A Study on Becker’s Univalence Criteria

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We study univalence properties for certain subclasses of univalent functions $K_1$, $K_2$, $K_2, \mu$, and $S(p)$, respectively. These subclasses are associated with a generalized integral operator. The extended Becker-typed univalence criteria will be studied for these subclasses.

1. Introduction and Preliminaries

Let $A$ denote the class of analytic functions $f$ in the open unit disk $\mathbb{U} = \{ z : |z| < 1 \}$ normalized by $f(0) = f'(0) - 1 = 0$. Thus, each $f \in A$ has a Taylor series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$  \hfill (1.1)

Let $A_2$ be the subclass of $A$ consisting of functions of the form

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k.$$ \hfill (1.2)

Let $K_2$ be the univalent subclass of $A$ which satisfies

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1, \quad z \in \mathbb{U}.$$ \hfill (1.3)
Let $\mathcal{R}_2$ be the subclass of $\mathcal{R}$ for which $f''(0) = 0$. Let $\mathcal{R}_n$ be the subclass of $\mathcal{R}_2$ consisting of functions of the form (1.2) which satisfy

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu, \quad 0 < \mu \leq 1, \ z \in \mathbb{U}. \quad (1.4)$$

Next, we define a subclass $S(p)$ of $A$ consisting of all functions $f(z)$ that satisfy

$$\left| \left( \frac{z}{(f(z))} \right)^p \right| \leq p, \quad 0 < p \leq 2, \ p \in \mathbb{R}, \ z \in \mathbb{U}. \quad (1.5)$$

For functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) $f \ast g$ is defined as usual by

$$(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \quad (1.6)$$

Define the function $\varphi(a, c; z)$ by

$$\varphi(a, c; z) = z \varphi(1, a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{-k-1}, \quad c \neq 0, -1, 2, \ldots, \quad (1.7)$$

where $(a)_k$ is the famous Pochhammer symbol defined in terms of Gamma function. It is easily seen that $\varphi(2 - a, 2; z)$ is a convex function, since $z \varphi''(z) = \varphi(2 - a, 1; z) \in \mathbb{V}(\alpha)$.

Using the fractional derivative of order $a$, $D_a^\alpha [1]$, Owa and Srivastava [2] introduced the operator $\Omega^a : A \rightarrow A$ which is known as an extension of fractional derivative and fractional integral, as follows:

$$\Omega^a f(z) = \Gamma(2 - a)z^a D_a^\alpha f(z), \quad \alpha \neq 2, 3, 4, \ldots$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-a)}{\Gamma(k+1-a)} a_k z^k$$

$$= \varphi(2, 2 - a; z) \ast f(z). \quad (1.8)$$

Note that $\Omega^0 f(z) = f(z)$.

For a function $f$ in $A$, we define $D^\alpha_1(\alpha, \beta, \mu) f(z) : A \rightarrow A$, the linear fractional differential operator, as follows:

$$I^\alpha_1(\alpha, \beta, \mu) f(z) = f(z),$$

$$I^\alpha_1(\alpha, \beta, \mu) f(z) = \left( \frac{\nu - \mu + \beta - \lambda}{\nu + \beta} \right) (\Omega^a f(z)) + \left( \frac{\mu + \lambda}{\nu + \beta} \right) z(\Omega^a f(z)).$$
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\[ I_{\lambda}^{\nu}(\alpha, \beta, \mu) f(z) = I_{\lambda}^{\nu} \left( I_{\lambda}^{\nu}(\alpha, \beta, \mu) f(z) \right) \]

\[ \vdots \]

\[ I_{\lambda}^{\nu}(\alpha, \beta, \mu) f(z) = I_{\lambda}^{\nu} \left( I_{\lambda}^{\nu-1}(\alpha, \beta, \mu) f(z) \right). \]

(1.9)

If \( f \) is given by (1.1), then by (1.8) and (1), we see that

\[ I_{\lambda}^{\nu}(\alpha, \beta, \mu) f(z) = z + \sum_{k=2}^{\infty} \left( \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \left( \frac{\nu + (\mu + \lambda)(k-1) + \beta}{\nu + \beta} \right) \right)^n a_k z^k. \]

(1.10)

