formula (8.10) we get for every $t \in [a,\infty)_{\mathbb{T}}$

$$\begin{aligned}\mathcal{K}_+^*(t,\nu)\mathcal{J}\widehat{z}(t,\lambda) &= -\mathcal{K}_+^*(t,\nu)\mathcal{J}\mathcal{K}_+(t,\lambda)\int_a^t \widetilde{Z}^{\sigma *}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^\sigma(s)\Delta s \\ &\quad - \mathcal{K}_+^*(t,\nu)\mathcal{J}\widetilde{Z}(t,\lambda)\int_t^\infty \mathcal{K}_+^{\sigma *}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^\sigma(s)\Delta s.\end{aligned} \tag{8.30}$$

The limit of the first term in (8.30) is zero because $\mathcal{K}_+^*(t,\nu)\mathcal{J}\mathcal{K}_+(t,\lambda)$ tends to zero for $t \to \infty$ by (7.4), and it is multiplied by a convergent integral as $t \to \infty$. Since the columns of $\widetilde{Z}(\cdot,\lambda)$ belong to $L^2_{\mathcal{W}}$, the function $\mathcal{K}_+^*(\cdot,\nu)\mathcal{J}\widetilde{Z}(\cdot,\lambda)$ is bounded on $[a,\infty)_{\mathbb{T}}$, and it is multiplied by an integral converging to zero as $t \to \infty$. Therefore, formula (8.14) follows. □

In the last result of this paper we construct another solution of the nonhomogeneous system (8.1) satisfying condition (8.14) and such that it starts with a possibly nonzero initial condition at $t = a$. It can be considered as an extension of Theorem 8.5.

Corollary 8.6. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Assume that $(\mathcal{S}_\lambda)$ is in the limit point or limit circle case. For $f \in L^2_{\mathcal{W}}$ and $v \in \mathbb{C}^n$ we define*

$$\widetilde{z}(t,\lambda,\alpha) := \mathcal{K}_+(t,\lambda,\alpha)v + \widehat{z}(t,\lambda,\alpha), \quad \forall t \in [a,\infty)_{\mathbb{T}}, \tag{8.31}$$

where $\widehat{z}(\cdot,\lambda,\alpha)$ is given in (8.9). *Then $\widetilde{z}(\cdot,\lambda,\alpha)$ solves the system* (8.1) *with $\alpha\widetilde{z}(a,\lambda,\alpha) = v$,*

$$\|\widetilde{z}(\cdot,\lambda,\alpha)\|_{\mathcal{W}} \le \frac{1}{|\mathrm{Im}(\lambda)|}\|f\|_{\mathcal{W}} + \|\mathcal{K}_+(\cdot,\lambda,\alpha)v\|_{\mathcal{W}}, \tag{8.32}$$

$$\lim_{t\to\infty}\mathcal{K}_+^*(t,\nu,\alpha)\mathcal{J}\widetilde{z}(t,\lambda,\alpha) = 0, \quad \textit{for every } \nu \in \mathbb{C}\setminus\mathbb{R}. \tag{8.33}$$

In addition, if the system $(\mathcal{S}_\lambda)$ is in the limit point case, then $\widetilde{z}(\cdot,\lambda,\alpha)$ is the only $L^2_{\mathcal{W}}$ solution of (8.1) *satisfying $\alpha\widetilde{z}(a,\lambda,\alpha) = v$.*

Proof. As in the previous proof we suppress the dependence on α. Since the function $\mathcal{K}_+(\cdot,\lambda)v$ solves $(\mathcal{S}_\lambda)$, it follows from Proposition 8.4 that $\widetilde{z}(\cdot,\lambda,\alpha)$ solves the system (8.1) and $\alpha\widetilde{z}(a,\lambda) = \alpha\mathcal{K}_+(a,\lambda)v = v$. Next, $\widetilde{z}(\cdot,\lambda) \in L^2_{\mathcal{W}}$ as a sum of two $L^2_{\mathcal{W}}$ functions. The limit in (8.33) follows from the limit (8.14) of Theorem 8.5 and from identity (7.4), because

$$\lim_{t\to\infty}\mathcal{K}_+^*(t,\nu)\mathcal{J}\widetilde{z}(t,\lambda) = \lim_{t\to\infty}\{\mathcal{K}_+^*(t,\nu)\mathcal{J}\mathcal{K}_+(t,\lambda)v + \mathcal{K}_+^*(t,\nu)\mathcal{J}\widehat{z}(t,\lambda)\} = 0. \tag{8.34}$$

Inequality (8.32) is obtained from estimate (8.13) by the triangle inequality.

Now we prove the uniqueness of $\tilde{z}(\cdot,\lambda)$ in the case of $(\mathcal{S}_\lambda)$ being of the limit point type. If $z_1(\cdot,\lambda)$ and $z_2(\cdot,\lambda)$ are two $L^2_{\mathcal{W}}$ solutions of (8.1) satisfying $\alpha z_1(a,\lambda) = v = \alpha z_2(a,\lambda)$, then their difference $z(\cdot,\lambda) := z_1(\cdot,\lambda) - z_2(\cdot,\lambda)$ also belongs to $L^2_{\mathcal{W}}$ and solves system $(\mathcal{S}_\lambda)$ with $\alpha z(\cdot,\lambda) = 0$. Since $z(\cdot,\lambda) = \Psi(\cdot,\lambda)c$ for some $c \in \mathbb{C}^{2n}$, the initial condition $\alpha z(\cdot,\lambda) = 0$ implies through (4.7) that $z(\cdot,\lambda) = \tilde{Z}(\cdot,\lambda)d$ for some $d \in \mathbb{C}^n$. If $d \neq 0$, then $z(\cdot,\lambda) \notin L^2_{\mathcal{W}}$, because in the limit point case the columns of $\tilde{Z}(\cdot,\lambda)$ do not belong to $L^2_{\mathcal{W}}$, which is a contradiction. Therefore, $d = 0$ and the uniqueness of $\tilde{z}(\cdot,\lambda)$ is established. □

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