From (1.8) and (1), \( D_{\mu, \nu}^{\alpha, \beta}(\alpha, \beta, \mu) f(z) \) can be written in terms of convolution as

\[ I_{\lambda}^{\nu}(\alpha, \beta, \mu) f(z) = \left\{ \varphi(2, 2 - \alpha; z) \ast g_{\mu, \nu}^{\alpha, \beta}(z) \cdots \varphi(2, 2 - \alpha; z) \ast g_{\mu, \nu}^{\alpha, \beta}(z) \right\} \ast f(z), \]

(1.11)

where

\[ g_{\mu, \nu}^{\alpha, \beta}(z) = \frac{z - ((\nu - \mu + \beta - \lambda)/(\nu + \beta))z^2}{(1 - z)^2} \]

\[ = z - \left( \frac{\nu - \mu + \beta - \lambda}{\nu + \beta} \right) z^2 \left( 1 + 2z + 3z^2 + \cdots \right) \]

\[ = z + \left( 1 + \frac{\mu + \lambda}{\nu + \beta} \right) z^2 + \left( 1 + 2\frac{\mu + \lambda}{\nu + \beta} \right) z^3 \cdots \]

(1.12)

\[ \vdots \]

\[ g_{\mu, \nu}^{\alpha, \beta}(z) = z + \sum_{k=2}^{\infty} \left( \frac{\nu + (\mu + \lambda)(k-1) + \beta}{\nu + \beta} \right) z^k, \]

\[ \varphi(2, 2 - \alpha; z) \ast g_{\mu, \nu}^{\alpha, \beta}(z) \cdots \varphi(2, 2 - \alpha; z) \ast g_{\mu, \nu}^{\alpha, \beta}(z) = n\text{-times product}, \]

(1.13)

which generalizes many operators. Indeed, if we choose suitably values of \( \alpha, \beta, \mu, \) and \( \nu \) in (1.12), we have the following.

(i) \( \beta = 1, \mu = 0, \) and \( \alpha = 0, \) we obtain \( D_{\alpha, \lambda}^{m}f(z) \) given by Aouf et al. [3].

(ii) \( \nu = 1, \beta = 0, \mu = 0, \) and \( \alpha = 0, \) we obtain \( D_{\lambda}^{m}f(z) \) given by Al-Oboudi [4].

(iii) \( \nu = 1, \beta = 0, \mu = 0, \lambda = 1, \) and \( \alpha = 0, \) we obtain \( D_{\lambda}^{m}f(z) \) given by Sălăgean [5].

(iv) \( \nu = 1, \beta = 1, \lambda = 1, \mu = 0, \) and \( \alpha = 0, \) we obtain \( I_{\lambda}^{m}f(z) \) given by Urалегадди and Somanath [6].

(v) \( \beta = 1, \lambda = 1, \mu = 0, \) and \( \alpha = 0, \) we obtain \( I_{\lambda}^{m}(\ell)f(z) \) given by Cho and Srivastava [7] and Cho and Kim [8].
(vi) \( \nu = 1, \beta = 0, \mu = 0, \lambda = 0, \) and \( n = 1, \) we obtain Owa and Srivastava differential operator [2].

(vii) \( \nu = 1, \beta = 0, \) and \( \mu = 0, \) we obtain \( D_\lambda^{n,\alpha} f(z) \) given by Al-Oboudi and Al-Amoudi [9, 10].

(viii) \( \beta = l, \mu = 0, \) and \( \alpha = p, \) we obtain \( I_p^n(\lambda, l) f(z) \) given by Catas [11].

(ix) \( \beta = l, \mu = 0, \alpha = p, \) and \( \lambda = 1, \) we obtain \( I_p^n(\lambda, l) f(z) \) given by Kumar et al. and Srivastava et al. respectively [12, 13].

Next, we introduce a new family of integral operator by using generalized differential operator already defined above.

For \( m \in \mathbb{N} \cup \{0\} \) and \( \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n, \rho \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, \) we define a family of integral operators \( Y_{\gamma, \eta, \lambda}(n, \rho, \nu, \alpha, \beta): A^m \to A^m \) by

\[
Y_{\gamma, \eta, \lambda}(n, \rho, \nu, \alpha, \beta : z) = \left\{ \rho \int_0^z t^{\rho-1} \prod_{i=1}^m \left( \frac{I_{\lambda_i}^{n,\nu}(\alpha_i, \beta_i, \mu_i, \eta_i)}{t} \right)^{\frac{1}{\alpha_i}} dt \right\}^{1/\rho}, \quad f_i \in A, \quad (1.14)
\]

which generalize many integral operators. In fact, if we choose suitable values of parameters in this type of operator, we get the following interesting operators.

(i) \( \nu = 1, \beta = 0, \mu = 0, \gamma_i = 1/\alpha_i, \) and \( \rho = 1, \) we obtain \( I(f_1, \ldots, f_m) \) given by Bulut [14].

(ii) \( n = 0, \nu = 1, \beta = 0, \mu = 0, \alpha = 0, \gamma_i = 1/(\alpha - 1), \) and \( \rho = n(\alpha - 1) + 1, \) we obtain \( F_{n,\alpha}(z) \) given by Breaz et al. [15].

(iii) \( n = 0, \nu = 1, \beta = 0, \mu = 0, \alpha = 0, \gamma_i = 1/\alpha_i, \) and \( \rho = 1, \) we obtain \( F_{\alpha}(z) \) given by D. Breaz and N. Breaz [16].

For our main result, we need the following lemmas.

**Lemma 1.1** (see [17, 18]). Let \( c \) be a complex number, \( |c| \leq 1, \) \( c \neq -1. \) If \( f(z) = z + az^2 + \cdots \) is a regular function in \( U \) and

\[
\left| c|z|^2 + \left( 1 - |z|^2 \right) \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in U,
\]

then the function \( f \) is regular and univalent in \( U. \)

**Lemma 1.2** (Schwarz Lemma). Let the function \( f(z) \) be regular in the disk \( U_R = \{ z \in \mathbb{C} : |z| < R \} \) with \( |f(z)| < M. \) If \( f(z) \) has one zero with multiply \( \geq m \) for \( z = 0, \) then

\[
|f(z)| \leq \frac{M}{R^m} |z|^m, \quad \forall z \in U_R,
\]

and equality holds only if \( f(z) = e^{i\theta}(M/R^m)|z|^m, \) where \( \theta \) is constant.
Lemma 1.3 (see [19]). Let $\delta$ be a complex number with $\Re \delta > 0$ such that $c \in \mathbb{C}$, $|c| \leq 1$, $c \neq -1$. If $f \in A$ satisfies the condition

$$
|cz^{2\delta} + (1 - |z|^{2\delta}) \frac{zf''(z)}{\delta f'(z)}| \leq 1, \quad \forall z \in \mathbb{U},
$$

(1.17)

then the function

$$
F_\delta(z) = \left\{ \delta \int_0^z t^{\delta-1} f'(t)dt \right\}^{1/\delta}
$$

(1.18)

is analytic and univalent in $\mathbb{U}$.

Lemma 1.4 (see [20]). If a function $f \in S(p)$, then

$$
\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leq p|z|^2, \quad \forall z \in \mathbb{U}.
$$

(1.19)

2. Univalence Properties

In this section, we will discuss the univalence properties of the new family of integral operators mentioned above.

Theorem 2.1. Let $c$ be a complex number, $|I_{\alpha,\beta,\mu,\eta}^{n,v}(\alpha, \beta, \mu, \eta) f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = 1, 2, 3, \ldots$ and $I_{\alpha,\beta,\mu,\eta}^{n,v}(\alpha, \beta, \mu, \eta) f_i(t) \in S(p_i)$ for $i = 1, 2, 3, \ldots$ such that

$$
\Re(\rho) \geq \sum_{i=1}^{\infty} \frac{((M_i - 1)p_i + 2)M_i - 1}{|\gamma_i|(M_i - 1)},
$$

(2.1)

where $\rho, \gamma_i$ are complex numbers. If

$$
|c| \leq 1 - \frac{1}{\Re(\rho)} \sum_{i=1}^{\infty} \frac{((M_i - 1)p_i + 2)M_i - 1}{|\gamma_i|(M_i - 1)}, \quad M_i \geq 1,
$$

(2.2)

then the family $\mathcal{Y}_{n,v,\lambda}(n, \rho, \nu, \alpha, \beta : z)$ is univalent.

Proof. Since $I_{\alpha,\beta,\mu,\eta}^{n,v}(\alpha, \beta, \mu, \eta) f_i(t) \in S(p_i)$, so by Lemma 1.4, we have

$$
\left| \frac{z^2(I_{\alpha,\beta,\mu,\eta}^{n,v}(\alpha, \beta, \mu, \eta) f_i(t))'}{(I_{\alpha,\beta,\mu,\eta}^{n,v}(\alpha, \beta, \mu, \eta) f_i(t))^2} - 1 \right| \leq p_i|z|^2, \quad \forall z \in \mathbb{U}.
$$

(2.3)

Now, by using hypothesis, we have

$$
|I_{\alpha,\beta,\mu,\eta}^{n,v}(\alpha, \beta, \mu, \eta) f_i(z)| \leq M_i,
$$

(2.4)
so by Lemma 1.3, we get

\[ |I_{k}(\alpha, \beta, \mu, \eta) f_i(z)| \leq M_i z, \quad \therefore R = 1. \]  

(2.5)

Let

\[ \frac{I_{k}^{\nu} (\alpha, \beta, \mu) f(z)}{z} = 1 + \sum_{k=2}^{\infty} \left( \frac{\Gamma(k + 1)\Gamma(2 - \alpha)}{\Gamma(k + 1 - \alpha)} \left( \frac{\nu + (\mu + \lambda)(k - 1) + \beta}{\nu + \beta} \right) \right) n_{k} z^{k-1} \neq 0 \]

\[ = 1 \quad \text{if} \ z = 0, \]

(2.6)

so

\[ \prod_{i=1}^{m} \left( \frac{I_{k}^{\nu} (\alpha, \beta, \mu) f_i(z)}{z} \right)^{1/y_i} = \left( \frac{I_{k}^{\nu} (\alpha, \beta, \mu, \eta) f_1(z)}{z} \right)^{1/y_1} \cdots \left( \frac{I_{k}^{\nu} (\alpha, \beta, \mu) f_m(z)}{z} \right)^{1/y_m} \]

\[ = 1. \]

(2.7)

Let

\[ F(z) = \int_{0}^{z} \left( \left( \frac{I_{k}^{\nu} (\alpha, \beta, \mu) f_1(t)}{t} \right)^{1/y_1} \cdots \left( \frac{I_{k}^{\nu} (\alpha, \beta, \mu) f_m(t)}{t} \right)^{1/y_m} \right) dt, \]

(2.8)

which implies that

\[ F'(z) = \left( \left( \frac{I_{k}^{\nu} (\alpha, \beta, \mu) f_1(t)}{t} \right)^{1/y_1} \cdots \left( \frac{I_{k}^{\nu} (\alpha, \beta, \mu) f_m(t)}{t} \right)^{1/y_m} \right), \]

\[ \frac{zF''(z)}{F'(z)} = \frac{1}{y_1} \left( \frac{z(I_{k}^{\nu} (\alpha, \beta, \mu, \eta) f_1(z))'}{I_{k}^{\nu} (\alpha, \beta, \mu, \eta) f_1(z)} - 1 \right) + \cdots + \frac{1}{y_m} \left( \frac{z(I_{k}^{\nu} (\alpha, \beta, \mu, \eta) f_m(z))'}{I_{k}^{\nu} (\alpha, \beta, \mu, \eta) f_m(z)} - 1 \right), \]

\[ \frac{zF''(z)}{F'(z)} = \sum_{i=1}^{m} \frac{1}{y_i} \left( \frac{z(I_{k}^{\nu} (\alpha, \beta, \mu, \eta) f_i(z))'}{I_{k}^{\nu} (\alpha, \beta, \mu, \eta) f_i(z)} - 1 \right). \]

(2.9)

(2.10)

This implies that

\[ \left| \frac{zF''(z)}{F'(z)} \right| \leq \sum_{i=1}^{m} \frac{1}{y_i} \left( \left| \frac{z(I_{k}^{\nu} (\alpha, \beta, \mu, \eta) f_i(z))'}{I_{k}^{\nu} (\alpha, \beta, \mu, \eta) f_i(z)} \right| + 1 \right), \]

(2.11)
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or

\[
\frac{|z^{F''}(z)|}{F'(z)} \leq \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( \left| \frac{z(I^n_{\alpha, \beta, \mu, \eta} f_i(z))'}{(I^n_{\alpha, \beta, \mu, \eta} f_i(z))^2} \right| (M_i + 1) \right). \tag{2.12}
\]

Using (2.5), we get

\[
\frac{|z^{F''}(z)|}{F'(z)} \leq \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( \frac{z(I^n_{\alpha, \beta, \mu, \eta} f_i(z))'}{(I^n_{\alpha, \beta, \mu, \eta} f_i(z))^2} - 1 \right) M_i + M_i + 1 \tag{2.13}
\]

This implies that

\[
\frac{|z^{F''}(z)|}{F'(z)} \leq \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( p_i |z|^2 M_i + M_i + 1 \right). \tag{2.14}
\]

By using (2.3), we get

\[
\frac{|z^{F''}(z)|}{F'(z)} \leq \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( p_i M_i + (M_i + M_i^2 + M_i^3 + \cdots) + 1 \right), \tag{2.15}
\]

which implies that

\[
\frac{|z^{F''}(z)|}{F'(z)} \leq \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( p_i M_i + (M_i + M_i^2 + M_i^3 + \cdots) + 1 \right), \tag{2.16}
\]

because \( M_i, M_i^2, M_i^3, \ldots \geq 1 \) implies that

\[
\frac{|z^{F''}(z)|}{F'(z)} \leq \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( p_i M_i + \left( M_i + \frac{M_i}{M_i - 1} \right) + 1 \right) = \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( p_i M_i + \left( \frac{2M_i - 1}{M_i - 1} \right) \right),
\]

\[
\frac{|z^{F''}(z)|}{F'(z)} \leq \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( p_i M_i^2 - p_i M_i + 2M_i - 1 \right) \leq \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( \left( p_i M_i - p_i + 2 \right) M_i - 1 \right),
\]

\[
\frac{|z^{F''}(z)|}{F'(z)} \leq \sum_{i=1}^{\infty} \frac{1}{|y_i|} \left( \left( (M_i - 1)p_i + 2 \right) M_i - 1 \right). \tag{2.17}
\]

Now, we calculate

\[
|cz^{2\rho} + (1 - |z|^{2\rho}) \frac{z^{F''}(z)}{\rho F'(z)}| \leq |c| + \frac{1}{|\rho|} \left| \frac{z^{F''}(z)}{F'(z)} \right| \leq |c| + \frac{1}{\Re(\rho)} \left| \frac{z^{F''}(z)}{F'(z)} \right|. \tag{2.18}
\]
This implies that
\[
|cz^{2\rho} + (1 - |z|^{2\rho}) \frac{z F''(z)}{\rho F'(z)}| < |c| + \frac{1}{\mathcal{R}(\rho)} \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|^{(M-1)}} \left( \frac{((M_i - 1)p_i + 2)M_i - 1}{M_i - 1} \right).
\]  
(2.19)

By using (2.46), we conclude that
\[
|cz^{2\rho} + (1 - |z|^{2\rho}) \frac{z F''(z)}{\rho F'(z)}| \leq |c| + \frac{1}{|\rho|} \frac{|z F''(z)|}{|F'(z)|} \leq 1.
\]  
(2.20)

Hence, by Lemma 1.3, the family of integral operators \( Y_{\gamma,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z) \) is univalent. \( \square \)

**Corollary 2.2.** Let \( c \) be a complex number, \(|I_{\lambda}^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(z)| \leq M, M \geq 1 \) for all \( i = \{1, 2, 3, \ldots\} \) and \( I_{\lambda}^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(t) \in S(p), M_i = M \geq 1, \) for all \( i = \{1, 2, 3, \ldots\} \) such that
\[
\mathcal{R}(\rho) \geq \sum_{i=1}^{\infty} \frac{((M - 1)p + 2)M - 1}{|\gamma_i|(M - 1)} 
\]  
(2.21)

where \( \rho, \gamma_i \) are complex numbers. If
\[
|c| \leq 1 - \frac{1}{\text{Re}(\rho)} \sum_{i=1}^{\infty} \frac{((M - 1)p_i + 2)M_i - 1}{|\gamma_i|(M - 1)}, \quad M \geq 1,
\]  
(2.22)

then the family \( Y_{\gamma,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z) \) is univalent.

**Corollary 2.3.** Let \( c \) be a complex number, \(|I_{\lambda}^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(z)| \leq M, M \geq 1, \) for all \( i = \{1, 2, 3, \ldots\} \) and the family \( I_{\lambda}^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(t) \in S(p), M_i = M \geq 1, |\gamma_i| = |\gamma|, \) for all \( i = \{1, 2, 3, \ldots\} \) such that
\[
\mathcal{R}(\rho) \geq \sum_{i=1}^{\infty} \frac{((M - 1)p + 2)M - 1}{|\gamma|(M - 1)} 
\]  
(2.23)

where \( \rho, \gamma_i \) are complex numbers. If
\[
|c| \leq 1 - \frac{1}{\text{Re}(\rho)} \sum_{i=1}^{\infty} \frac{((M - 1)p_i + 2)M_i - 1}{|\gamma|(M - 1)}, \quad M \geq 1,
\]  
(2.24)

then the family \( Y_{\gamma,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z) \) is univalent.

Using the method given in the proof of Theorem 2.1, one can prove the following results.
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**Theorem 2.4.** Let $c$ be a complex number, $|I_3^{n,v}(\alpha, \beta, \mu, \eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = 1, 2, 3, \ldots$ and the family $I_3^{n,v}(\alpha, \beta, \mu, \eta)f_i(t) \in S(p_i)$ for $i = 1, 2, 3, \ldots$ and $c$ such that

$$\Re(\rho) \geq \sum_{i=1}^{\infty} \frac{(p_iM_i - 1)(M_i + 1)}{|y_i|p_iM_i},$$  \hspace{1cm} (2.25)

where $\rho, y_i$ are complex numbers. If

$$|c| \leq 1 - \frac{1}{\Re(\rho)} \sum_{i=1}^{\infty} \frac{(p_iM_i - 1)(M_i + 1)}{|y_i|(p_iM_i)}, \hspace{1cm} M_i \geq 1,$$

then the family $Y_{\nu,\eta,\alpha}(n, \rho, \nu, \alpha, \beta : z)$ is univalent.

**Theorem 2.5.** Let $c$ be a complex number, $|I_3^{n,v}(\alpha, \beta, \mu, \eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = 1, 2, 3, \ldots, n$ and $I_3^{n,v}(\alpha, \beta, \mu, \eta)f_i(t) \in S(p_i)$ for $i = 1, 2, 3, \ldots, n$ such that

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(p_i(M_i - 1) + M_i^n - 2)M_i + 1}{|y_i|(M_i - 1)},$$  \hspace{1cm} (2.27)

where $\rho, y_i$ are complex numbers. If

$$|c| \leq 1 - \frac{1}{\Re(\rho)} \sum_{i=1}^{n} \frac{(p_i(M_i - 1) + M_i^n - 2)M_i + 1}{|y_i|(M_i - 1)}, \hspace{1cm} M_i \geq 1,$$

then the family $Y_{\nu,\eta,\alpha}(n, \rho, \nu, \alpha, \beta : z)$ is univalent.

**Theorem 2.6.** Let $c$ be a complex number, $|I_3^{n,v}(\alpha, \beta, \mu, \eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = 1, 2, 3, \ldots, n$ and $I_3^{n,v}(\alpha, \beta, \mu, \eta)f_i(t) \in S(p_i)$ for $i = 1, 2, 3, \ldots, n$ such that

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(p_i + (n(n + 1)/2))M_i - 1}{|y_i|},$$  \hspace{1cm} (2.29)

where $\rho, y_i$ are complex numbers. If

$$|c| \leq 1 - \frac{1}{\Re(\rho)} \sum_{i=1}^{n} \frac{(p_i + (n(n + 1)/2))M_i - 1}{|y_i|}, \hspace{1cm} M_i \geq 1,$$

then the family $Y_{\nu,\eta,\alpha}(n, \rho, \nu, \alpha, \beta : z)$ is univalent.

**Theorem 2.7.** Let $c$ be a complex number, $|I_3^{n,v}(\alpha, \beta, \mu, \eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = 1, 2, 3, \ldots, n$ and $I_3^{n,v}(\alpha, \beta, \mu, \eta)f_i(t) \in R_{2,\nu},$ for $i = 1, 2, 3, \ldots, n$ such that

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i + n(n + 1))M_i}{|y_i|},$$  \hspace{1cm} (2.31)
where $\rho$, $\gamma_i$ are complex numbers. If

$$|c| \leq 1 - \frac{1}{\text{Re}(\rho)} \sum_{i=1}^{n} \left( \frac{\mu_i + n(n + 1)}{|\gamma_i|} \right) M_i, \quad M_i \geq 1, \quad (2.32)$$

then the family $\nu_1,\nu_\lambda(n, \rho, \nu, \alpha, \beta : z)$ is univalent.

**Proof.** Using the proof of Theorem 2.1, we have

$$\left| \frac{zF''(z)}{F'(z)} \right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} \left( \left| \frac{z(I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(z))'}{(I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(z))'} - 1 \right| M_i + M_i + 1 \right), \quad (2.33)$$

Since $I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(t) \in R_{2,\mu}$, so by using (1.4), we get

$$\left| \frac{z^2 (I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(z))'}{(I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(z))'} - 1 \right| < \mu_i, \quad 0 < \mu \leq 1, \quad z \in \mathbb{U}. \quad (2.34)$$

So from (2.33), we get

$$\left| \frac{zF''(z)}{F'(z)} \right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} \left( \left| \frac{z(I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(z))'}{(I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(z))'} - 1 \right| M_i + M_i + 1 \right), \quad (2.35)$$

or

$$\left| \frac{zF''(z)}{F'(z)} \right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} (\mu_i M_i + 2M_i), \quad M_i > 1, \quad (2.36)$$

$$\left| \frac{zF''(z)}{F'(z)} \right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} \left( \mu_i M_i + 2M_i + 4M_i + \cdots + n\text{-times} \right), \quad M_i > 1, \quad (2.36)$$

$$\left| \frac{zF''(z)}{F'(z)} \right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} (\mu_i M_i + n(n + 1)M_i), \quad M_i > 1. \quad (2.36)$$

Now, we evaluate the expression

$$\left| c |z|^{2\rho} + \left( 1 - |z|^{2\rho} \right) \frac{zF''(z)}{\rho F'(z)} \right| \leq |c| + \frac{1}{|\rho|} \left| \frac{zF''(z)}{F'(z)} \right| \leq |c| + \frac{1}{\Re(\rho)} \left| \frac{zF''(z)}{F'(z)} \right|, \quad (2.37)$$

$$\left| c |z|^{2\rho} + \left( 1 - |z|^{2\rho} \right) \frac{zF''(z)}{\rho F'(z)} \right| \leq |c| + \frac{1}{\Re(\rho)} \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} (\mu_i M_i + n(n + 1)M_i).$$
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Using (2.45) and (2.46), we conclude that

$$
|c|z|^{2p} + \left(1 - |z|^{2p}\right)\frac{zF'(z)}{\rho F(z)} \leq 1.
$$

(2.38)

Hence by using Lemma 1.3, the family \( \mathcal{Y}_{n, \rho, \nu}(\alpha, \beta, \mu, \eta : z) \) is univalent.

\( \square \)

**Corollary 2.8.** Let \( c \) be a complex number, \( |I^{(n,v)}_\lambda(\alpha, \beta, \mu, \eta) f_i(z)| \leq M, \) \( M \geq 1 \) for all \( i = 1, 2, 3, \ldots, n \) and \( I^{(n,v)}_\lambda(\alpha, \beta, \mu, \eta) f_i(t) \in \mathcal{S}_{2, \mu}, \) for \( i = 1, 2, 3, \ldots, n \) such that

$$
\Re(\rho) \geq \sum_{i=1}^{n} \frac{\mu_i + n(n+1)M}{|\gamma_i|},
$$

(2.39)

where \( \rho, \gamma_i \) are complex numbers. If

$$
|c| \leq 1 - \frac{1}{\text{Re}(\rho)} \sum_{i=1}^{n} \frac{\mu_i + n(n+1)M}{|\gamma_i|}, \quad M \geq 1,
$$

(2.40)

then the family \( \mathcal{Y}_{n, \rho, \nu}(\alpha, \beta, \mu, \eta : z) \) is univalent.

**Corollary 2.9.** Let \( c \) be a complex number, \( |I^{(n,v)}_\lambda(\alpha, \beta, \mu, \eta) f_i(z)| \leq M, \) \( M \geq 1, \) \( |\gamma_i| = |\gamma| \) for all \( i = 1, 2, 3, \ldots, n \) and \( I^{(n,v)}_\lambda(\alpha, \beta, \mu, \eta) f_i(t) \in \mathcal{S}_{2, \mu}, \) for \( i = 1, 2, 3, \ldots, n \) such that

$$
\Re(\rho) \geq \sum_{i=1}^{n} \frac{\mu_i + n(n+1)M}{|\gamma|},
$$

(2.41)

where \( \rho, \gamma, \gamma_i \) are complex numbers. If

$$
|c| \leq 1 - \frac{1}{\text{Re}(\rho)} \sum_{i=1}^{n} \frac{\mu_i + n(n+1)M}{|\gamma|}, \quad M \geq 1,
$$

(2.42)

then the family \( \mathcal{Y}_{n, \rho, \nu}(\alpha, \beta, \mu, \eta : z) \) is univalent.

Using a similar method as in the proof of Theorem 2.7, one can prove the following results.

**Theorem 2.10.** Let \( c \) be a complex number, \( |I^{(n,v)}_\lambda(\alpha, \beta, \mu, \eta) f_i(z)| \leq M_i, \) \( M_i \geq 1 \) for all \( i = 1, 2, 3, \ldots, n \) and \( I^{(n,v)}_\lambda(\alpha, \beta, \mu, \eta) f_i(t) \in \mathcal{S}_{2, \mu}, \) for \( i = 1, 2, 3, \ldots, n \) such that

$$
\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i M_i - 1) M_i + M_i^n M_i}{|\gamma_i|(M_i - 1)},
$$

(2.43)
where \( \rho, \gamma_i \) are complex numbers. If
\[
|c| \leq 1 - \frac{1}{\text{Re}(\rho)} \sum_{i=1}^{n} \frac{1}{|\gamma_i|} \left( \mu_i M_i - 1 \right) M_i + M_i^n M_i,
\]
where \( M_i \geq 1 \),
\( \rho, \gamma_i \) are complex numbers. If
\[
|c| \leq 1 - \frac{1}{\text{Re}(\rho)} \sum_{i=1}^{n} \frac{1}{|\gamma_i|} \left( \mu_i M_i - 1 \right) M_i + M_i^n M_i,
\]
then the family \( \Upsilon_{\gamma_i, \eta, \lambda}(n, \rho, \nu, \alpha, \beta : z) \) is univalent.

Theorem 2.11. Let \( c \) be a complex number, \( |I_{\nu}^{\mu}(\alpha, \beta, \mu, \eta) f_i(z)| \leq M_i, M_i \geq 1 \) for all \( i = \{1, 2, 3, \ldots\} \) and \( I_{\nu}^{\mu}(\alpha, \beta, \mu, \eta) f_i(t) \in \mathbb{R}_{2, \rho}, \) for \( i = \{1, 2, 3, \ldots\} \) such that
\[
\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i M_i - \mu_i + 2) M_i - 1}{|\gamma_i| (M_i - 1)},
\]
where \( \rho, \gamma_i \) are complex numbers. If
\[
|c| \leq 1 - \frac{1}{\text{Re}(\rho)} \sum_{i=1}^{n} \frac{1}{|\gamma_i|} \left( \mu_i M_i - 1 \right) M_i + M_i^n M_i,
\]
then the family \( \Upsilon_{\gamma_i, \eta, \lambda}(n, \rho, \nu, \alpha, \beta : z) \) is univalent.

Note that some other related work involving integral operators regarding univalence criteria can also be found in [21–23].

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References
